Directed random polymers and Macdonald processes

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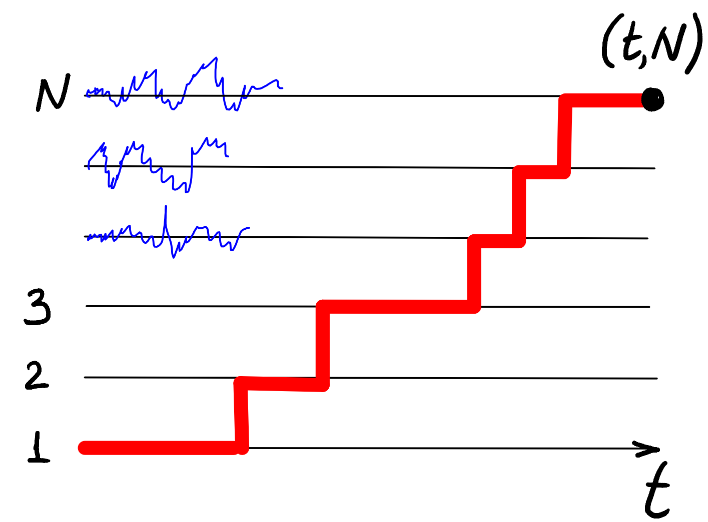
Partition function for a semi-discrete directed random polymer

$$Z_t^N = \int_{0<s_1<\dots<s_{N-1}<t} e^{B_1(0,s_1)+B_2(s_1,s_2)+\dots+B_N(s_{N-1},t)} \, ds_1 \dots ds_{N-1}$$

$B_1, \dots, B_N$ are independent Brownian motions

$$B_k(\alpha,\beta) := B_k(\beta) - B_k(\alpha) = \int_\alpha^\beta \dot{B}_k(x)\,dx$$



[O'Connell-Yor 2001]

$$u(t,N) := e^{-3t/2} Z_t^N = e^{-3t/2} \int\limits_{0<s_1<\dots<s_{N-1}<t} e^{B_1(0,s_1)+\dots+B_N(s_{N-1},t)}\, ds$$

satisfies

$$\frac{\partial u(t,N)}{\partial t} = \big(u(t,N-1) - u(t,N)\big) + \dot{B}_N(t)\cdot u(N,t)$$

with $u(0,N) = \delta_{1N}$.

This is a discrete analog of the <u>stochastic heat equation</u>

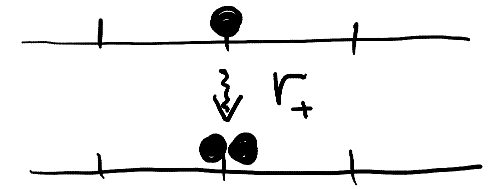
$$u_t = \frac{1}{2}\Delta u + \dot{W}\cdot u$$

where $\dot{W}$ is the space-time white noise.

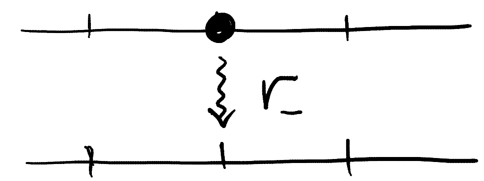
The path integral is the <u>Feynman-Kac solution</u>

Consider an ensemble of particles in $\mathbb{Z}$ that have masses and evolve according to the following rules

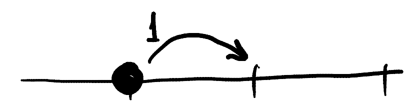
- At each time t any particle at location x <u>splits</u> into two identical particles of the same mass with rate $r_+(t,x)$



- Each particle <u>dies</u> with rate $r_-(t,x)$



- Each particle <u>jumps</u> to the right with rate 1



Then the expected total mass $m(t,x)$ satisfies

$$m_t = \big(m(N-1) - m(N)\big) + (r_+ - r_-)\,m.$$

White noise models rapidly oscillating medium

Solutions of stochastic heat equations are <u>intermittent</u>.

Define <u>moment</u> Lyapunov exponents

$$\gamma_p := \lim_{t\to\infty} \frac{1}{t} \ln \left\langle \left(u(t, N(t))\right)^p \right\rangle$$

