Perturbation of the dynamics of C^1-diffeomorphisms

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Recent advances in modern dynamics
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Survey (in french) ArXiv:0912.2896

Differentiable dynamics

Consider:

- M: compact boundaryless manifold,
- $\mathrm{Diff}^r(M)$, $r \geq 1$.

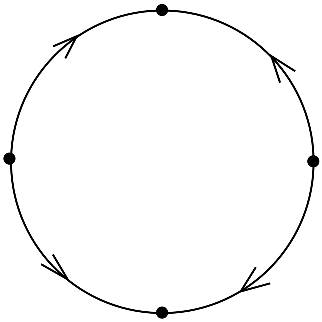
Goal 1: *understand the dynamics of "most" $f \in \mathrm{Diff}(M)$.*
"Most": at least a dense part.

Our viewpoint: describe a *generic* subset of $\mathrm{Diff}^1(M)$.
Generic (Baire): countable intersection of open and dense subsets.

Goal 2: *identify regions of $\mathrm{Diff}(M)$ with different dyn. behavior.*

Examples (1), in dimension 1

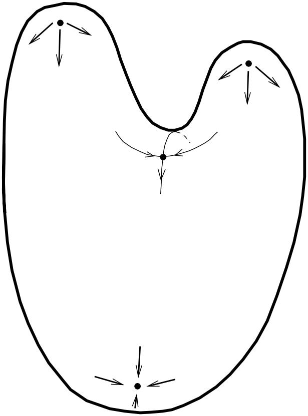
On $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$.



Morse-Smale dynamics are open and dense in $\mathrm{Diff}^r(\mathbb{T}^1)$.

Examples (2), in any dimension

time-one map of the gradient flow of a Morse function.



Definition

A Morse-Smale diffeomorphism:

- *finitely many hyperbolic periodic orbits,*
- *any other orbit is trapped: it meets $U \setminus f(\bar{U})$ where U open satisfies $f(\bar{U}) \subset U$.*

- Stable under perturbations.
- Zero topological entropy.

Examples (3): Hyperbolic diffeomorphisms

$f \in \mathrm{Diff}(M)$ is *hyperbolic* if there exists $K_0, \dots, K_d \subset M$ s.t.:

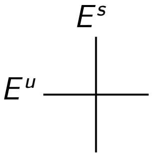
- each K_i is a hyperbolic invariant: $T_K M = E^s \oplus E^u$,
- any orbit in $M \setminus (\bigcup_i K_i)$ is trapped.

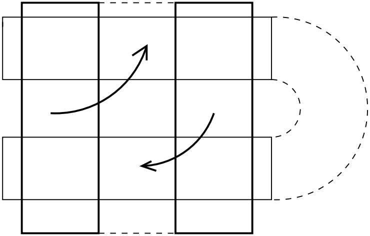
Good properties: Ω-stability, coding, physical measures,...

The set $hyp(M) \subset \mathrm{Diff}^r(M)$ of hyperbolic dynamics is open.

Examples (3): Hyperbolic diffeomorphisms

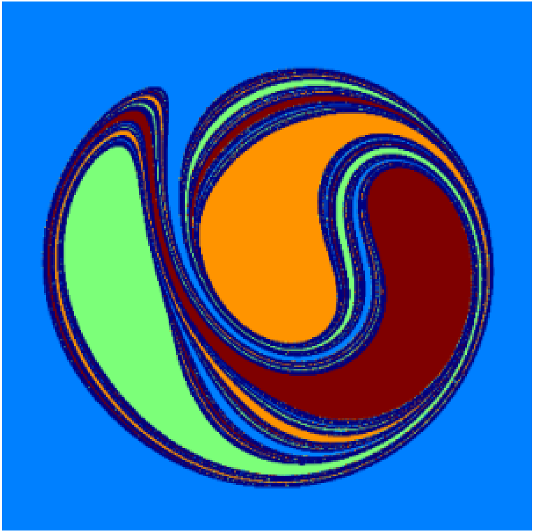
The Smale's horseshoe.





