Statistical properties of one-dimensional maps

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Goal

Existence and statistical properties of absolutely continuous invariant measures of:

- `Non-degenerate smooth interval maps`. We will assume them to be topologically exact and with all cycles hyperbolic repelling.
  - The reference measure will be the Lebesgue measure on the interval domain.
- `A complex rational map` f, viewed as dynamical system acting on its Julia set $J(f)$.
  - When $J(f) = \overline{\mathbb{C}}$, the reference measure will be spherical measure.
  - When $J(f) \neq \overline{\mathbb{C}}$, the reference measure will be a conformal measure of minimal exponent.

Overview

Main goal:

Large Derivatives $\Rightarrow$ super-polynomially mixing a.c.i.p.

First lecture:

Large Derivatives

$\Rightarrow$ Backward Contraction

$\Rightarrow$ Super-polynomial Shrinking of Components

Second lecture:

- inducing scheme;
- reduction of the exponential case to the "Key Lemma".

Goals for today:

- complete the proof of the exponential case, by showing the "Key Lemma";
- tail estimates for non-exponential rates.

Pull-backs

f real or complex one-dimensional map;
V subset of the ambient space ($[0, 1]$ or $\overline{\mathbb{C}}$);
$m \geq 1$ an integer.

Definition

- A pull-back of V by f^m is a connected component of $f^{-m}(V)$.
- A pull-back W of V by f^m is diffeomorphic if the restriction of f^m to W is a diffeomorphism onto its image.

Observations:

- In this definition V could be disconnected.
- If W is a diffeomorphic pull-back of V by f^m, then $f^m(W)$ is a connected component of V.

Nice sets

f real or complex one-dimensional map.

Nice set for f: an open neighborhood V of $\mathrm{Crit}'(f)$ such that every connected component of V contains precisely a critical point of f and such that for $n \geq 1$

$$f^n(\partial V) \cap V = \emptyset.$$

For $c \in \mathrm{Crit}'(f)$ we denote by V^c the connected component of V containing c, so that

$$V = \bigsqcup_{c \in \mathrm{Crit}(f)} V^c.$$

Markovian property:
For each pull-back W of V

$$W \cap V = \emptyset \text{ or } W \subset V;$$

Nice couples

f a real or complex one-dimensional map.

Nice couple for f: a pair of nice sets $(\widehat{V}, V)$ such that

$$\overline{V} \subset \widehat{V}$$

and such that for every $n \geq 1$

$$f^n(\partial V) \cap \widehat{V} = \emptyset.$$

Joint Markovian property: *for each pull-back W of $\widehat{V}$*

$$W \cap V = \emptyset \text{ or } W \subset V.$$

Induced map

f a real or complex one-dimensional map;

$(\widehat{V}, V)$ nice couple for.

A good time for $x \in V$: an integer $m \geq 1$ such that

$$f^m(x) \in V$$

and such that the pull-back of $\widehat{V}$ by f^m containing x is diffeomorphic;

D: the set of points in V having a good time;

$m(x)$: the least good time of $x \in D$;

canonical induced map associated to $(\widehat{V}, V)$:

$$\begin{array}{rccc} F: & D & \longrightarrow & V \\ & x & \longmapsto & f^{m(x)}(x) \end{array}$$

Induced map

- The set D is a disjoint union of diffeomorphic pull-backs of V.
- In each connected component W of D the return time m is constant on W.
  We will denote this number by $m(W)$ and we will denote by $c(W)$ the critical point such that $f^{m(W)}(W) \subset V^{c(W)}$.
  Then the map
  $$f^{m(W)} : W \to V^{c(W)}$$
  extends to a diffeomorphism from a neighborhood of W onto $\widehat{V}^{c(W)}$.

