Infinitely many non-intersecting random walks

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[1] This talk is based on the joint work with Alexei Borodin

Lozenge tilings and *finitely* many non-intersecting random walks.

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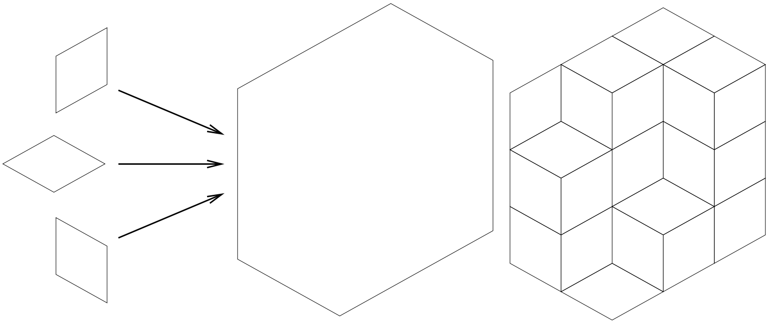
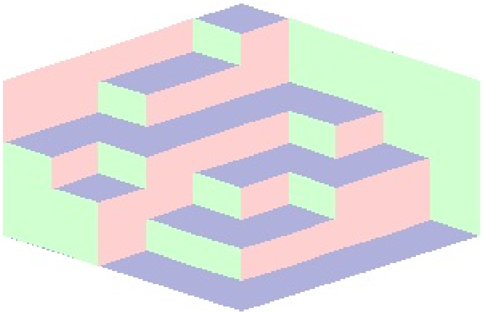
We are interested in tilings by rhombi with angles $\pi/3$ and $2\pi/3$ and side lengths 1 (lozenges).
The simplest tileable domain is an equi-angular hexagon of side lengths a, b, c, a, b, c.

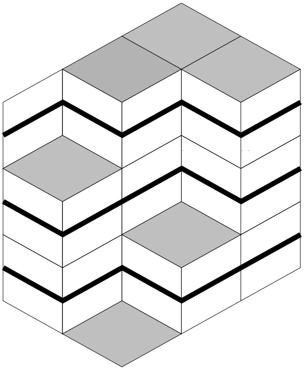


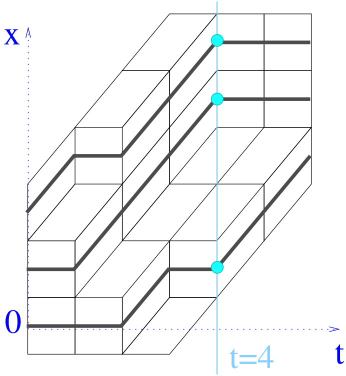
We are interested in *uniformly random* tilings of a *fixed* $a \times b \times c$ hexagon



This model has a very interesting limit behavior as $a, b, c \to \infty$.

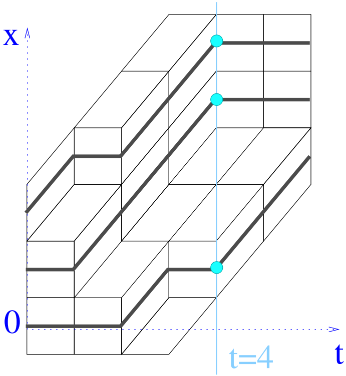
Non-intersecting walks





Tilings are in bijection with families of non-intersecting paths with fixed starting and ending points.

N non-intersecting random walks.

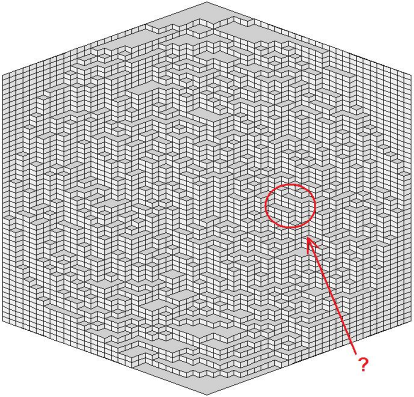


For the uniform measure vertical sections of paths produce a *Markov chain*.