A hyperbolic diffeomorphism has positive topological entropy, iff there is a *transverse homoclinic orbit*

the Plykin attractor.



Examples (4): robust non-hyperbolic diffeomorphisms

The set $hyp(M) \subset \mathrm{Diff}^r(M)$ of hyperbolic dynamics is *not dense*,

when $\dim(M) = 2$, $r \geq 2$ (*Newhouse*)

or when $\dim(M) > 2$ and $r \geq 1$ (*Abraham-Smale*),

Smale's Conjecture:

The set $hyp(M) \subset \mathrm{Diff}^r(M)$ is *dense*, when $\dim(M) = 2$, $r = 1$.

Goal. Describe a **dense** set of diffeomorphisms $\mathcal{G} \subset \text{Diff}^1(M)$.

Definition. $\mathcal{G}$ is *generic* (Baire) if it contains a dense G_δ set (i.e. a countable intersection of open and dense subsets) of $\text{Diff}^1(M)$.

Rk. $\text{Diff}^1(M)$ is a Baire space.

Properties.

- $\mathcal{G}$ is generic $\Rightarrow$ $\mathcal{G}$ is dense.
- $\mathcal{G}_1$ and $\mathcal{G}_2$ are generic $\Rightarrow$ $\mathcal{G}_1 \cap \mathcal{G}_2$ is generic

Example: Kupka-Smale's Theorem.
Genericaly in $\text{Diff}^r(M)$, *the periodic orbits are hyperbolic.*

$$Per(f) \subset Rec^+(f) \subset L^+(f) \subset \Omega(f) \subset \mathcal{R}(f).$$

Definition. x is *chain-recurrent* iff for every $\varepsilon > 0$ it belongs to a periodic ε-pseudo-orbit.
The *chain-recurrent set* $\mathcal{R}(f)$ is the set of chain-recurrent points.

Property (Conley).
$M \setminus \mathcal{R}(f)$ is the set of points that are *trapped*.

Definition. $x \sim y$ is the equivalence relation on $\mathcal{R}(f)$:
"$\forall \varepsilon > 0$, x, y belong to a same periodic ε-pseudo-orbit".
The chain-recurrence classes are the equivalence classes of $\sim$.

Property (Conley).
– The chain-recurrence classes are compact and invariant.
– For any classes $K \neq K'$, there exists U open such that $K \subset U$, $K' \subset M \setminus U$ and either $f(\overline{U}) \subset U$ or $f^{-1}(\overline{U}) \subset U$.

Definition. A *quasi-attractor* is a class having arbitrarily small neighborhoods U s.t. $f(\overline{U}) \subset U$.

- There always exists a quasi-attractor.

C^1-perturbation lemmas (1)

For hyperbolic diffeomorphisms, pseudo-orbits are *shadowed*.
For arbitrary diffeomorphisms, this becomes false.

- Try to get it after a perturbation of the diffeomorphism!

With C^0**-small perturbations**, this is easy.

With C^1**-small perturbations**, this is much more difficult.

With C^r**-small perturbations,** $r > 1$, this is unknown.

Theorem (Pugh's closing lemma).
For any diffeomorphism f and any $x \in \Omega(f)$, there exists g close to f in $\mathrm{Diff}^1(M)$ such that x is periodic.

Theorem (Hayashi's connecting lemma).
For any f and any non-periodic x, y, z, if z is accumulated by forward iterates of x and by backwards iterates of y, then there are g close to f in $\mathrm{Diff}^1(M)$ and $n \geq 1$ such that $y = g^n(x)$.

Theorem [Bonatti – C] (Connecting lemma for pseudo-orbits).
For any f whose periodic orbits are hyperbolic and any x, y, if there exist ε-pseudo-orbits connecting x to y for any $\varepsilon > 0$, then there are g close to f in $\mathrm{Diff}^1(M)$ and $n \geq 1$ s.t. $y = g^n(x)$.