and the <u>almost sure</u> Lapunov exponent

$$\tilde{\gamma}_1 := \lim_{t\to\infty} \frac{1}{t} \ln u(t, N(t))$$

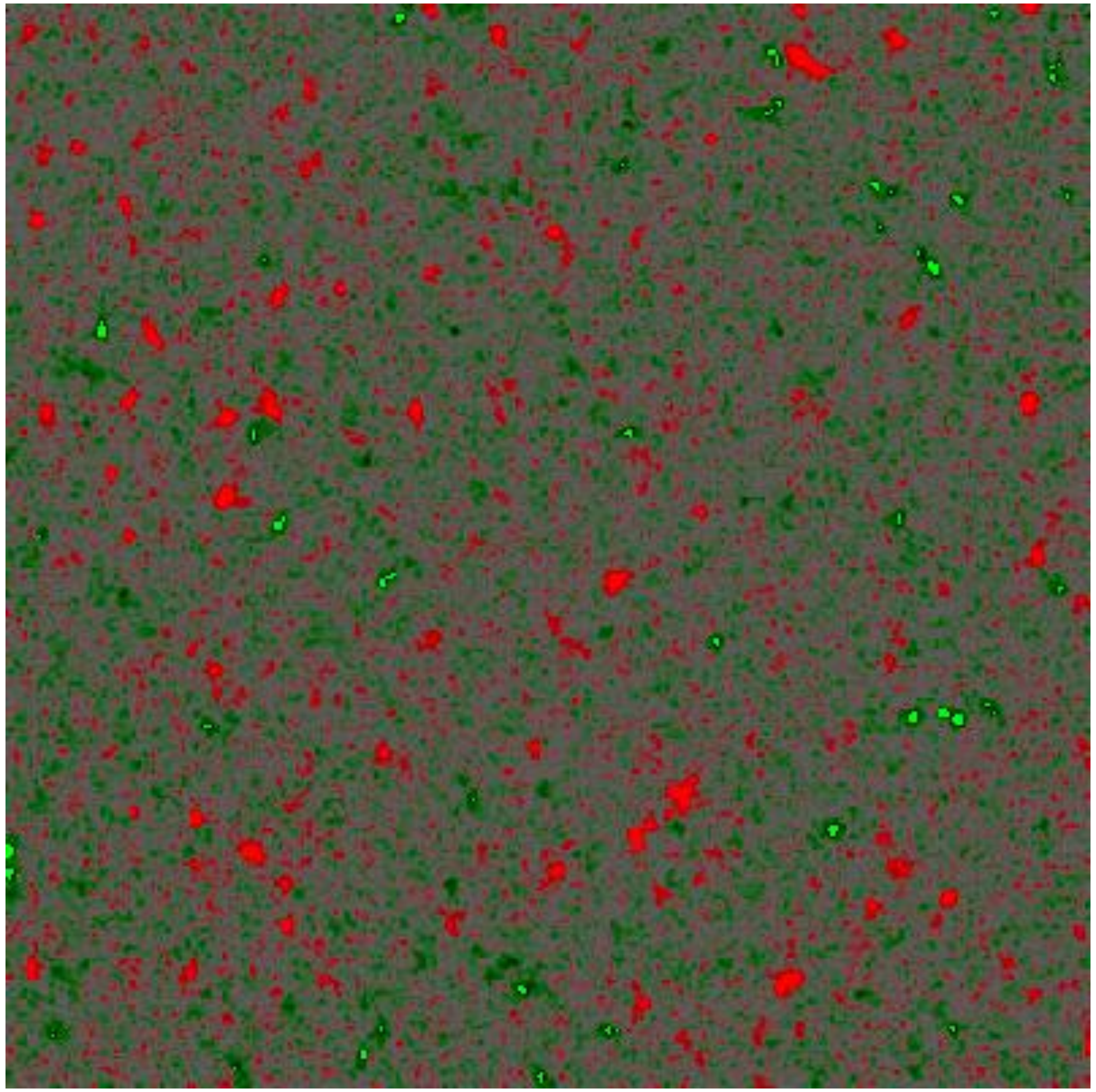
Intermittency means

$$\tilde{\gamma}_1 < \gamma_1 < \frac{\gamma_2}{2} < \frac{\gamma_3}{3} < \dots$$

Moments are dominated by higher and higher peaks of smaller and smaller probabilities

Zel'dovich et al. argued in the 1980's that qualitatively similar intermittency phenomenon arises in magnetic fields in turbulent flows, like those on the surface of the Sun.