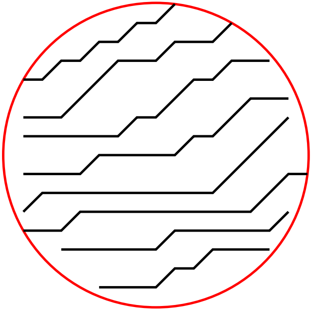
This chain is identified with $N = a$ independent simple random walks conditioned to finish after time $T = b + c$ at prescribed points $c, \dots, c + N - 1$ without collisions.

Local limits and paths

We enlarge the hexagon and observe the picture near a fixed point



Locally we still see paths.

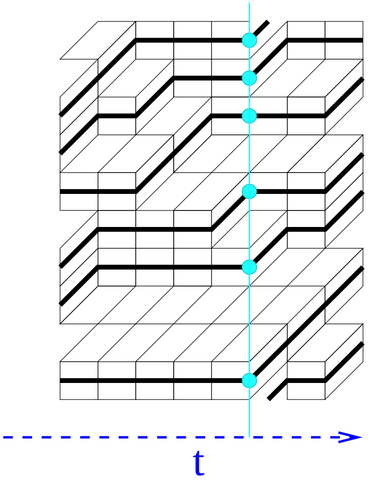


Local limit theorem

Theorem. (G.-2007) Fix a point inside a hexagon and let $a, b, c \to \infty$ so that their ratios tend to finite limits. In the neighborhood of the point we get a well-defined limit, which is a measure on lozenge tilings of the plane (or, equivalently, on infinite families of non-intersecting paths).

The limit measures posses lots of interesting properties and, in a sense, are completely explicit.

Infinitely-many paths?



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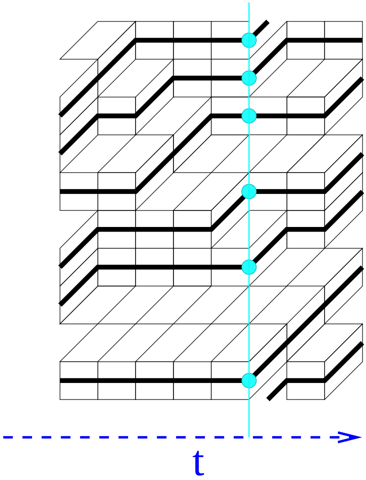
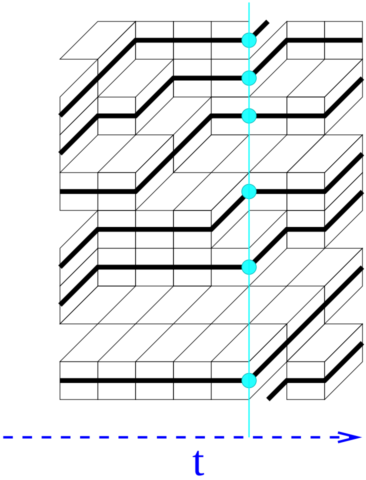


Q1. Is the limit object still a Markov chain?

Infinitely-many paths?



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Q2. How to define and deal with such *infinite-dimensional* Markov chains?

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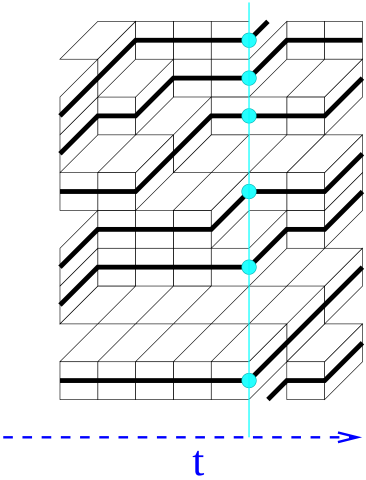

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Problem. The convergence theorem deals with *finite-dimensional* distributions, which are only tangentially related to the global Markov property.