Induced map

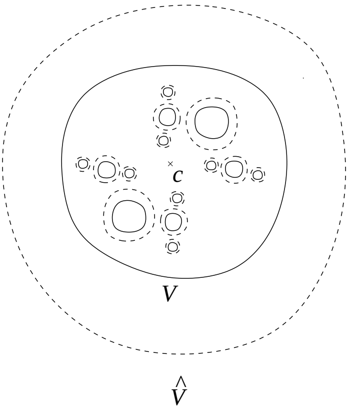

Figure: The domain of the induced map associated to $(\widehat{V}, V)$

Bad pull-backs

Definition

Given an integer $m \geq 1$ we will say that a pull-back $\widetilde{W}$ of $\widehat{V}$ by f^m is a bad pull-back of $\widehat{V}$ of order m if for every

$$m' \in \{1, \ldots, m\}$$

such that $f^{m'}(\widetilde{W}) \subset \widehat{V}$, the pull-back of $\widehat{V}$ containing $\widetilde{W}$ is not diffeomorphic.

Inducing scheme

Lemma

Let $(\widehat{V}, V)$ be a sufficiently small nice couple and let $F : D \to V$ be the corresponding induced map. Suppose that for some $\alpha \in (0, 1$ or $\mathrm{HD}(J(f)))$ we have

$$\sum_{\widetilde{W} \text{ bad pull-back of } \widehat{V}} \mathrm{diam}(\widetilde{W})^{\alpha} < +\infty.$$

Then

$$\mathrm{HD}((V \cap J(f)) \setminus J(F)) < \mathrm{HD}(J(f)).$$

In particular,

$$\mathrm{HD}(J(F)) = \mathrm{HD}(J(f)).$$

Inducing scheme

Applying results of Mauldin–Urbanski we obtain:

Theorem (Conformal measure)

Let $(\widehat{V}, V)$ be a sufficiently small nice couple and let $F : D \to V$ be the corresponding induced map. Suppose that for some $\alpha \in (0, 1$ or $\mathrm{HD}(J(f)))$,

$$\sum_{\widetilde{W} \text{ bad pull-back of } \widehat{V}} \mathrm{diam}(\widetilde{W})^{\alpha} < +\infty,$$

$$\sum_{W \text{ connected component of } D} \mathrm{diam}(W)^{\alpha} < +\infty.$$

Then there is a conformal measure μ for f of exponent $\mathrm{HD}(J(f))$ satisfying $\mathrm{HD}(\mu) = \mathrm{HD}(J(f))$ and such that $\mu(J(F)) > 0$.

Inducing scheme

Applying Young's theorem we obtain:

Theorem (A.c.i.p.)

Let $(\widehat{V}, V)$ be a sufficiently small nice couple and let $F : D \to V$ be the corresponding induced map. Assume that the hypotheses of the previous theorem are satisfied. If

$$\sum_{\substack{W \text{ connected component of } D \\ m(W) \geq n}} \operatorname{diam}(W)^{1 \text{ or } \operatorname{HD}(J(f))}$$

is exponentially small with n, then there is an exponentially mixing a.c.i.p. If it decreases super-polynomially, then there is a super-polynomially mixing a.c.i.p.

Exponential shrinking of components

Exponential Shrinking of Components:
There are constants $\theta \in (0,1)$, $C > 0$ and $\delta_0 > 0$ such that for every x, $\delta \in (0, \delta_0)$, every integer $m \geq 1$ and every connected component W of $f^{-m}(B(x,\delta))$ we have,

$$\mathrm{diam}(W) \leq C\theta^{-m}.$$

Exponential Shrinking of Components

$\Leftrightarrow$ Topological Collet–Eckmann condition

Exponential case

Exponential Shrinking of Components

$\Rightarrow$ existence of arbitrarily small nice couples

Furthermore,

$$\sum_{\widetilde{W} \text{ bad pull-back of } \widehat{V}} \text{diam}(\widetilde{W})^{\alpha} < +\infty.$$

follows from the counting estimate:

for a given $n \geq 1$ the number of bad pull-backs of V is sub-exponential in n.