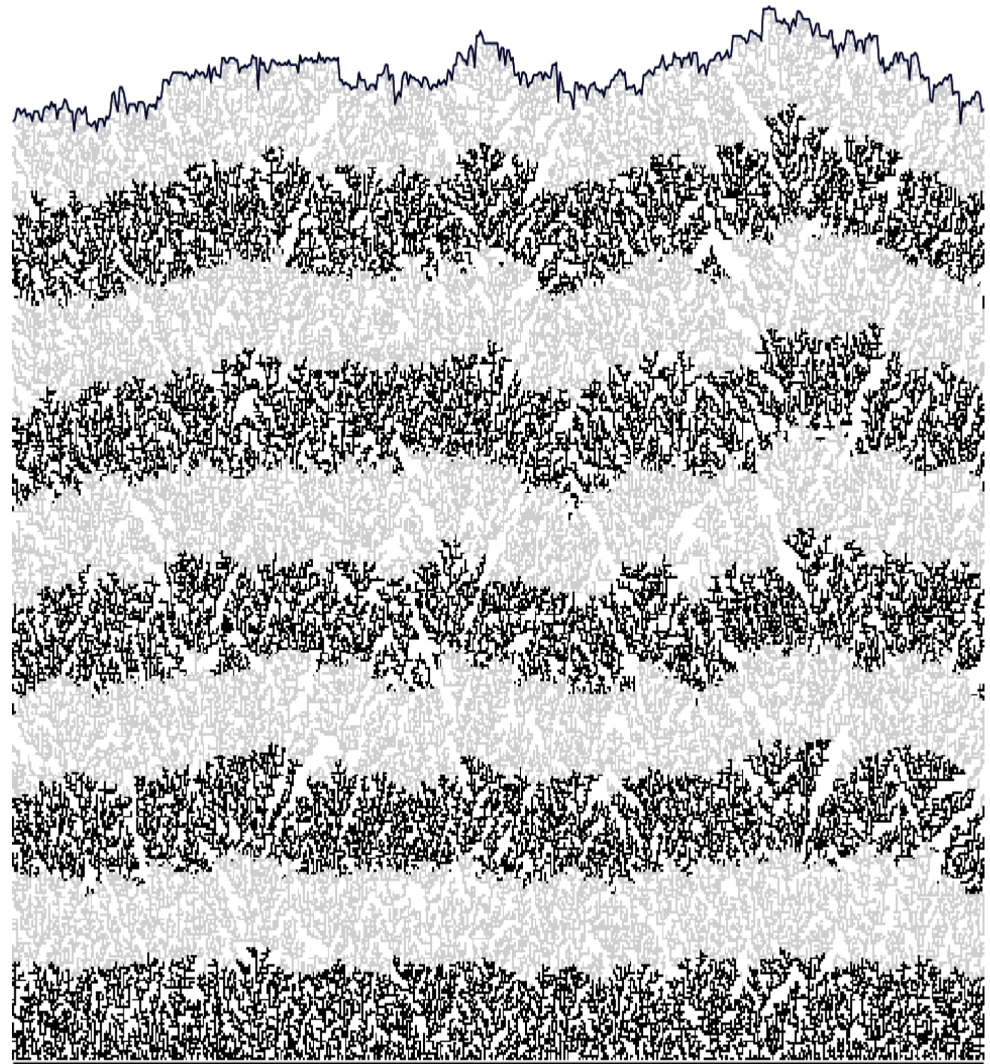
This is a high resolution magnetogram of a quiet Sun region (SOHO-MDI image)



On the other hand, if $u(t,x)$ satisfies $u_t = \frac{1}{2}\Delta u + \dot{W}\cdot u$ then $h := \log(u)$ formally satisfies

$$h_t = \frac{1}{2}\Delta h + \frac{1}{2}(\nabla h)^2 + \dot{W}$$

For the space-time white noise $\dot{W}$ this is the Kardar-Parisi-Zhang (KPZ) equation invented in 1986 to describe random interface growth. Thus, log(u) and random interfaces have to be in the same universality class.



Ballistic deposition

The semi-discrete Brownian directed polymer is exactly solvable.

Theorem (Borodin-Corwin, 2011) The Laplace transform of the polymer partition function Z_t^N can be written as a Fredholm determinant

$$\left\langle e^{-u Z_t^N} \right\rangle = \det\left(\mathbb{1} + K_u\right)_{L^2(\circlearrowleft)}$$

where

$$K_u(v, v') = \frac{i}{2} \int_{-i\infty+\frac{1}{2}}^{i\infty+\frac{1}{2}} \left(\frac{\Gamma(v-1)}{\Gamma(s+v-1)} \right)^N \frac{u^s e^{vts + \frac{ts^2}{2}}}{s+v-v'} \frac{ds}{\sin \pi s} .$$

Corollary (B-C, B-C-Ferrari, 2011-12) Set $F_t^N = \log Z_t^N$. For any $\varkappa > 0$

$$\lim_{N\to\infty} \mathbb{P}\left\{ \frac{F_{\varkappa N}^N - N \bar{f}_{\varkappa}}{N^{1/3}} \leq r \right\} = F_{GUE}\left(\left(\frac{\bar{g}_{\varkappa}}{2} \right)^{-1/3} r \right)$$

$$\lim_{N\to\infty} \mathbb{P}\left\{ \frac{F^N_{\varkappa N} - N\bar{f}_\varkappa}{N^{1/3}} \le r \right\} = F_{GUE}\left(\left(\frac{\bar{g}_\varkappa}{2}\right)^{-1/3} r \right)$$