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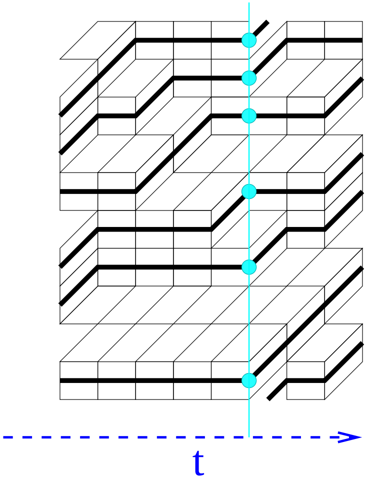

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This is still open!

Plan.

Having identified the problem we will now generalize and simplify the model as much as possible.
Aim 1: Remove the technical obstacles leaving the main questions unaffected.
Aim 2: Try to find a related model that have some additional structures which might help.

Step 1: generalization.

The construction starts with a 1D random process $X(t)$ taking values in $\mathbb{Z}$.

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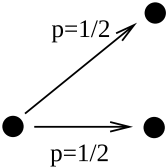
1. Simple discrete-time random walk



At each time step the particle either (with probability $1/2$) jumps by 1 step or stays put.

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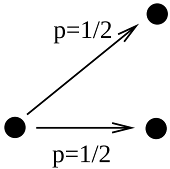

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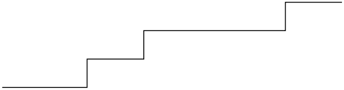
2. Poisson process



Continuous time process with independent increments.
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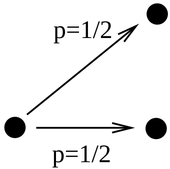

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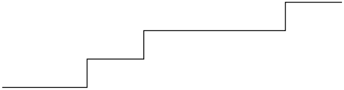

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3. More complicated versions
   RW with more complicated jump rules, birth-death processes, etc.

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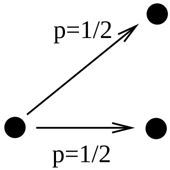

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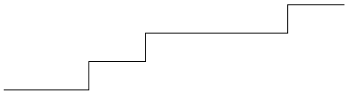
2. Poisson process $\longleftarrow$ we stick to this case



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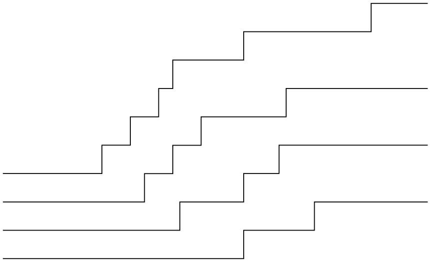
Step 2: simplification

Let us remove the condition that at time T the particles are at the prescribed positions.

We want to define N-dimensional Markov process $(X_1(t), \ldots, X_N(t))$ with non-intersecting paths.



These are N independent processes distributed as $X(t)$ conditioned never to collide.

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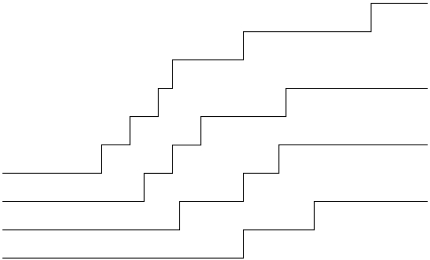

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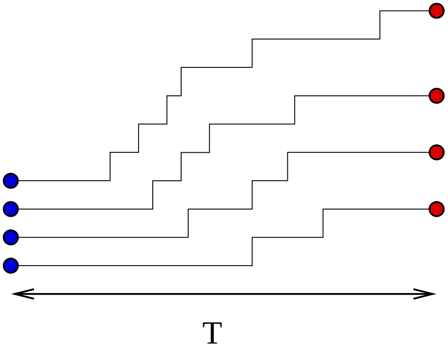
How to make this definition rigorous?