Key Lemma

Key Lemma *Suppose f has the Exponential Shrinking of Components property. Then there is* $\alpha \in (0, 1$ *or* $\mathrm{HD}(J(f)))$ *such that*

$$\sum_{W \text{ connected component of } D} \mathrm{diam}(W)^{\alpha} < +\infty.$$

As we saw in the previous lecture,

$$\text{Key Lemma} \;\Rightarrow\; \text{exponential tail estimate}$$

combined with the previous results we get

Key Lemma ⇒ existence of an exponentially mixing a.c.i.p.

Structure of the induced map

f real or complex one-dimensional map;

$(\widehat{V}, V)$ nice couple for f;

$F : D \to V$ induced map associated to $(\widehat{V}, V)$.

$\mathfrak{D}$: the collection of connected components of D;

$\mathfrak{D}_0$: the collection of those $W \in \mathfrak{D}$ such that $m(W)$ is the first return time of W to V;

$\mathfrak{D}_{\widetilde{W}}$, for a bad pull-back $\widetilde{W}$ of $\widehat{V}$: the collection of those $W \in \mathfrak{D}$ contained in $\widetilde{W}$ such that $f^{m(\widetilde{W})}(W) \in \mathfrak{D}_0$.

Lemma

$$\mathfrak{D} = \mathfrak{D}_0 \sqcup \bigsqcup_{\widetilde{W} \text{ bad pull-back of } \widehat{V}} \mathfrak{D}_{\widetilde{W}}.$$

Structure of the induced map

f real or complex one-dimensional map;
$(\widehat{V}, V)$ nice couple for f;
$F : D \to V$ induced map associated to $(\widehat{V}, V)$.

$\mathfrak{L}_V :=$ connected components of the complement of $K(V)$.

It is the collection of connected components of the first landing map to V.

The collection of first return domains $\mathfrak{D}_0$ is equal to the pull-back of $\mathfrak{L}_V$ by $f|V$.

For a bad pull-back $\widetilde{W}$ of $\widehat{V}$, the collection $\mathfrak{D}_{\widetilde{W}}$ is the pull-back of $\mathfrak{L}_V$ by $f^{m(\widetilde{W})+1}|\widetilde{W}$.

Discrete density

Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ be the projective real line, so we have a natural action of group of (real) Möbius maps on $\overline{\mathbb{R}}$.

Given $\alpha > 0$ and a family $\mathfrak{F}$ of subsets of $\overline{\mathbb{R}}$ or $\overline{\mathbb{C}}$ we put

$$\|\mathfrak{F}\|_\alpha := \sup \left\{ \sum_{W \in \mathfrak{F}} \mathrm{diam}(\varphi(W))^\alpha : \varphi \text{ Möbius map} \right\}.$$

$$\mathrm{supp}(\mathfrak{F}) := \bigcup_{W \in \mathfrak{F}} W.$$

Lemma

$$\sum_{W \in \mathfrak{F}} \mathrm{diam}(W)^\alpha \leq \|\mathfrak{F}\|_\alpha \, \mathrm{diam}(\mathrm{supp}(\mathfrak{F}))^\alpha.$$

Discrete density

f real or complex one-dimensional map;
V nice set for f.

Lemma

If f is uniformly expanding on $K(V)$, then there is α between $\mathrm{HD}(K(V))$ and 1 or $\mathrm{HD}(J(f))$ such that

$$\sum_{W \in \mathfrak{L}_V} \mathrm{diam}(W)^{\alpha} < +\infty.$$

Furthermore,

$$\|\mathfrak{L}_V\|_{\alpha} < +\infty.$$

Discrete density

f real or complex one-dimensional map;
$(\widehat{V}, V)$ nice couple for f.