- $F_{GUE}(x) = \det(\mathbb{1} - K_{Airy})_{L^2(x,+\infty)}$ is the GUE Tracy-Widom distribution.
- At $\varkappa = \infty$, rescaled $F^N_{\varkappa N}$ is distributed as the largest eigenvalue of the Gaussian Hermitian random matrix.
- Variance $= O(N^{2/3})$ obtained by [Seppäläinen-Valko, 2010]
- F_{GUE}, $N^{1/3}$ were expected by KPZ universality

$$\lim_{N\to\infty} \mathbb{P}\left\{ \frac{F^N_{\varkappa N} - N\bar{f}_\varkappa}{N^{1/3}} \le r \right\} = F_{GUE}\left(\left(\frac{\bar{g}_\varkappa}{2}\right)^{-1/3} r \right)$$

- $\bar{f}_\varkappa = \inf_{y>0} (\varkappa y - \Psi(y))$, $\Psi(y) = (\log \Gamma(y))'$, is the almost sure Lyapunov exponent, conjectured in [O'Connell-Yor, 2001], proved in [Moriarty, O'Connell, 2007].
- $\bar{g}_\varkappa = -\Psi''(y_\varkappa)$, $y_\varkappa = \operatorname{arginf}(\varkappa y - \Psi(y))$

$\bar{f}_\varkappa, \bar{g}_\varkappa$ a posteriori confirmed in [Spohn, 2011] via more advanced KPZ related arguments.

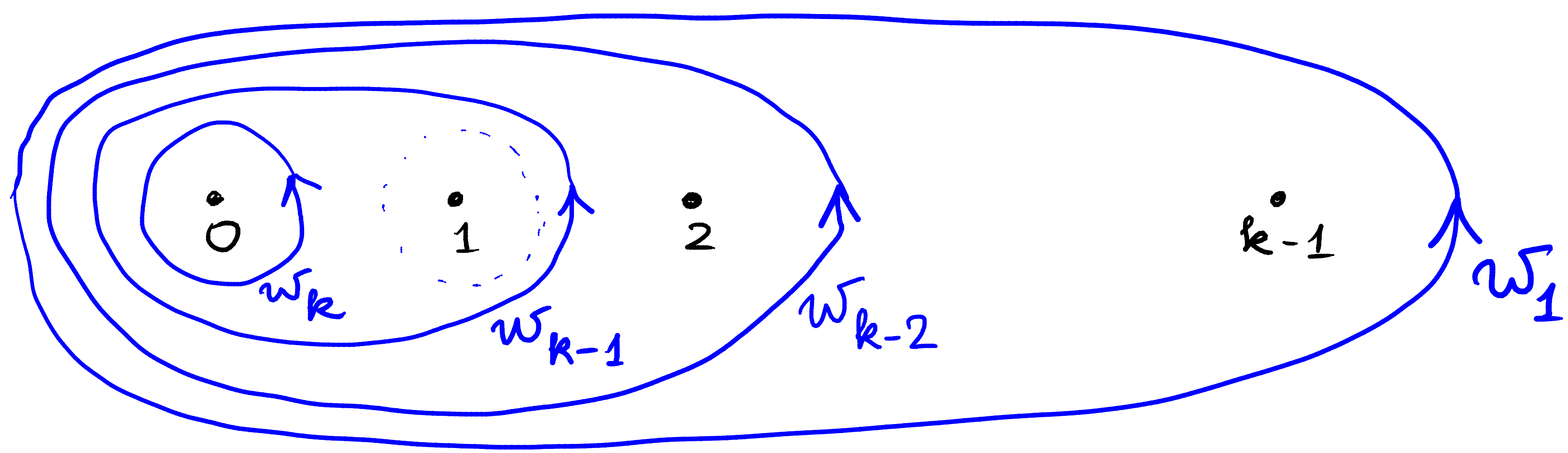
The semi-discrete Brownian directed polymer is exactly solvable.

Theorem (Borodin-Corwin, 2011) The moments of the polymer partition function have the following integral representation

$$\left\langle Z_t^{N_1} \cdots Z_t^{N_k} \right\rangle = \frac{e^{\frac{kt}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{1 \le A < B \le k} \frac{w_A - w_B}{w_A - w_B - 1} \prod_{j=1}^{k} \frac{e^{t w_j}}{w_j^{N_j}} \, dw_j$$

where $N_1 \ge N_2 \ge \dots \ge N_k \ge 1$.

The contours are such that w_A - contour contains 0 and $\{w_B + 1\}_{B>A}$



In the (hierarchically lower) case of the partition function for fully continuous directed Brownian polymer

$$\mathcal{Z}(t,x) = \int\limits_{\substack{\text{Brownian paths} \\ B(0)=0,\ B(t)=x}} e^{\int_0^t W(s,B(s))\,ds}$$

a similar formula for the distribution was obtained in 2010 in two different ways:

- Using analysis of ASEP in [Tracy-Widom, 2008-09] and the limit the weak asymmetry limit to polymers [Bertini-Giacomin, 1997]. This is due to [Amir-Corwin-Quastel], [Sasamoto-Spohn].
- Using quantum delta Bose gas and replica trick [Dotsenko], [Calabrese-Le Doussal-Rosso].