(We want to condition on the event of probability 0 and also to keep Markov property)

Theorem/Definition.[Konig-O'Connell-Roch] Fix T and numbers $y_1(T) < \cdots < y_N(T)$. Let $Z^T(t)$ be N independent processes distributed as $X(t)$, started from points $(1, \ldots, N)$, and conditioned to finish at time T in points $(y_1(T), \ldots, y_N(T))$ without collisions.

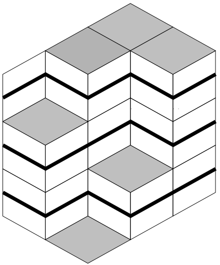


As $T \to \infty$ and $y_i(T)/T \to \beta_i$, the processes $Z^T(t)$ converge to a Markov process $\mathcal{Z}_N^{\beta_1,\ldots,\beta_N}(t)$.

Background

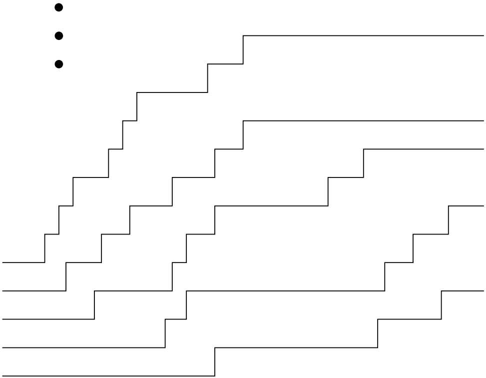
Distinguished case $\beta_i = 1$, $Z_N^{1,\dots,1}(0) = (1, \dots, N)$

1. Limit of uniformly random lozenge tilings of hexagons



2. For fixed t_0, the probability distribution $\mathcal{Z}_N^{1,\dots,1}(t_0)$ also arises in representation theory of infinite-dimensional unitary group $U(\infty)$.
3. For fixed t_0, the probability distribution of $\mathcal{Z}_N^{1,\dots,1}(t_0)$ is described by the so-called Charlier orthogonal ensemble — discrete random matrix-type distribution.

Question: How to define a $N \to \infty$ limit of such processes?



Informally we want to have countably many independent processes distributed as $X(t)$ and conditioned never to collide.

(Note a nontrivial behavior at zero time)

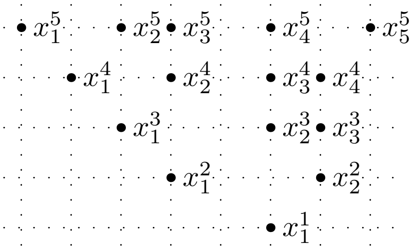
What's new?

From the first sight, nothing changed in the problem.

However, now there IS an additional structure!

Namely, the processes for different N are *related*.

Interlacing particles on $\mathbb{Z} \times \mathbb{Z}_+$:



one on the first horizontal line,
two on the second line, etc,
subject to the conditions
$x_i^{j+1} < x_i^j \le x_{i+1}^{j+1}$.

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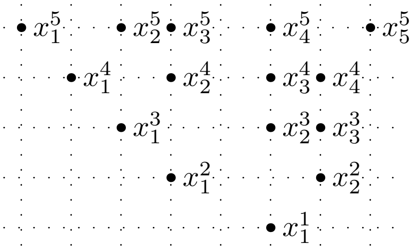

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Suppose that each particle has an exponential clock. All clocks are independent and the rate for particles at line j (i.e. $x_1^j, \dots, x_j^j$) is α_j. When the clock rings particle attempts to jump to the right.

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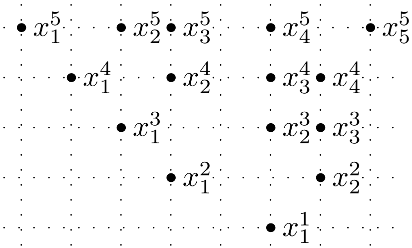

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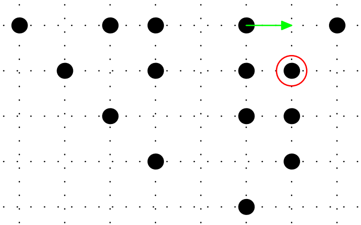
The interlacing conditions are preserved by the rule "if higher, then lighter".

