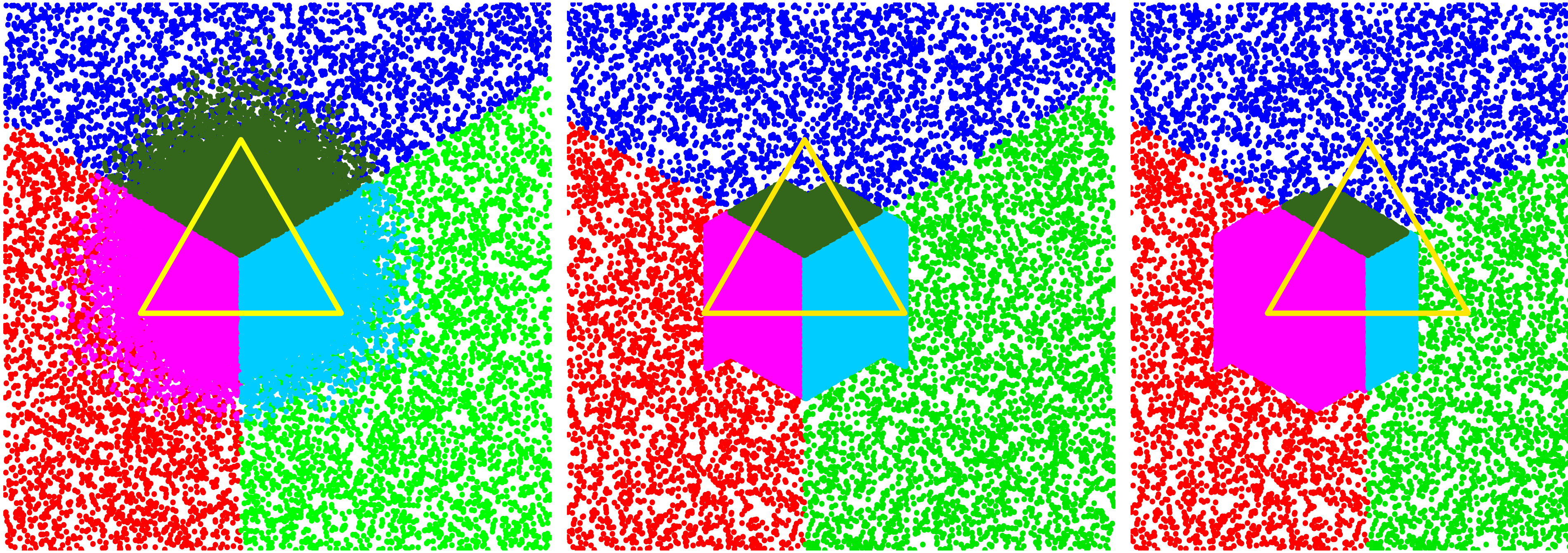
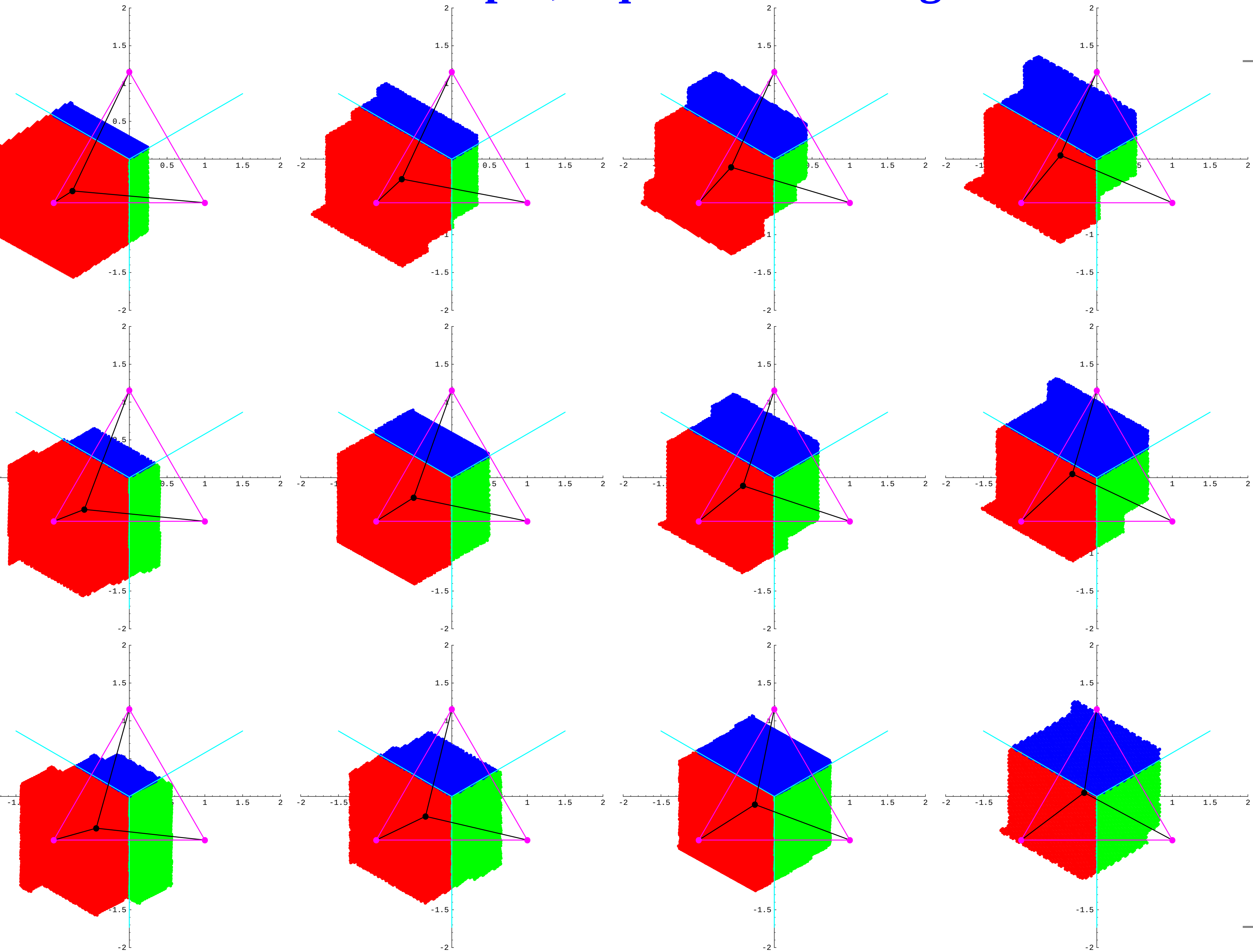
Dynamics and Voronoi regions