The replica approach is based on showing that

$$\overline{Z}(t, x_1, \dots, x_k) = \langle Z(t, x_1) \cdots Z(t, x_k) \rangle$$

satisfies

$$\frac{\partial \overline{Z}}{\partial t} = H \overline{Z}, \qquad H = \frac{1}{2} \Delta + \frac{1}{2} \sum_{i \neq j} \delta(x_i - x_j),$$

quantum Bose-gas

finding eigenbasis of H via Bethe ansatz [Lieb-Liniger, 1963], [McGuire, 1964], and using the expansion

$$\langle e^{-u Z(t,x)} \rangle \text{"="} \sum_{k=0}^{\infty} \frac{(-u)^k}{k!} \langle Z^k(t,x) \rangle$$

Such series always diverge due to intermittency! Drawing conclusions is risky, originally an incorrect answer was obtained.

We use a totally different approach. Comparison with Bose gas yields

<u>Theorem (Borodin-Corwin, 2011)</u> For $x_1 < x_2 < \dots < x_k$, the integral

$$\mathcal{U}(t, x_1, \dots, x_k) := \int \cdots \int \prod_{A<B} \frac{z_A - z_B}{z_A - z_B - c} \prod_{j=1}^{k} e^{\frac{t}{2} z_j^2 + x_j z_j} \frac{dz_j}{2\pi i}$$

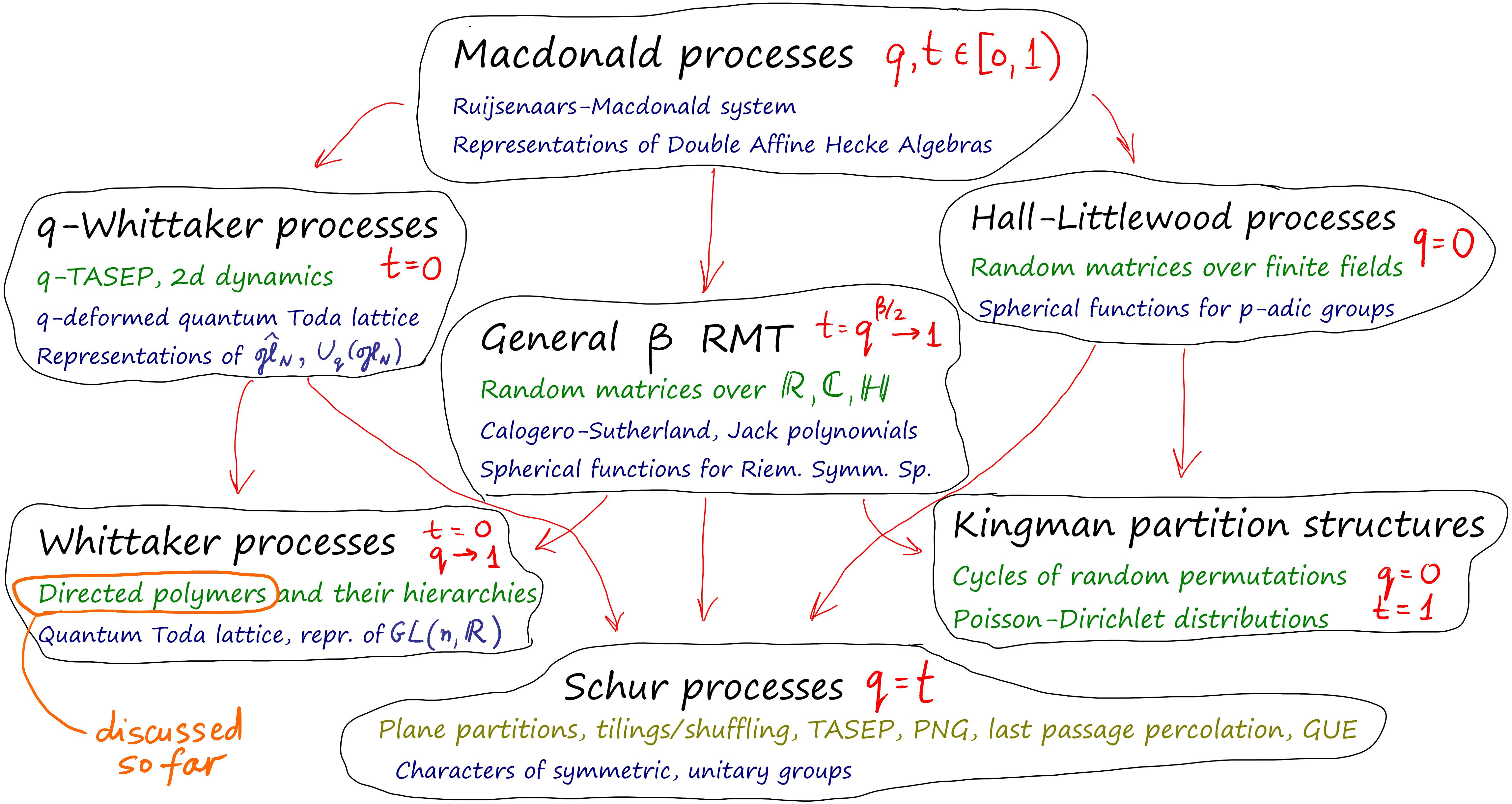
where the z_j – integration is over $\alpha_j + i\mathbb{R}$ with $\alpha_1 > \alpha_2 + c > \alpha_3 + 2c > \dots$

