Subtractive algorithms

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The map

$$
\tau: \mathbf{x} = (x_1, x_2, \ldots, x_n) \mapsto \text{sort}(x_1, x_2 - x_1, \ldots, x_n - x_1)
$$

is defined on ordered $n$-tuples, all $x_i \geq 0$.

Note that $\mathbf{x}_\infty = \lim_{n \to \infty} \tau^n(\mathbf{x})$ exists. It is a fixed point of $\tau$ and therefore the first coordinate of $\mathbf{x}_\infty$ is zero.

If all coordinates of $\mathbf{x}$ are rationally independent then the second coordinate of $\mathbf{x}_\infty$ is zero as well.
A pedestrian on a line

A pedestrian walks up and down on a line, taking steps of length $x_1, \ldots, x_n$, all rationally independent. Find the length of a minimal interval that enables an infinite walk that does not visit any point twice.

For instance, if there are only two steps $x_1, x_2$ then the length is $x_1 + x_2$ and the walk is an irrational rotation on the circle.
An algorithm to solve this problem

Sort the steps in increasing order $x_1 < x_2 < \cdots < x_n$. Let $I = [0, y]$ be a minimal interval. Partition it into $[0, y - x_1] \cup (y - x_1, y)$.

On the subinterval $[0, y - x_1]$ there is an infinite walk with steps $x_1, x_2 - x_1, \ldots, x_n - x_1$. This is the subtractive algorithm, proposed by Meester.

A pedestrian walks up and down on \( \mathbb{Z} \), taking integral steps of length \( p_1, \ldots, p_n \) such that gcd is one. Find the length of a maximal interval \( I \) such that the pedestrian cannot visit all points of \( I \).

For instance, if there are only two steps \( p_1, p_2 \) then the length is \( p_1 + p_2 - 2 \). Again the solution is by the subtraction operation.

Source: Tijdeman and Zamboni, Fine-Wilf words, Indag Math 2003
Does the subtraction terminate?

Consider Meester’s problem on a triple \((x_1, x_2, x_3)\). If \(x_3 > x_1 + x_2\) then steps of size \(x_3\) do not help. The minimal length is \(x_1 + x_2\) by irrational rotation.

For general \(n\)-tuples, Meester’s algorithm iterates

\[
\tau : x = (x_1, x_2, \ldots, x_n) \mapsto \text{sort}(x_1, x_2 - x_1, \ldots, x_n - x_1)
\]

until \(x_1 + x_2 < x_3\).

**Question:** does this algorithm terminate almost surely?  
**Answer:** yes

source: for triples, Meester and Nowicki, Israel J 1989; general case: Kraaikamp and Meester, ETDS 1995
Equivalent question: is it true that $x_\infty = \lim_{n \to \infty} \tau^n(x)$ has third coordinate $> 0$ almost surely?

Observe that $\tau(x_1, x_2, \ldots, x_n) = \text{sort}(x_1, x_2 - x_1, \ldots, x_n - x_1)$ respects projective coordinates, which reduces the degrees of freedom by one. If we normalize the third coordinate to 1, then ordered triples can be depicted by the triangle $0 < x < y < 1$:

The algorithm terminates as soon as $\tau^n(x) \in L$. 
In his monograph on continued fractions Schweiger generalizes $\tau$ and considers the **fully subtractive algorithm**:

$$\tau: (x_1, \ldots, x_a, \ldots, x_n) \mapsto \text{sort}(x_1, \ldots, x_a, x_{a+1} - x_a, \ldots, x_n - x_a)$$

Again, it is easy to show that $x_\infty$ has first $a + 1$ coordinates equal to zero a.s. Schweiger presents two conjectures:

1. The $a + 2$ coordinate of $x_\infty$ is positive a.s.
2. $\tau$ is ergodic, i.e, invariant sets are null sets or co-null sets.

1 is true and 2 is false, but conjecture 2 may be true if $n = a + 2$.

As always, accelerate the algorithm

\[ \tau : (x_1, \ldots, x_a, \ldots, x_n) \mapsto \text{sort}(x_1, \ldots, x_a, x_{a+1} - k x_a, \ldots, x_n - k x_a) \]

with \( k = \left\lfloor \frac{x_{a+1}}{x_a} \right\rfloor \). Observe that the permutation on the coordinates is a ‘rifle shuffle’.

**Lemma**

*All cylinders are full*

**Lemma**

*The set \( L = \{x_1 + \cdots + x_{a+1} < x_{a+2}\} \) is invariant.*

The proof is by bounded distortion: in each iteration a positive fraction of \( U \), the complement of \( L \), enters \( L \).
A sketch of the principal cylinders for the algorithm on triples:

Points that never enter $L$ are those that return infinitely often to the cylinder that is entirely contained in $U$. 
Observe that \( \tau(x) = \text{sort}(x_1, x_2 - x_1, \ldots, x_n - x_1) \) is linear and has determinant 1.

Now we normalize the \( n^{th} \) coordinate to 1, so writing \( y = \tau(x) \), in normalized coordinates the map is \( T(x) = \frac{1}{y_n} y \), where \( y_n \) is the final coordinate of \( y \). Therefore \( DT(x) \) has determinant \( \left( \frac{1}{y_n} \right)^n \).

To bound distortion on an \( m \)-cylinder \( \Delta \) we have to bound \( y_n \) away from zero for all \( y = T^m(x) \) in that cylinder.
An principal cylinder $\Delta_{(k,\pi)}$ is given by the acceleration $k$ and the rifle shuffle $\pi$. If $\pi(a) = n$ then $1 - x_a < x_a$. So $y_n$ is bounded away from zero, since $y_n = x_a$.

More generally, an $m$-cylinder $\Delta_{(k_1,\pi_1)(k_2,\pi_2)\cdots(k_m,\pi_m)}$ has bounded distortion if $\pi_m(a) = n$. All elements that remain in $U$ are contained in such $m$-cylinders for arbitrary large $m$.

Since cylinders are full, by bounded distortion any such $m$-cylinder loses a proportion to $L$. Therefore $L$ is an absorbing set and points that remain in $U$ have measure zero. This proves Schweiger’s first conjecture.
Now we know that $x_\infty$ has a positive $a + 2$-nd coordinate $x_{a+2}^\infty$ a.s. Define $f(x) = x_{a+2}^\infty$. Then $f$ is $\tau$-invariant and non-constant if $n > a + 2$ so $\tau$ is not ergodic.

The remaining case $n = a + 2$ is non-trivial.

Points in $L$ zigzag down slowly. Is there a non-trivial invariant set?
Exotic invariant sets

The subset of triples $x$ such that $x_\infty = 0$ is a Sierpinski triangle:

Such complex-dynamic like fractals occur in general subtractive maps. Nogeira and Schweiger have found a Cantor fan, which is known from the exponential family in complex dynamics, in the Poincaré algorithm

$$(x_1, x_2, x_3) \mapsto \text{sort}(x_1, x_2 - x_1, x_3 - x_2)$$

It is natural to consider for $b \leq a$

$$\tau: (x_1, \ldots, x_a, \ldots, x_n) \mapsto \text{sort}(x_1, \ldots, x_a, x_{a+1} - x_b, \ldots, x_n - x_b)$$

Again, it is easy to show that $x_\infty$ has $a + 1$ coordinates that are equal to zero. Numerical experiments suggest that almost surely $x_\infty$ has $2a - b + 1$ coordinates that are equal to zero.

Unfortunately, this $\tau$ admits no Markov partition. A proof for these numerical results seems difficult.
Thank you