$$F(x) = x + \gamma - \mathrm{Vor}(x)$$

Random and constant inputs

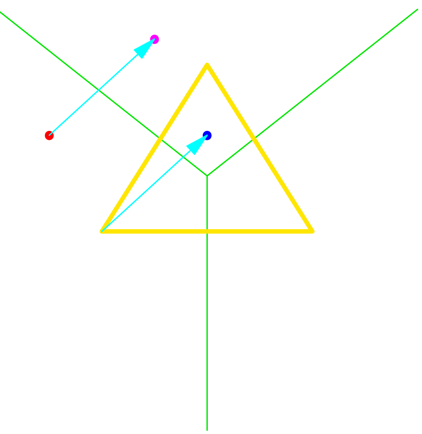


Constant Input, Equilateral Triangle



Dynamics and Voronoi regions

- Let $\mathbb{R}^d = \bigcup V_i$ be a (finite) partition of the affine space, and t_i vectors in $\mathbb{R}^d$ Define $F(x) = x + t$, where $t = t_i$, whenever $x \in V_i$.
- We will consider mostly the Error Diffusion generated by
  - a polytope $\mathrm{conv}\{c_i\} = \mathcal{P} \subset \mathbb{R}^d$ with
  - the partition defined by the Voronoi regions of the corners c_i, $\bar{V}_i = \{y : ||y - c_i||_2 \leq ||y - c_j||_2, \forall j\}$ and
  - the translations defined by a point $\gamma \in \mathcal{P}^\circ$, $t_i = \gamma - c_i$,
  - $\gamma = \sum \gamma_i c_i$, $\gamma_i > 0$, $\sum \gamma_i = 1$.
  - $F(x) = x + \gamma - c_i$, whenever $x \in V_i$ (with some tie-breaking rules).
- The polytope $\mathcal{P}$ will be a (full dimensional) simplex $\Delta(c_i)$ with $d+1$ corners $c_0, \ldots, c_d$.
- The point (input) γ will be constant.



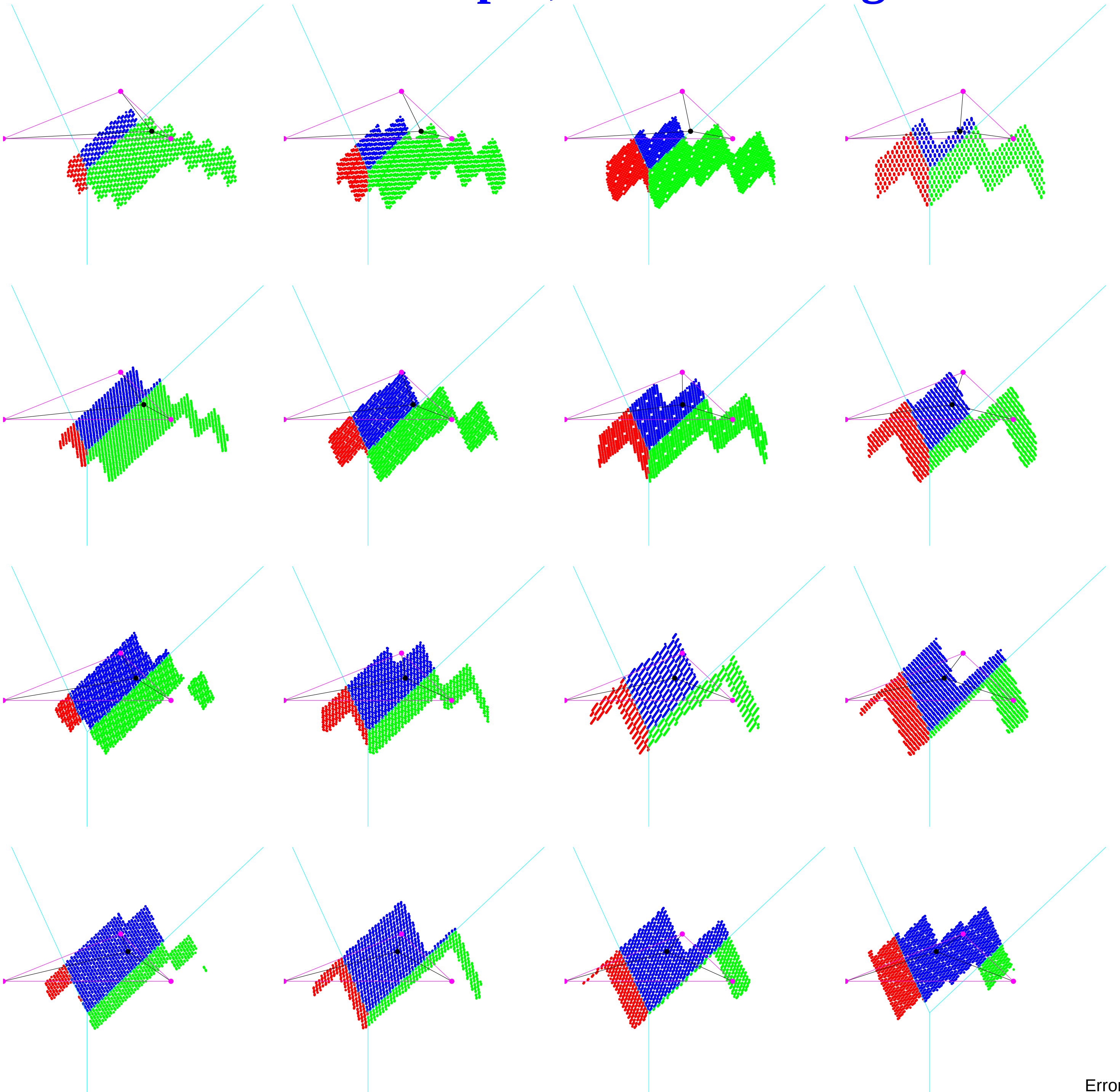
Frequencies for a constant input system in a simplex