Theorem [C] (Global connecting lemma).
For any f whose periodic orbits are hyperbolic and any $x_0, \dots, x_k$, if there exist ε-pseudo-orbits connecting $x_0, \dots, x_k$ for any $\varepsilon > 0$, then there is g close to f in $\mathrm{Diff}^1(M)$ such that $x_0, \dots, x_k$ belong to a same orbit.

For C^1-generic diffeomorphisms:

- $\overline{Per(f)} = \mathcal{R}(f)$.
- Any chain-recurrence class is the Hausdorff limit of a sequence of periodic orbits.
- *Weak shadowing lemma*: for any $\delta > 0$, there exists $\varepsilon > 0$ such that any ε-pseudo-orbit $\{x_0, \dots, x_k\}$ is δ-close to a segment of orbit $\{x, f(x), \dots, f^n(x)\}$ for the Hausdorff distance.
- For any x in a dense G_δ set $X \subset M$, the accumulation set of its forward orbit is a quasi-attractor.

Let O be a hyperbolic periodic orbit.

Definition. The *homoclinic class* $H(O)$ is the closure of the set of transverse homoclinic orbits of O.

$$H(O) = \overline{W^s(O) \pitchfork W^u(O)}.$$

- It is a transitive set. Periodic points are dense.
- For hyperbolic diffeomorphisms,
  "homoclinic classes = chain-recurrence classes = basic sets."

Theorem [B – C] For C^1-generic f, the homoclinic classes are the chain-recurrence classes which contain a periodic orbit.

- Homoclinic classes may be described by their periodic orbits.
- The other chain-recurrence classes are called *aperiodic classes*.

Theorem [Bonatti – Díaz]. When $\dim(M) \geq 3$, there exists $\mathcal{U} \neq \emptyset$ open such that generic diffeomorphisms $f \in \mathcal{U}$:

- have aperiodic classes (carrying odometer dynamics),
- have uncountable many chain-recurrence classes,
- exhibit *universal dynamics*.

One expects [Potrie]: $\mathcal{U}'$ open s.t. generic diffeomorphisms $f \in \mathcal{U}'$ have infinitely many homoclinic classes and no aperiodic classes.

A pathology [B – C – Shinohara]. Pesin theory becomes false. There exists $\mathcal{U}''$ open such that generic diffeomorphisms $f \in \mathcal{U}$ have *hyperbolic* ergodic measures whose stables/unstable manifolds are reduced to points, a.e.

Perturbation of the dynamics of C^1-diffeomorphisms

1. General C^1-generic properties.
2. **Role of the homoclinic tangencies.**
3. Role of the heterodimensional cycles.

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Goal. *Split the space* Diff(M) *according to the dynamical behavior.*

- We look for subclasses of systems which:
  - either can be globally well described (phenomenon),
  - or exhibit a very simple local configuration, that generates rich instabilities (mecanisms).

- We are mostly interested by classes of systems that are **open**.

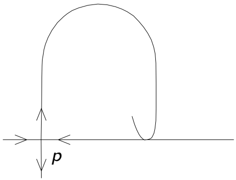
Example of decomposition:

Theorem. *There exists two disjoint open sets* $\mathcal{MS}, \mathcal{H} \subset \mathrm{Diff}^1(M)$ *whose union is dense:*

- $\mathcal{MS}$*: Morse-Smale diffeomorphisms,*
- $\mathcal{H}$*: diffeomorphism exhibiting a transverse homoclinic intersection.*

$\dim(M) = 2$: Pujals-Sambarino,
$\dim(M) = 3$: Bonatti-Gan-Wen,
$\dim(M) \geq 4$: C.

Example of mechanism: homoclinic tangencies.



Homoclinic tangency associated to a hyperbolic periodic point p.