yields the solution of the quantum many body system

$$\frac{\partial \mathcal{U}}{\partial t} = \frac{1}{2}\Big(\Delta + c \sum_{i \neq j} \delta(x_i - x_j)\Big)\mathcal{U}$$

with the delta initial condition $\mathcal{U}(0, x) = \delta(x)$.

Note the symmetry between the attractive and repulsive cases (positive–negative c). Bethe eigenstates are very different!


Macdonald processes $q, t \in [0,1)$
Ruijsenaars-Macdonald system
Representations of Double Affine Hecke Algebras
q-Whittaker processes $t=0$
q-TASEP, 2d dynamics
q-deformed quantum Toda lattice
Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$
Hall-Littlewood processes $q=0$
Random matrices over finite fields
Spherical functions for p-adic groups
General β RMT $t=q^{\beta/2}\to 1$
Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$
Calogero-Sutherland, Jack polynomials
Spherical functions for Riem. Symm. Sp.
Whittaker processes $t=0$, $q\to 1$
Directed polymers and their hierarchies
Quantum Toda lattice, repr. of $GL(n,\mathbb{R})$
discussed so far
Kingman partition structures
Cycles of random permutations $q=0$, $t=1$
Poisson-Dirichlet distributions
Schur processes $q=t$
Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE
Characters of symmetric, unitary groups

(Ascending) Macdonald processes are probability measures on interlacing triangular arrays (Gelfand–Tsetlin patterns)

$$\begin{array}{ccccccccc}
\lambda^{(N)}_N & & \lambda^{(N)}_{N-1} & & \cdots & & \lambda^{(N)}_2 & & \lambda^{(N)}_1 \\
& \lambda^{(N-1)}_{N-1} & & \lambda^{(N-1)}_{N-2} & & \cdots & & \lambda^{(N-1)}_1 & \\
& & \ddots & & \cdots & & \cdot^{\cdot^{\cdot}} & & \\
& & & \lambda^{(2)}_2 & & \lambda^{(2)}_1 & & & \\
& & & & \lambda^{(1)}_1 & & & &
\end{array}$$

$$\lambda^{(m)}_j \in \mathbb{Z}_{\geq 0}$$

Macdonald polynomials

$$\mathbb{P}\left(\lambda^{(k)}\right) = \frac{P_{\lambda^{(k)}}(a_1, \dots, a_k)\, Q_{\lambda^{(k)}}(b_1, \dots, b_M)}{\Pi(\underbrace{a_1, \dots, a_k}; \underbrace{b_1, \dots, b_M})}$$

normalization constant

two groups of parameters

Macdonald polynomials $P_\lambda(x_1,\dots,x_N)\in\mathbb{Q}(q,t)[x_1,\dots,x_N]^{S(N)}$ with partitions $\lambda=(\lambda_1\geq\lambda_2\geq\dots\geq\lambda_N\geq 0)$ form a basis in symmetric polynomials in N variables over $\mathbb{Q}(q,t)$. They diagonalize

$$\left(\mathcal{D}_1 f\right)(x_1,\dots,x_N)=\sum_{i=1}^{n}\prod_{j\neq i}\frac{t x_i-x_j}{x_i-x_j}\, f(x_1,\dots,q x_i,\dots,x_N)$$

with (generically) pairwise different eigenvalues

$$\mathcal{D}_1 P_\lambda=\left(q^{\lambda_1}t^{N-1}+q^{\lambda_2}t^{N-2}+\dots+q^{\lambda_N}\right)P_\lambda .$$

They have many remarkable properties that include orthogonality (dual basis Q_λ), simple reproducing kernel (Cauchy type identity), Pieri and branching rules, index/variable duality, explicit generators of the algebra of (Macdonald) operators commuting with $\mathcal{D}_1$, etc.

We are able to do two basic things:

- Construct relatively explicit Markov operators that map Macdonald processes to Macdonald processes;
- Evaluate averages of a broad class of observables.