- When $\mathcal{P}$ is a simplex for any x we have $\frac{\#\{n<N:\mathrm{Vor}(F_\gamma^n(x))=c_i\}}{N} \to_N \gamma_i$
-

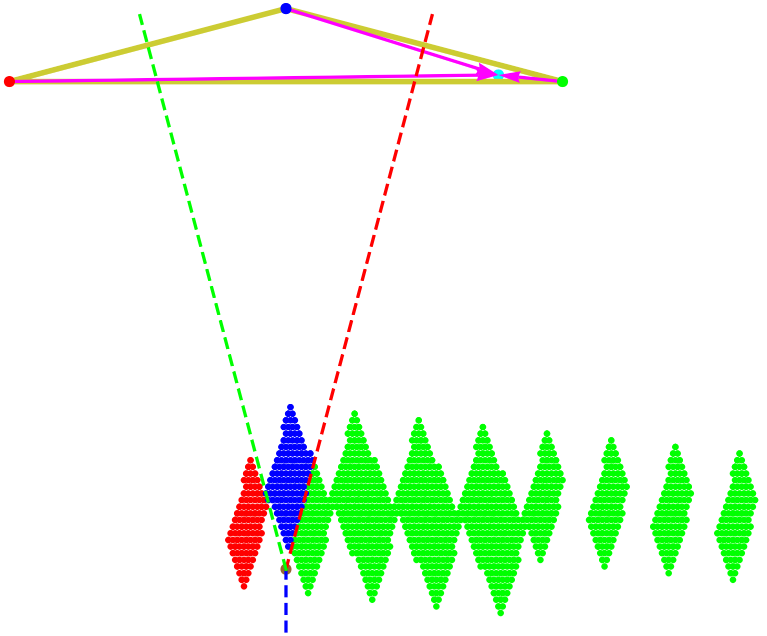
$$
\begin{aligned}
0 \quad \leftarrow \quad \frac{F_\gamma^N(x)-x}{N} &= \frac{1}{N}\sum_{n<N}(\gamma - \mathrm{Vor}(F^n(x))) \\
&= \gamma - \sum_{i=0}^{d}\frac{n_i}{N}c_i = \sum_{i=0}^{d}\gamma_i c_i - \sum_{i=0}^{d}\frac{n_i}{N}c_i \\
&= \sum_{i=0}^{d}(\gamma_i - \frac{n_i}{N})c_i \\
&\quad \frac{n_i}{N} \to \gamma_i \quad n_i = \#\{n : \mathrm{Vor}(x_n) = c_i\}
\end{aligned}
$$

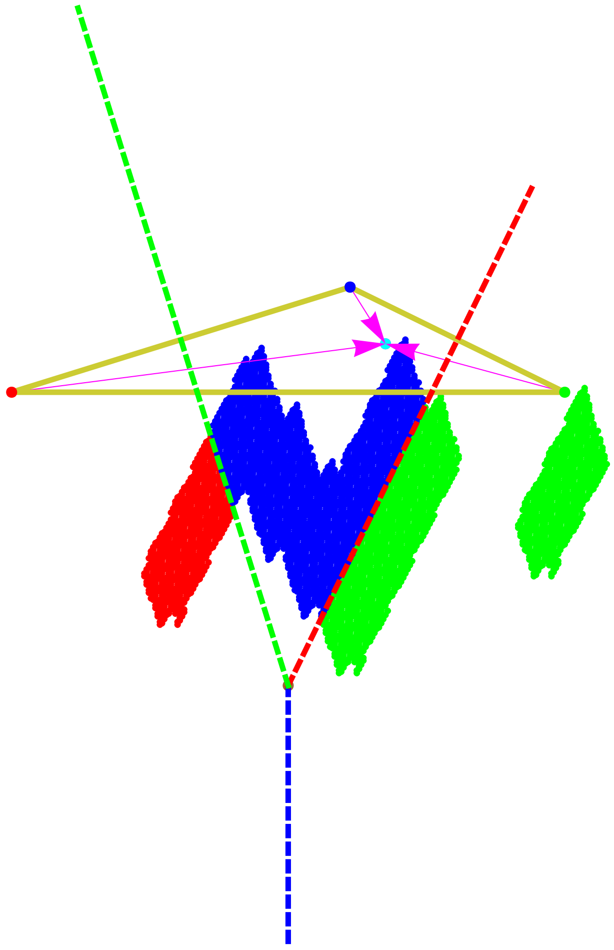
by the uniqueness of baricentric coordinates.

Constant Input, Obtuse Triangle



Constant Input, Obtuse Triangle





Acuteness (simplex)

1. Supporting planes: each Voronoi region V_i lies inside a halfspace parallel to the face opposite to c_i
2. Facewise: the outward normal vectors to the faces form obtuse angles
3. Edgewise: the edges form acute angles
4. Inverted cones: The inverted vertex cone fits inside its (dual) Voronoi cone.
5. Center inside: The center of the simplex lies inside the simplex.

- $(1) \equiv (2) \Rightarrow (3) \equiv (4)$. (5) neither implies nor is implied by $(1) \ldots (4)$

Structure of the Invariant Set (simplex with constant input)