- This mechanism is *fragile* (one-codimensional).

Definition. $f \in \mathrm{Diff}^r(M)$ exhibits a *C^r-robust homoclinic tangency* if there is a transitive hyperbolic set K s.t. for any g C^r-close to f, $W^s(x)$ and $W^u(y)$ have a tangency for some $x, y \in K_g$.

Theorem (Newhouse). *C^r-robust homoclinic tangency exist when:*

– $\dim(M) = 2$ *and* $r \geq 2$,

– $\dim(M) \geq 3$ *and* $r \geq 1$.

Property (Newhouse, Palis-Viana).

– *When* $\dim(M) = 2$, *for any open set* $\mathcal{U} \subset \mathrm{Diff}^r(M)$ *exhibiting a robust homoclinic tangency, generic diffeomorphisms in* $\mathcal{U}$ *have infinitely many sinks (hence chain-recurrence classes).*

– *When* $\dim(M) \geq 3$, *still true if the tangency is "sectionally dissipative".*

Rk (Bonatti-Viana). When $\dim(M) \geq 3$, there can exist simultaneously (other kind of) robust tangencies and only finitely many classes.

Homoclinic tangencies generate wild dynamics (2): universal dynamics

Definition. $f \in \mathrm{Diff}^r(M^d)$ is *C^r-universal*, if for any orientation preserving C^r embedding $g\colon B^d \to int(B^d)$, there exists:
– g' close to g,
– a ball $B \subset M$ and $n \geq 1$, such that $f^n(B) \subset B$,
satisfying $f^n_{|B} = g'$.

Theorem (Bonatti-Díaz). *Assume $d \geq 3$ and $r = 1$.*
Any f exhibiting "enough" C^1-robust homoclinic tangencies admits a C^1-neighborhood where C^1-universal dynamics is generic.

Theorem (Turaev). *Assume $d = 2$ and $r \geq 2$.*
Any f with a transitive hyperbolic set K such that:
– K has C^r-robust homoclinic tangency,
– K contains periodic points with Jacobian > 0 and < 0,
admits a C^r-neighborhood where C^r-universal dynamics is generic.

Definition. $f \in \mathrm{Diff}^r(M^d)$ is *C^r-universal*, if for any orientation preserving C^r embedding $g\colon B^d \to int(B^d)$, there exists:
– g' close to g,
– a ball $B \subset M$ and $n \geq 1$, such that $f^n(B) \subset B$,
satisfying $f^n_{|B} = g'$.

- Produces:
  - uncountable many chain-recurrence classes,
  - aperiodic classes (odometer type).

Weak form of hyperbolicity

Consider an invariant set K.

Definition. An invariant splitting $T_K M = E \oplus F$ is *dominated* if there is $N \geq 1$ s.t. for any $x \in K$ and any unitary $u \in E_x, v \in F_x$,

$$\|D_x f^N.u\| \leq \frac{1}{2}\|D_x f^N.v\|.$$

Properties. – still holds on the closure of K,

– still holds for invariant sets K' in a neighborhood U of K,

– prevents the existence in U of a periodic orbit O with stable dimension $= \dim(E)$ exhibiting a homoclinic tangency.

$\mathcal{T}$: the set of diffeomorphisms having a homoclinic tangency.

Theorem [C – Sambarino – D.Yang]. *For generic $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathcal{T}}$, each chain-recurrence class Λ admits a dominated splitting*

$$T_\Lambda M = E^s \oplus E^c_1 \oplus \cdots \oplus E^c_k \oplus E^u,$$

where: – each E^c_i is one-dimensional,
– E^s is uniformly contracted,
– E^u is uniformly expanded.

Theorem [C – Pujals – Sambarino]. *Under the same setting, if Λ is not a sink or a source, then E^s, E^u are non-degenerated.*

Consequence.