Given a bad pull-back $\widetilde{W}$ of $\widehat{V}$ of order n, let $\ell(\widetilde{W})$ be the number of those $m \in \{1, \ldots, n\}$ such that

$$f^m(\widetilde{W}) \subset \widehat{V}.$$

Geometric Lemma

- $$\|\mathfrak{D}_0\|_\alpha < +\infty.$$
- *There is a constant $C > 0$ such that for every bad pull-back $\widetilde{W}$ of $\widehat{V}$*
$$\|\mathfrak{D}_{\widetilde{W}}\|_\alpha \le C^{\ell(\widetilde{W})} \|\mathfrak{D}_0\|_\alpha.$$

Proof of the Key Lemma

$$\sum_{W \in \mathfrak{D}} \operatorname{diam}(W)^\alpha$$

$$\leq \sum_{W \in \mathfrak{D}_0} \operatorname{diam}(W)^\alpha + \sum_{\widetilde{W} \text{ bad pull-back of } \widehat{V}} \sum_{W \in \mathfrak{D}_{\widetilde{W}}} \operatorname{diam}(\widetilde{W})^\alpha$$

$$\leq \|\mathfrak{D}_0\|_\alpha \sum_c \operatorname{diam}(V^c)^\alpha$$

$$+ \|\mathfrak{D}_0\|_\alpha \sum_{\widetilde{W} \text{ bad pull-back of } \widehat{V}} C^{\ell(\widetilde{W})} \operatorname{diam}(\widetilde{W})^\alpha < +\infty.$$

Badness exponent

f real or complex one-dimensional map.

Badness exponent of a nice $\widehat{V}$ for f:

$$\delta_{\mathrm{bad}}(\widehat{V}) = \inf \left\{ t > 0 : \sum_{\widetilde{W} \text{ bad pull-back of } \widehat{V}} \mathrm{diam}(\widetilde{W})^t < +\infty \right\}.$$

Badness exponent of f:

$$\delta_{\mathrm{bad}}(f) = \inf \left\{ \delta_{\mathrm{bad}}(\widehat{V}) : \widehat{V} \text{ nice set for } f \right\}.$$

Badness exponent

Theorem

If f is backward contracting, then f has arbitrarily small nice couples and

$$\delta_{bad}(f) = 0.$$

Non-exponential tail estimates

We will say that a sequence of positive numbers $(\theta_m)_{m=1}^{+\infty}$ is slowly varying if

$$\lim_{n \to +\infty} \frac{\theta_{m+1}}{\theta_m} = 1.$$

Definition

Let $\Theta := (\theta_m)_{m=1}^{+\infty}$ be a non-increasing and slowly varying sequence of positive numbers. Then we will say that a map f has the Θ-Shrinking of Components property if there are constants $C > 0$ and $\delta_0 > 0$ such that for every x, every $\delta \in (0, \delta_0)$, every m and every pull-back W of $B(x, \delta)$ by f^m,

$$\operatorname{diam}(W) \leq C\theta_m.$$

Non-exponential tail estimates

Theorem

Let f be such that $\delta_{bad}(f) < 1$ or $\mathrm{HD}(J(f))$ and fix

$$t \in (\delta_{bad}(f), 1 \text{ or } \mathrm{HD}(J(f))).$$

Put $\Theta := (\theta_n)_{n=1}^{+\infty}$ a slowly varying and non-increasing sequence of positive numbers and assume that f has the Θ-Shrinking of Components property.
Then for every sufficiently small nice couple $(\widehat{V}, V)$ there is $C > 0$ and $\alpha \in (0, 1 \text{ or } \mathrm{HD}(J(f)))$ such that for every $m \geq 1$

$$\sum_{\substack{W \in \text{ connected component of } D \\ m(W) \geq m}} \mathrm{diam}(W)^{\alpha} \leq C \sum_{n=m}^{+\infty} \theta_n^{\alpha - t}.$$

Polynomially mixing a.c.i.p.s

Taking $\Theta := \left(n^{-\beta}\right)_{n=1}^{+\infty}$ in the previous theorem we obtain:

Corollary

Let f have Polynomial shrinking of components of exponent $\beta > 0$. If

$$\beta(1 - \delta_{bad}(f)) > 2$$

then for every

$$p < \beta(1 - \delta_{bad}(f)) - 2$$

the map f possesses a polynomially mixing a.c.i.p. of exponent p.

Pollynomially mixing a.c.i.p.s

Question

Does

Super-polynomial Shrinking of Components

$$\Rightarrow \ \delta_{bad}(f) = 0 \ ?$$