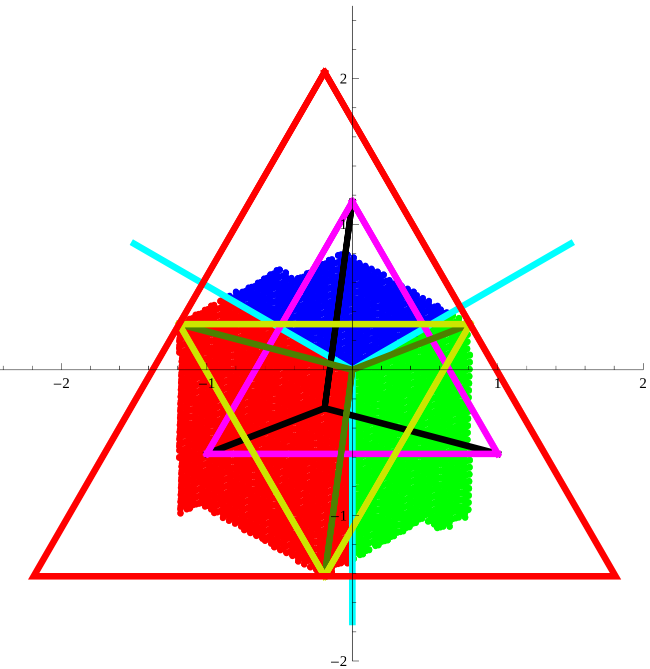
- The minimal absorbing set $\mathcal{Q}$ of F_γ with fixed $\gamma \in \mathcal{P}^0$ is a tile with respect to the lattice $L = \{\sum_{i,j} n_{ij}(c_i - c_j)\,, n_{ij} \in \mathbb{Z}\}$.
- This is a
  - **Theorem** for $\mathcal{P}$ non-obtuse triangle.
  - **Theorem** for $\mathcal{P}$ an acute simplex with typical (ergodic) input.
  - **Work in progress** for general (obtuse) triangle.
  - **Work in progress** for acute simplices with general input.
  - **Conjecture** for general simplices with general input.
  - **Unknown** for general polytopes with all the corners on some lattice.
- Tiles
  - $Q \subset \mathbb{R}^d$ is a tile with respect to the lattice $L = \mathbb{Z}(w_1, \ldots, w_d) = \{\sum_{i=1}^d n_i w_i, n_i \in \mathbb{Z}\}$, $w_i \in \mathbb{R}^d$, independent, if the map $T : Q \times L \to \mathbb{R}^d, \quad T(q, w) = q + w$ is 1-1 and onto.
  - The points c_i generate a lattice $L = \sum m_i c_i$, with $m_i \in \mathbb{Z}$, $\sum m_i = 0$. This is the lattice $L(c_1 - c_0, \ldots, c_d - c_0)$.

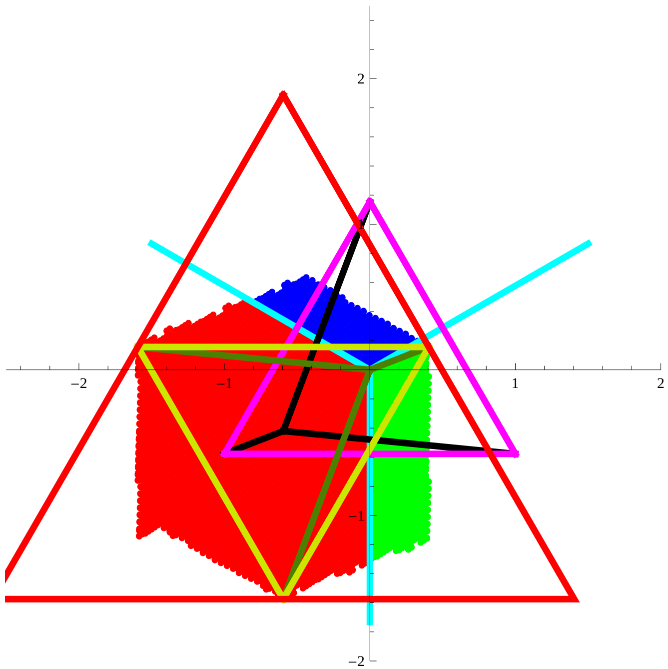
Structure of the Invariant Set (simplex with constant input)

- Let V_i be any partition of $\mathbb{R}^d$, c_i be a collection of $d+1$ independent points and $t_i = \gamma - c_i$ with some fixed point γ.
  Define $F(x) = x + t_i = x + \gamma - c_i$ for $x \in V_i$.
- If F admits a bounded invariant set Q which is a tile for the lattice $L(\{c_i - c_0\})$ then
- For any subset of indices $0 \in I \subset \{0, 1, \ldots, d\}$ the set $Q_I = \bigcup_{i \in I}(Q \cap V_i)$ is a tile for the lattice
- $L_I = L(c_i - c_0, \ldots, c_j - \gamma), \quad i \in I, \quad j \notin I$
- $\mathrm{Vol}(Q_I) = |\det(L_I)| = \mathrm{Vol}(Q) \cdot \sum_{\text{ß} \in I} \gamma_i$, where $\mathrm{Vol}(Q) = |\det(L)|$.
- Weaker version:
  - If $Q + L$ is onto $\mathbb{R}^d$ then $Q_I + L_I$ is onto $\mathbb{R}^d$
  - If $Q + L$ is 1-1 the $Q_I + L_I$ is 1-1.

Acute simplex with ergodic input

- Define $\mathcal{O}$ as a point equidistant to all the vertices.
- Define $w_i = \mathcal{O} + \gamma - c_i$ and $\mathcal{R} = \Delta(w_i)$
- Define $u_i = \mathcal{O} + \gamma - c_i + \sum_j (c_i - c_j)$ and $\mathcal{B} = \Delta(u_i)$
- We have w_i lies on the i-th face of $\mathcal{B}$.
- We have $u_i \in V_i$
- There are no other points in $\mathcal{B}$ equivalent to $\mathcal{R}^\circ$.
- $F(\mathcal{B}) \subset \mathcal{B}$. Moreover $\mathcal{B}$ absorbs everything.