Any C^1-generic diffeomorphism which admits infinitely many sinks or sources is limit in $\mathrm{Diff}^1(M)$ *of diffeomorphisms exhibiting a homoclinic tangency.*

Finiteness conjecture (Bonatti).

Any C^1-generic diffeomorphism which admits infinitely many chain-recurrence classes is limit in $\mathrm{Diff}^1(M)$ of diffeomorphisms exhibiting a homoclinic tangency.

Far from homoclinic tangencies (1): invariant measures

Assume that f is not limit in $\mathrm{Diff}^1(M)$ of diffeomorphisms exhibiting a homoclinic tangency.

Theorem (Mañé-Wen-Gourmelon).
Any limit set K of a sequence of periodic orbits (O_n) with stable dimension s has a dominated splitting

$$T_K M = E \oplus F, \quad \dim(E) = s.$$

Corollary.
The support of any ergodic measure μ has a dominated splitting:

$$T_{supp(\mu)} M = E_{cs} \oplus E_c \oplus E_{cu},$$

Along E_{cs}, E_c, E_{cu} the Lyapunov exponents of μ are < 0, 0, > 0,
The dimension of E_c is 0 or 1.

Far from homoclinic tangencies (2): minimal sets

Theorem (Gan–Wen–D.Yang). *Consider $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathcal{T}}$. Any minimal set K has a dominated splitting*

$$T_K M = E^s \oplus E^c_1 \oplus E^c_2 \oplus \cdots \oplus E^c_k \oplus E^u,$$

each E^c_i has dimension 1 and E^s, E^u are uniform.

Proved by *interpolation* of the dominated splittings on K, using:

Theorem (Liao). *Consider any $f \in \mathrm{Diff}^1(M)$ and K invariant s.t.*
– K has a dominated splitting $T_K M = E \oplus F$,
– E is not uniformly contracted,
– on any $K' \subset K$, the function $\log |Df|$ has negative average for some invariant measure μ on K',
then any neighborhood of K contains periodic orbits whose maximal Lyapunov exponent along E is < 0 and close to 0.

Far from homoclinic tangencies (3): chain-recurrence classes

Theorem [C – Sambarino – D.Yang]. *For generic* $f \in \mathrm{Diff}^1(M) \setminus \overline{\mathcal{T}}$, *each chain-recurrence class* Λ *admits a dominated splitting*

$$T_\Lambda M = E^s \oplus E^c_1 \oplus \cdots \oplus E^c_k \oplus E^u,$$

where: – *each* E^c_i *is one-dimensional,*
– E^s *is uniformly contracted,*
– E^u *is uniformly expanded.*

Proved by *extension* of the dominated splittings of subsets.

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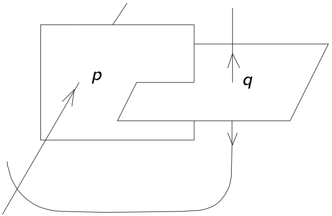


Heterodimensional cycle associated to hyperbolic periodic point p, q.

- This mechanism is *fragile* (one-codimensional).

Definition. f exhibits a *robust heterodimensional cycle* associated to p, q if there are transitive hyperbolic sets K_p, K_q containing p, q s.t. for any g C^1-close to f, $W^s(x) \cap W^u(y) \neq \emptyset$ for some $(x, y) \in K_p \times K_q$ and also for some $(x, y) \in K_q \times K_p$.

- Robust heterodimensional cycles do exist when $dim(M) \geq 3$.

Consider a C^1-generic f and two hyperbolic periodic points p, q with different stable dimension inside a same chain-recurrence class.

- Genericity $\Rightarrow$ robustness (Bonatti-Díaz). For any diffeomorphism C^1-close to f one has $H(p) = H(q)$ and there exists a robust heterodimensioonal cycle associated to p, q.
- Non-hyperbolic measures (Díaz-Gorodetsky). f has an *ergodic* measure with one Lyapunov exponent equal to 0.