The construction is based on commutativity of Markov operators

$$\mathbb{P}(\lambda \to \mu) = \frac{P_\mu(x_1,\dots,x_{n-1})}{P_\lambda(x_1,\dots,x_n)} P_{\lambda/\mu}(x_n), \qquad \mathbb{P}(\lambda \to \nu) = \frac{1}{\Pi(x;u)} \frac{P_\nu(x_1,\dots,x_m)}{P_\lambda(x_1,\dots,x_m)} P_{\nu/\lambda}(u),$$

skew Macdonald polynomials

normalization

additional parameter

an idea from [Diaconis-Fill, 1990], and Schur process dynamics from [Borodin-Ferrari, 2008].

Evaluation of averages is based on the following observation. Let $\mathcal{D}$ be an operator that is diagonalized by the Macdonald polynomials (for example, a product of Macdonald operators),

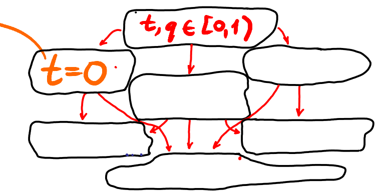
$$\mathcal{D} P_\lambda = d_\lambda P_\lambda .$$

Applying it to the Cauchy type identity $\sum_\lambda P_\lambda(a) Q_\lambda(b) = \Pi(a;b)$ we obtain

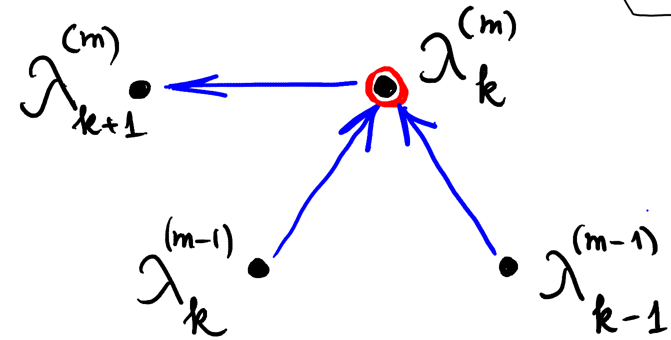
$$\langle d_\lambda \rangle = \frac{\mathcal{D}^{(a)} \Pi(a;b)}{\Pi(a;b)} .$$

If all the ingredients are explicit (as for products of Macdonald operators), we obtain meaningful probabilistic information. This should be contrasted with the lack of explicit formulas for the Macdonald polynomials.

Here is an example of a Markov process preserving the class of the q-Whittaker processes (Macdonald processes with $t=0$).

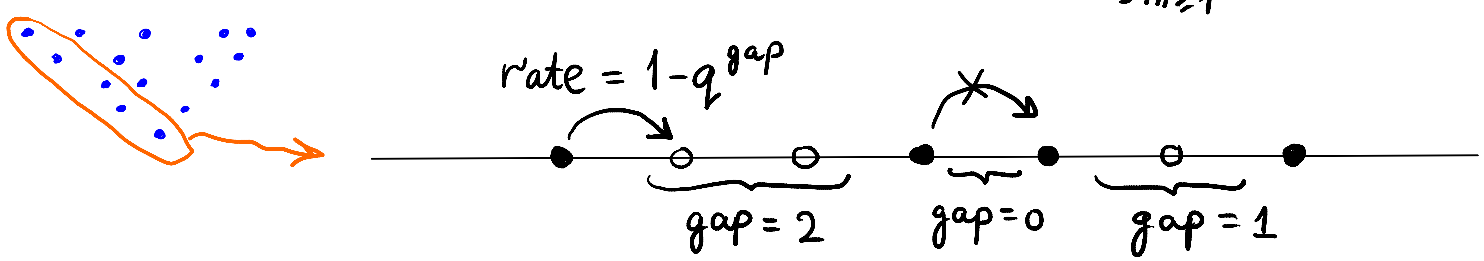


Each coordinate of the triangular array jumps by 1 to the right independently of the others with