Block:



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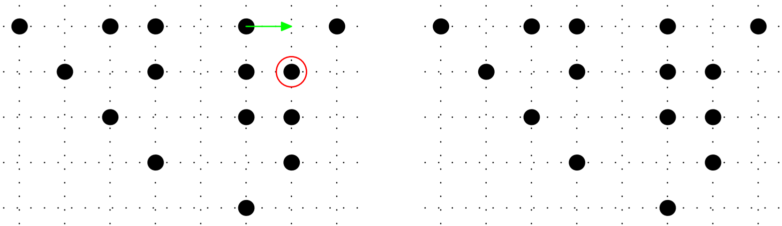

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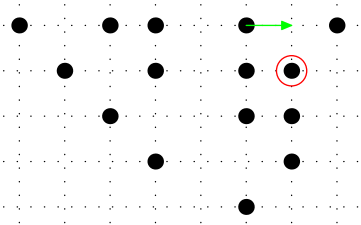

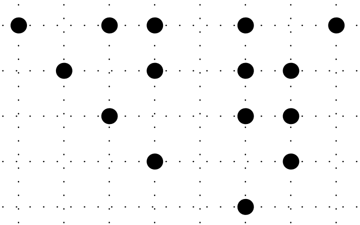

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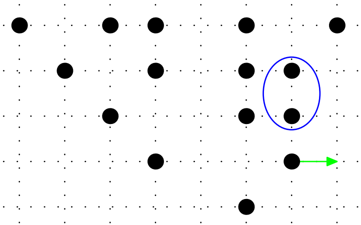

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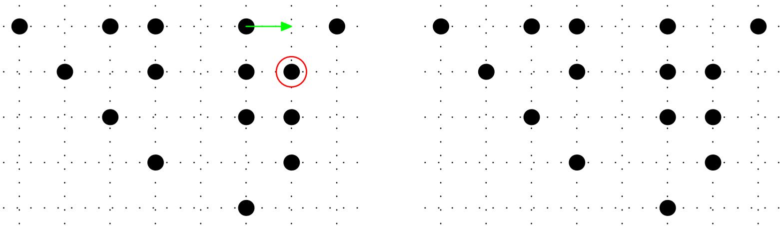

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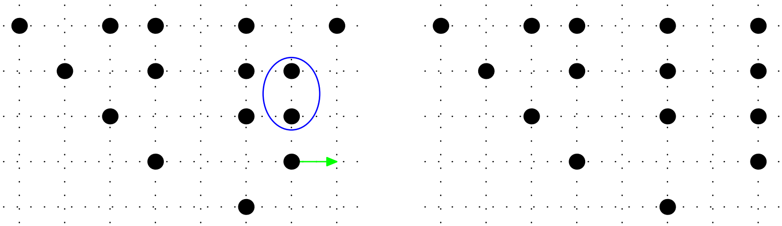
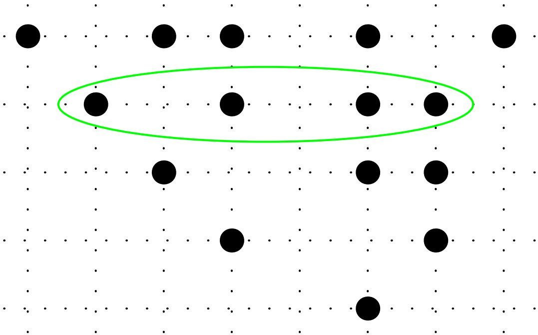


Markovian projection



Proposition. For every N the projection of the dynamics to N particles on the Nth horizontal line $(x_1^N, \ldots, x_N^N)$ is a Markov chain.