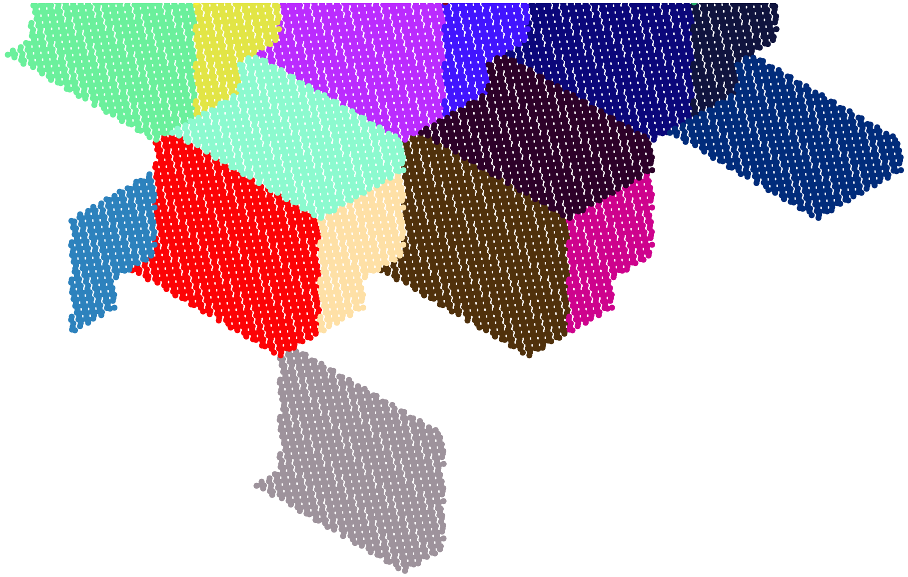


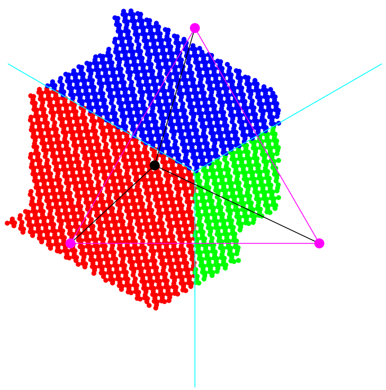


Structure of the Invariant Set (simplex with constant input)

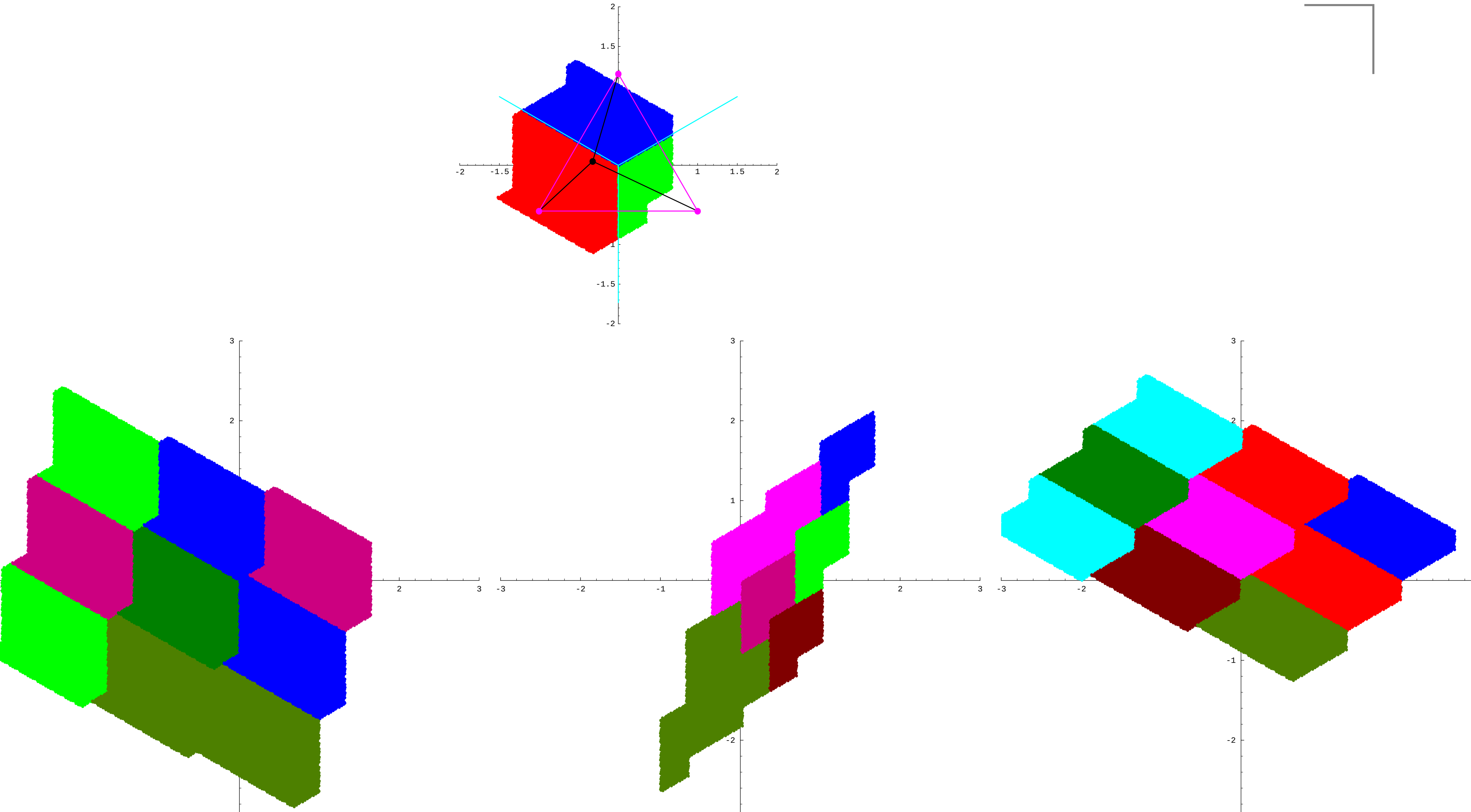
- Let V_i be any partition of $\mathbb{R}^d$, c_i be a collection of $d+1$ independent points and $t_i = \gamma - c_i$ with some fixed point γ.
  Define $F(x) = x + t_i = x + \gamma - c_i$ for $x \in V_i$.
- If F admits a bounded invariant set Q which is a tile for the lattice $L(\{c_i - c_0\})$ then
- For any subset of indices $0 \in I \subset \{0, 1, \ldots, d\}$ the set $Q_I = \bigcup_{i \in I}(Q \cap V_i)$ is a tile for the lattice
- $L_I = L(c_i - c_0, \ldots, c_j - \gamma), \quad i \in I, \quad j \notin I$
- $\mathrm{Vol}(Q_I) = |\det(L_I)| = \mathrm{Vol}(Q) \cdot \sum_{ß \in I} \gamma_i$, where $\mathrm{Vol}(Q) = |\det(L)|$.
- Weaker version:
  - If $Q + L$ is onto $\mathbb{R}^d$ then $Q_I + L_I$ is onto $\mathbb{R}^d$
  - If $Q + L$ is 1-1 the $Q_I + L_I$ is 1-1.