Conjecture (Palis). Any $f \in \mathrm{Diff}^r(M)$ can be approximated by a hyperbolic diffeomorphism or by a diffeomorphism exhibiting a homoclinic bifurcation (tangency or cycle).

This holds when $\dim(M) = 1$. (Morse-Smale systems are dense.)

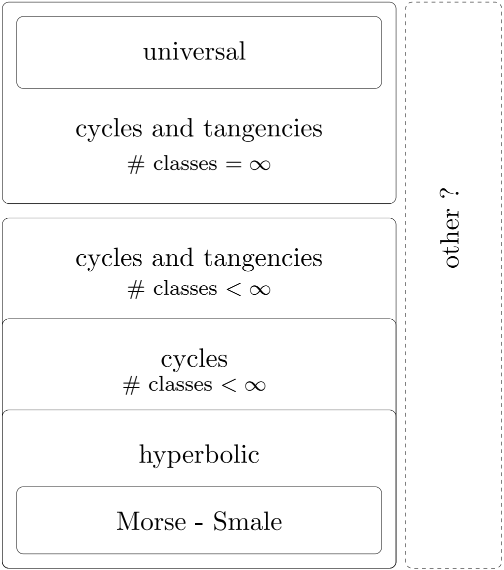
Theorem (Pujals-Sambarino).
The conjecture holds for C^1-diffeomorphisms of surfaces.

Conjecture (Bonatti-Díaz). Any $f \in \mathrm{Diff}^1(M)$ can be approximated by a hyperbolic diffeomorphism or by a diffeomorphism exhibiting a heterodimensional cycle.

This would imply Smale's conjecture on surfaces.

Theorem (C). The conjecture holds for volume-preserving diffeomorphisms in dimension ≥ 3.

Conjectured panorama of C^1-dynamics



Theorem (C-Pujals-Sambarino). Consider $f \in \mathrm{Diff}^2(M)$.
Let K with a dominated splitting $T_K M = E \oplus F$, $dim(F) = 1$ s.t.

– all periodic points in K are hyperbolic, no sink,
– there is no invariant curve in K tangent to F,

then F is uniformly expanded.

Goes back to a theorem by Mañé, for one-dimensional dynamics.

Theorem (C).
For $f \in \mathrm{Diff}^1(M)$ generic, not limit of a homoclinic bifurcation:

- Any aperiodic class has a dominated splitting

$$T_K M = E^s \oplus E^c \oplus E^u, \quad dim(E^c) = 1,$$

  and the Lyapunov exponent along E^c is 0 for any measure.

- Any homoclinic class has a dominated splitting

$$T_K M = E^s \oplus E^c_1 \oplus E^c_2 \oplus E^u, \quad dim(E^c_i) \leq 1.$$

  All periodic orbits have stable dimension $dim(E^s + E^c_1)$.
  If $dim(E^c_i) = 1$, there exists periodic orbits in K with a Lyapunov exponent along E^c_i close to 0.

Theorem (C-Pujals).
Any C^1 generic diffeomorphism that can not be approximated by a homoclinic bifurcation is essentially hyperbolic.

Definition of essential hyperbolicity. There exist hyperbolic attractors $A_1, \dots, A_k$ and repellors $R_1 \dots, R_\ell$ s.t.:

- the union of the basins of the A_i is (open and) dense in M,
- the union of the basins of the R_i is (open and) dense in M,

Theorem (Bonatti-C-Pujals).
Consider f and a hyperbolic set K with a dominated splitting

$$T_K M = (E^{ss} \oplus E^c) \oplus E^u.$$

Then,

- either K is contained in a submanifold tangent to $E^c \oplus E^u$,
- or there are g $C^{1+\alpha}$-close to f and $p \in K_g$ periodic with a *strong connection*:

$$W^{ss}(p) \cap W^{uu}(p) \setminus \{p\} \neq \emptyset.$$