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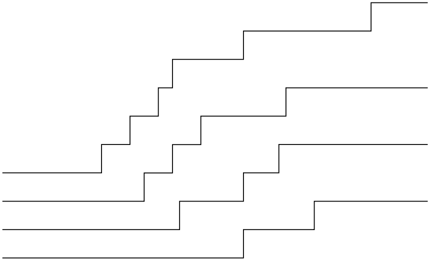
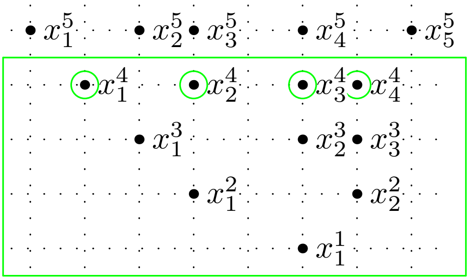


The set of speeds $\beta_1, \ldots, \beta_N$ is the set $\{\alpha_i\}$ *rearranged* in the increasing order.

More hidden structures

Definition. The probability measure P on the set of families of interlacing particles is α-Gibbs if



for any N and any fixed $x_1^N, \ldots, x_N^N$ the conditional distribution of interlacing particles on horizontal lines $1 \ldots N-1$ is

$$\frac{1}{M} \prod_{j=1}^{N} \alpha_j^{|x^j| - |x^{j-1}|}, \quad |x^j| = x_1^j + \cdots + x_j^j, \quad |x_0^j| = 0.$$

Why are α–Gibbs measures important?

Proposition 1. The above Markov dynamics on interlacing particles *preserves* the convex set of α–Gibbs measures. In other words, if the distribution of family of interlacing particles is α-Gibbs at zero time, then it is α–Gibbs at all times.

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Proposition 2. For any sequence $\alpha = \{\alpha_i\}$ there exists a space $\mathcal{X}_\alpha$ such that the set Ω of all α–Gibbs probability measures is homeomorphic to the set $\mathcal{M}_p(\mathcal{X}_\alpha)$ of all probability measures on $\mathcal{X}_\alpha$.

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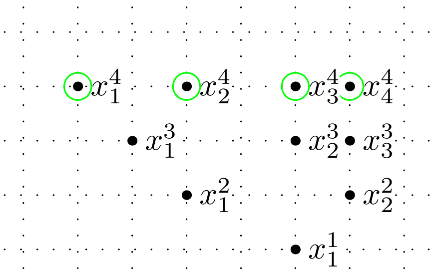
This is a general statement of convex analysis. It tells *nothing* about the actual structure of the space $\mathcal{X}$. And, indeed, this structure can be *very* different.

One example

Consider the set of α–Gibss measures on $4(4+1)/2 = 10$ interlacing particles on first 4 horizontal lines:



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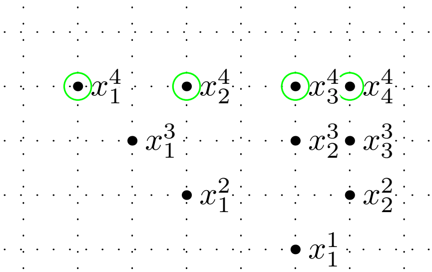

The space of α–Gibbs probability measures on the *first* 4 *lines* is homeomorphic to the set of all probability measures on 4-particle configurations (i.e. 4th horizontal line)

Construction of $\mathcal{Z}_{\infty}^{\alpha_1,\alpha_2,\cdots}(t)$

We are ready to define transitional probability (measure) $P_t^{\alpha}(x \to dy)$, $x \in \mathcal{X}_{\alpha}$, of the desired $N \to \infty$ limit of the processes on Nth horizontal line.

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Start the dynamics from this measure and wait time t.

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We *define* $P_t^\alpha(x \to dy)$ to be equal to this measure.

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Of course, the actual Markov chain strongly depends on the choice of α_i. The set $\mathcal{X}_\alpha$ is currently known only in two special cases.