Tiles, Equilateral Triangle

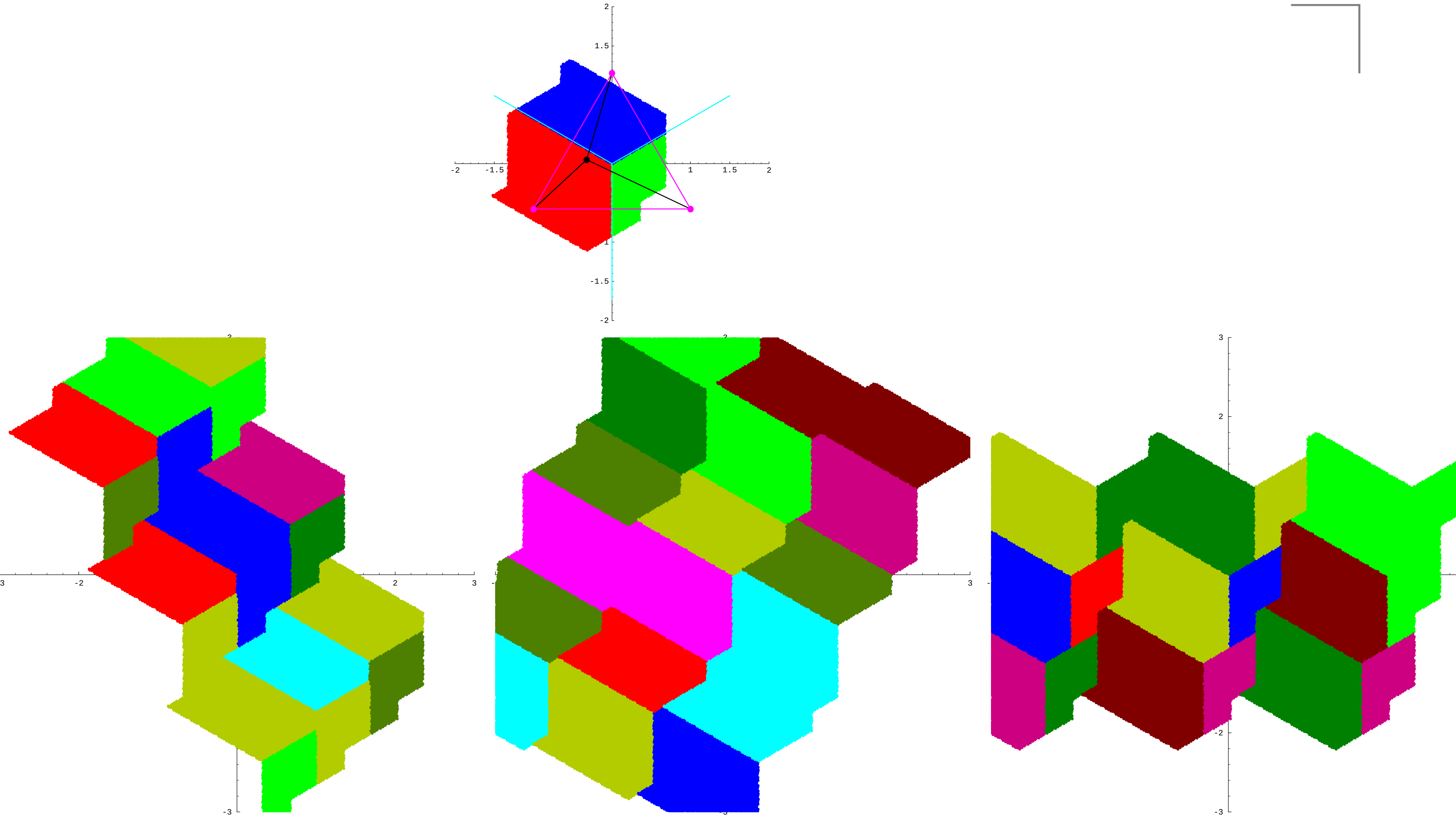




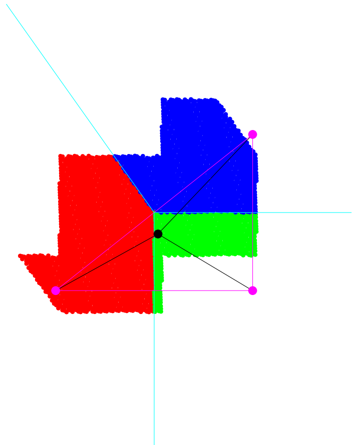
Sub-Tiles, Equilateral Triangle

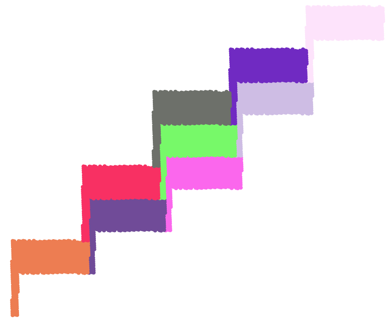


Multi-Tiles, Equilateral Triangle

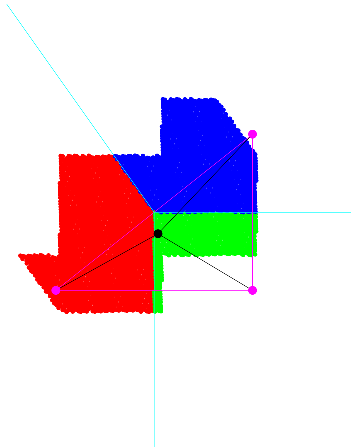


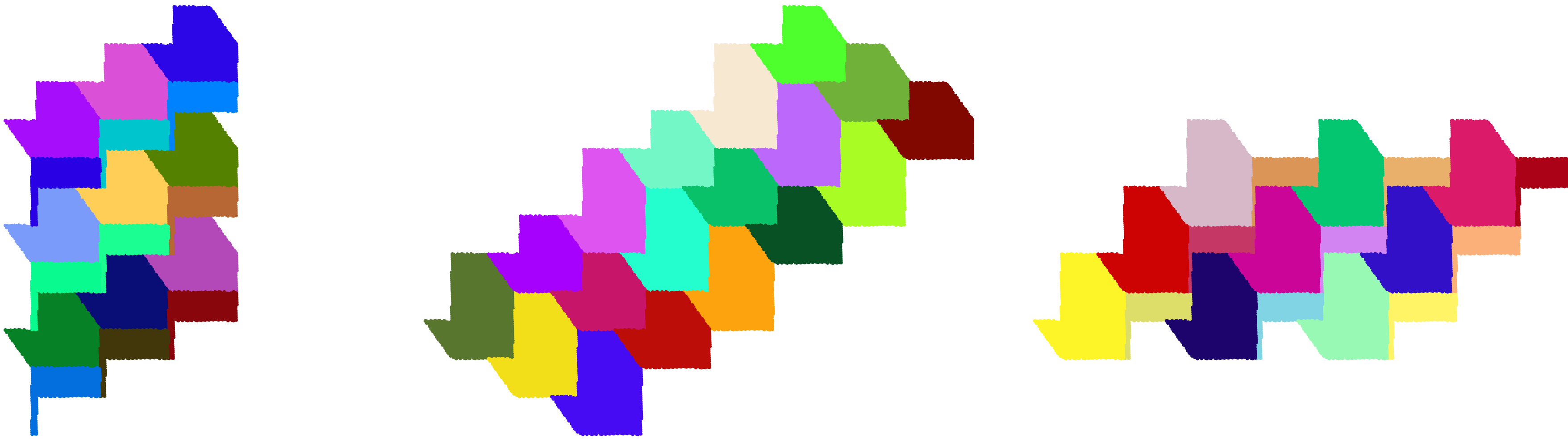
Sub-Tiles, Right Triangle



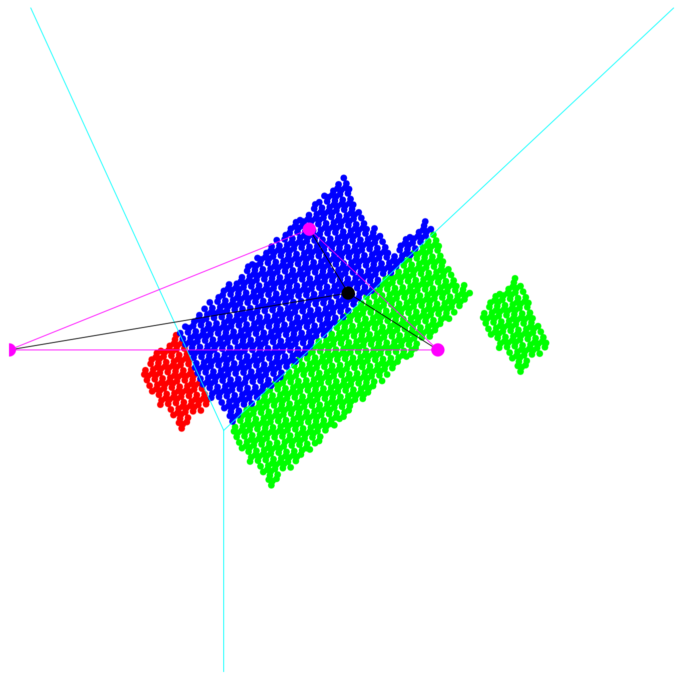


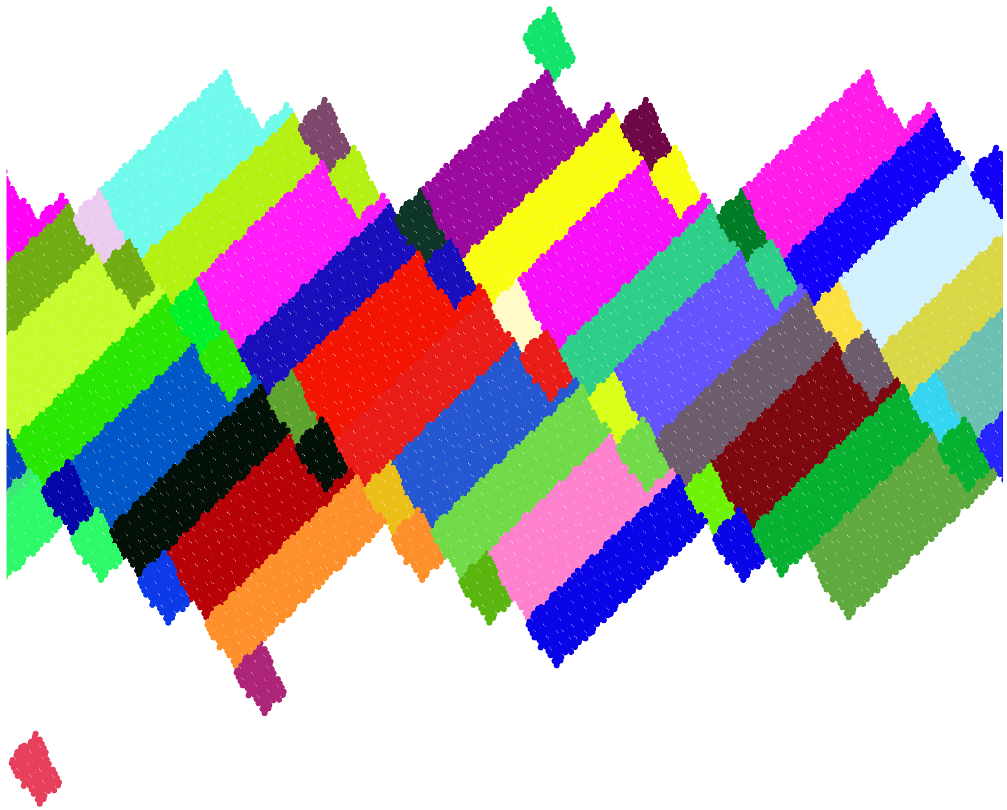
Multi-Tiles, Right Triangle



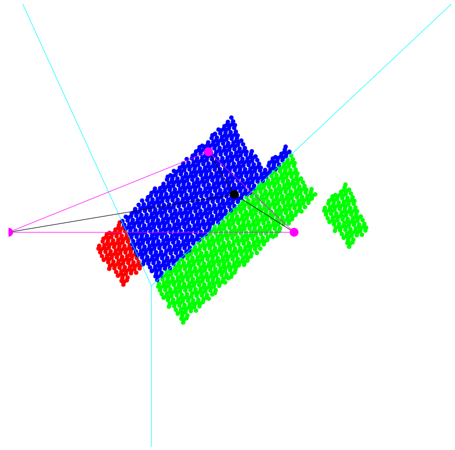


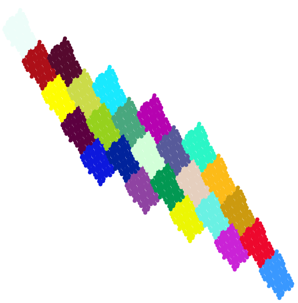
Tiles, Obtuse Triangle

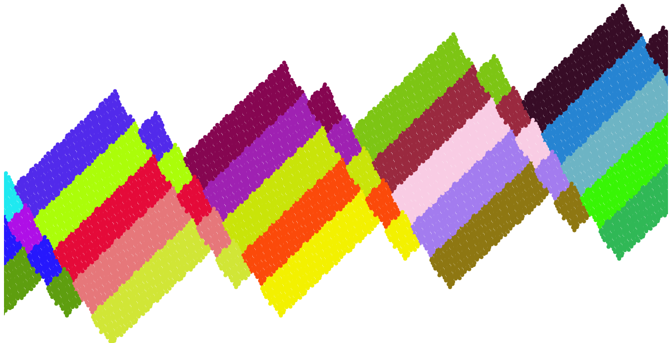