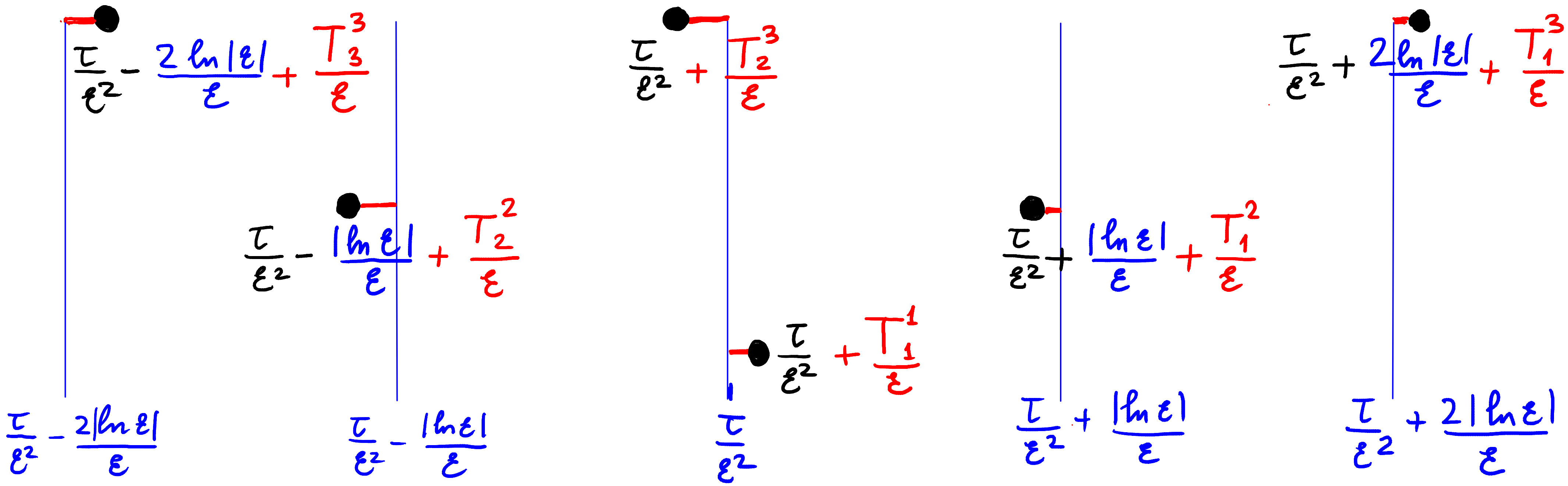
$$\mathrm{rate}\left(\lambda_k^{(m)}\right) = \frac{\left(1-q^{\lambda_{k-1}^{(m-1)}-\lambda_k^{(m)}}\right)\left(1-q^{\lambda_k^{(m)}-\lambda_{k+1}^{(m)}+1}\right)}{\left(1-q^{\lambda_k^{(m)}-\lambda_k^{(m-1)}}\right)}.$$

The set of coordinates $\left\{\lambda_m^{(m)}-m\right\}_{m\geq 1}$ forms q-TASEP



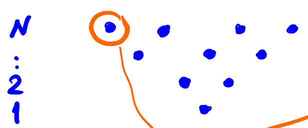
SIMULATION

As $q = e^{-\varepsilon} \to 1$, at large times τ/ε^2, with zero initial conditions, low rows of the triangular array behave as

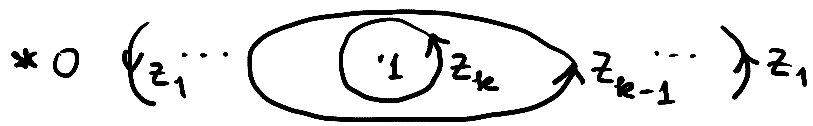


The real array $\{T_j^m\}_{1\le j\le m}$ is distributed according to the Whittaker process, and T_1^N or $-T_N^N$ is distributed as $\log Z_\tau^N$. The Whittaker process and its connection to polymers is due to [O'Connell, 2009].

Taking the observables corresponding to powers of the first Macdonald operator yields



$$\mathbb{E}\left(q^{\lambda_N^{(N)}(\tau)}\right)^k = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint\cdots\oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)\tau z_j}}{(1-z_j)^N} \frac{dz_j}{z_j}$$



$$\underbrace{\mathbb{E}\sum_{k=0}^{\infty} \frac{\left(q^{\lambda_N^{(N)}}\right)^k \zeta^k}{(1-q)\cdots(1-q^k)}}_{q\text{-Laplace transform of } q^{\lambda_N^{(N)}}} = \mathbb{E}\,\frac{1}{(\zeta q^{\lambda_N^{(N)}};q)_\infty} = \det(\mathbb{1}+K)_{L^2(\mathbb{N}\times \circlearrowleft)}$$

with

$$K(n_1, w_1; n_2, w_2) = \frac{f(w_1)\cdots f(q^{n_1-1}w_1)\,\zeta^{n_1}}{q^{n_1}w_1 - w_2}, \qquad f(w) = \frac{e^{(q-1)\tau w}}{(1-w)^N}$$

This is a perfectly legal q-version of the replica trick.

To summarize:

- The Macdonald processes form a new class of exactly solvable probabilistically meaningful measures on sequence of partitions
- They generalize the Schur processes but they are not determinantal; integrability comes from structural properties of the Macdonald poly's
- Several directed polymer models are obtained as limits, new algebraic and analytic properties follow
- Massive amounts of joint moment formulas are available while many-point limiting distributions are still resisting
- A new integral ansatz for solving quantum many body problems applies in other settings
- Many things remain to be investigated as Macdonald processes have a variety of degenerations