Case $\alpha_1 = \alpha_2 = \cdots = 1.$

$$\mathcal{X}_\alpha \subset \mathbb{R}^{4\infty+2} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \times \mathbb{R},$$

is the set of sextuples

$$(a^+, b^-, a^+, b^-; c^+, c^-)$$

such that

$$a^\pm = (a_1^\pm \geq a_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty, \quad b^\pm = (b_1^\pm \geq b_2^\pm \geq \cdots \geq 0) \in \mathbb{R}^\infty,$$

$$\sum_{i=1}^{\infty} (a_i^\pm + b_i^\pm) \leq c^\pm, \quad b_1^+ + b_1^- \leq 1.$$

This is related to the representation theory of $U(\infty)$.

Case $\alpha_1 = \alpha_2 = \cdots = 1$.

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However, our dynamics boils down to the *deterministic* shift of δ^+.
[Note, that this is the closest case to the problem we started from!]

Case $\alpha_i = q^{1-i}$, $0 < q < 1$.

$\mathcal{N} = \mathcal{X}_\alpha$ is the set of monotonous sequences of integers:

$$\mathcal{N} = \{\nu = (\nu_1 < \nu_2 < \nu_3 < \dots) \in \mathbb{Z}^\infty\}.$$

For a family of interacting particles tiling distributed according to α–Gibbs measure $\mathcal{E}^\nu$ almost surely for every j.

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In other words, this is a semi-infinite point configuration read from left to right.

Denote the limit process started from $0 < 1 < 2 < 3 < \dots$ through $\mathcal{Z}_\infty^{1,q^{-1},\dots}(t)$.

Properties of $\mathcal{Z}_{\infty}^{1,q^{-1},\cdots}(t)$.

1. Finite-dimensional distributions of $\mathcal{Z}_{\infty}^{1,q^{-1},\cdots}(t)$ are $N \to \infty$ limits of distributions of the process $\mathcal{Z}_{N}^{1,q^{-1},\cdots}(t)$ on Nth horizontal line.
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2. $\mathcal{Z}_\infty^{1,q^{-1},\cdots}(t)$ is naturally extended to a Feller Markov process on $\overline{\mathcal{N}}$ — local compactification of $\mathcal{N}$.
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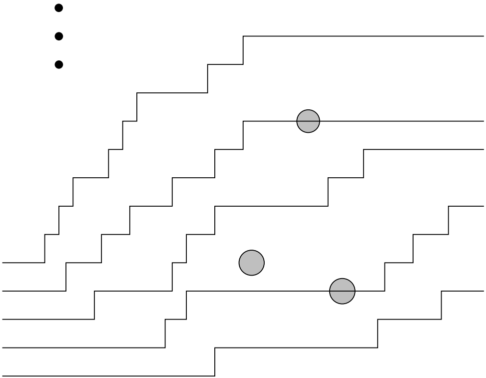
Determinantal point process: correlation functions



$$\rho_n(t_1, x_1; \ldots t_n, x_n)$$
$$= \mathrm{Prob}(\text{paths go through points } (t_1, x_1), \ldots, (t_n, x_n))$$

Determinantal point process: kernel

For any $n \geq 1$, the nth correlation function ρ_n of process $\mathcal{Z}_\infty^{1,q^{-1},\dots}(t)$ has the form

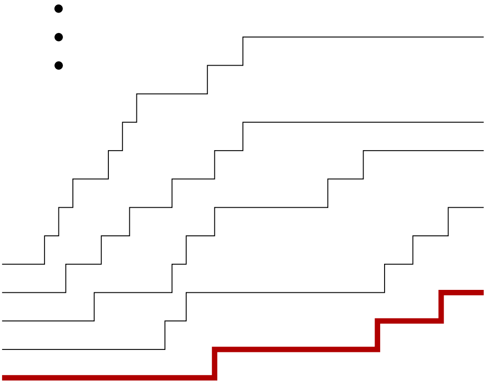
$$\rho_n(x_1, t_1; x_2, t_2; \dots; x_n, t_n) = \det_{i,j=1,\dots,n}[K(x_i, t_i; x_j, t_j)],$$

where

$$K(x_1, t_1; x_2 t_2) = -\frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{dw}{w^{x_1 - x_2 + 1}} e^{w(t_1 - t_2)} \mathbf{1}_{t_1 > t_2} + \frac{1}{(2\pi i)^2}\oint_{\mathcal{C}} dw \oint_{\mathcal{C}'} dz e^{wt_1 - zt_2} \frac{(w;q)_\infty}{(z;q)_\infty} \frac{z^{x_2}}{w^{x_1+1}} \frac{1}{w-z},$$

$\mathcal{C}$ is positively oriented and includes only the pole 0 of the integrand; $\mathcal{C}'$ goes from $+i\infty$ to $-i\infty$ between $\mathcal{C}$ and point 1.