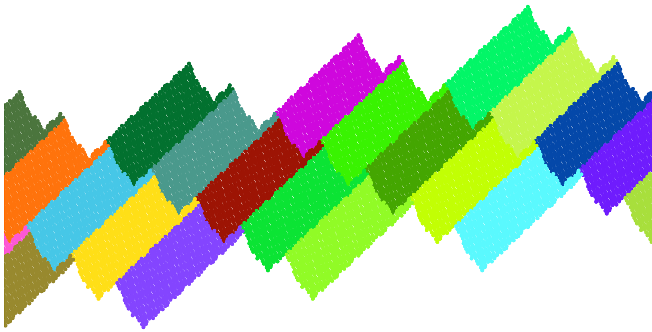


Sub-Tiles, Obtuse Triangle

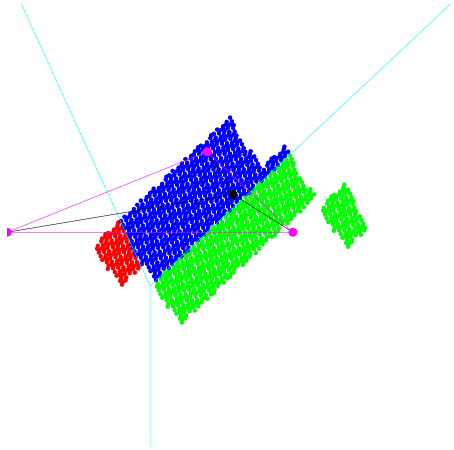


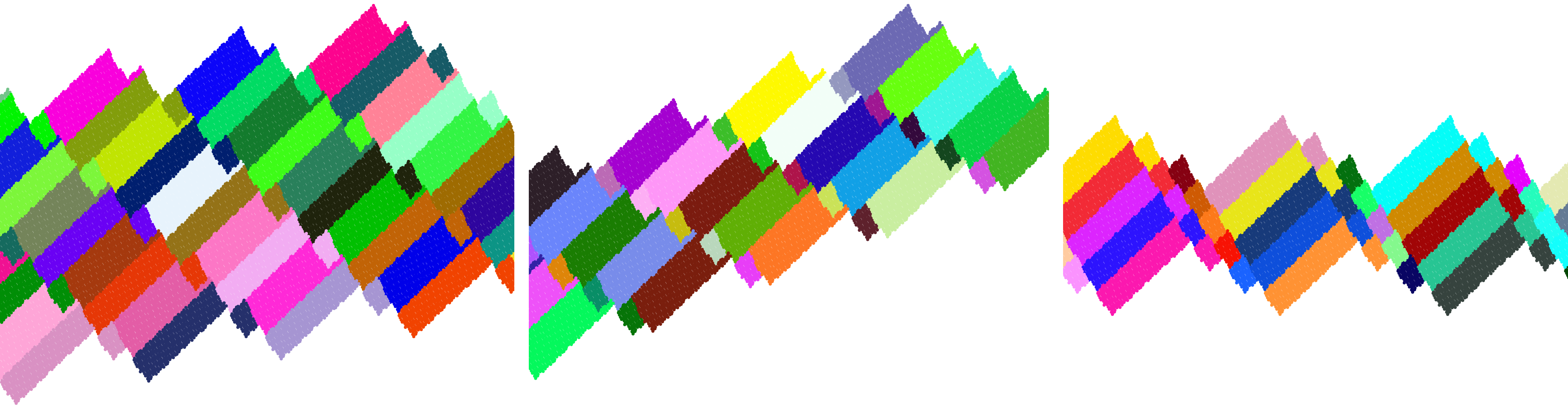






Multi-Tiles, Obtuse Triangle





Subtiles

- If the set Q and the lattice L are onto $\mathbb{R}^d \subset Q + L$ AND
- $\forall q \in Q \forall N \exists n^+, n^- \geq N \exists q^+, q^- \in Q_I \quad F^{n^+}(q) = q^+, F^{n^-}(q^-) = q$ then
- $Q_I + L_I$ is onto $\mathbb{R}^d$.
-
- If the Q and the lattice L are 1-1 to $\mathbb{R}^d \supset Q + L$ AND
- $\forall N \geq 0 F^N(Q_I) \subset Q$ then
- $Q_I + L_I$ are 1-1 in $\mathbb{R}^d$.
  - Represent any point in $\mathbb{R}^d$ as a sum $q + \sum n_i(c_i - c_0)$, using the assumption drive q into Q_I, and then manipulate the summation and obtain the representation of x in terms of $Q_I + L_I$.
  - If $Q_I + L_I$ is not 1-1 then we have some $q_I, q'_I \in Q_I$ with $q_I - q'_I \in L_I$. This element of L_I has a non zero components $c_j - \gamma$, with $\sum_{j \notin I} n_j > 0$. Take $Q \ni q = F^N(q'_I)$ and consider $q - q_I$, both in Q. Manipulate the sums and get $q - q_I \in L$ implying $q = q_I$. Manipulate again and obtain $N = 0$, that is $q_I = q'_I$.