Bottommost particle of the process $\mathcal{Z}_{\infty}^{1,q^{-1}\cdots}(t)$.



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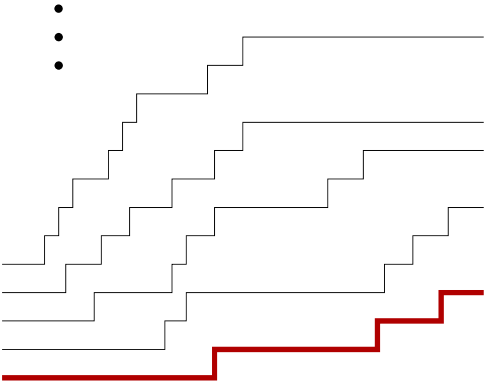
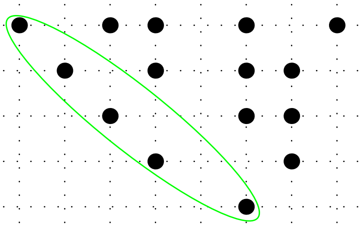


We will give an independent description of the stochastic evolution of the smallest particle.

Bottommost particle of the process $\mathcal{Z}_{\infty}^{1,q^{-1}\cdots}(t)$.



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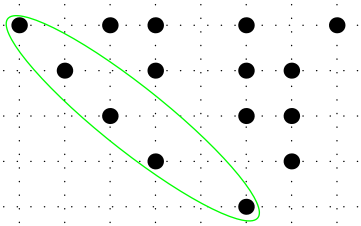
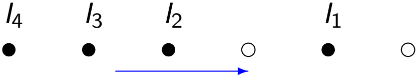
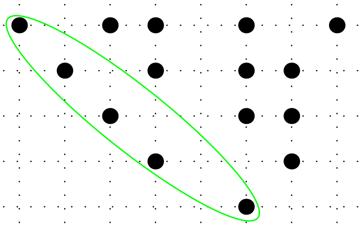
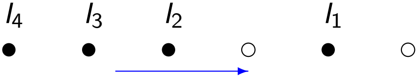




TASEP with rate$(l_k) = q^{1-k}$.

Bottommost particle of the process $\mathcal{Z}_{\infty}^{1,q^{-1}\cdots}(t)$.





TASEP with rate$(l_k) = q^{1-k}$.

Proposition Stochastic evolution of "densely packed group" $\lim_{k\to\infty}[x(l_k)+k-1] = \lim_{k\to\infty}[x_1^k+k-1]$ is the same as the evolution of the bottommost particle of $\mathcal{Z}_{\infty}^{1,q^{-1}\cdots}(t)$.

Related questions:

Our dynamics ends up being deterministic for $\alpha_i = 1$. However, there is a way to introduce *different* natural stochastic dynamics which will also preserve the Gibbs measures.
[Borodin-Olshanski,2010] This leads to a non-trivial Markov process on the limit (infinite-dimensional with continuous coordinates) space with invariant distribution given by the so-called (z, w)–measures related to the problem of harmonic analysis on the infinite-dimensional unitary group $U(\infty)$.

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Further open questions:

1. Infinite-dimensional dynamics for general sequence α_i?
2. Macdonald-like deformations?
3. What is the answer in the original lozenge tilings settings?

Literature:

1. A. Borodin, V. Gorin, *Markov processes of infinitely many nonintersecting random walks*, to appear in Probability Theory and Related Fields. arXiv: 1106.1299.
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