The bulk scaling limit of the Laguerre ensemble

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Objective of the talk:

- Family of random matrices: $(M_n)_{n\geq 1}$; M_n : $n \times n$ matrix with random entries.
- AIM: To study the structure of the spectrum (set of eigenvalues) for large random matrices.
- Ensemble of random matrices considered: the β-Laguerre ensemble, $\beta > 0$, a generalization of the Wishart ensembles.

β-ensemble: the β-Laguerre ensemble, $\beta > 0$

Fix $\beta > 0$. For $(\lambda_1, \cdots, \lambda_n) \in (\mathbb{R}^+)^n$, define

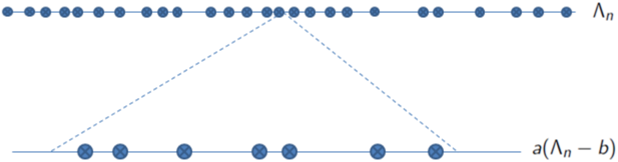
$$\mathbb{P}_{\beta}(\lambda_1, \cdots, \lambda_n) = \frac{1}{Z^{\beta}_{n,m+1}} \prod_{1 \le j < k \le n} |\lambda_j - \lambda_k|^{\beta} \prod_{\ell=1}^{n} \lambda_\ell^{\frac{\beta}{2}(m-n)-1} e^{-\frac{\beta}{2}\lambda_\ell}$$

where $m - 1 \ge n$ and $Z^{\beta}_{n,m-1}$ is a normalizing constant.

- $\beta = 1$: Wishart real ensemble
- $\beta = 2$: Wishart complex ensemble
- $\beta = 4$: Wishart quaternion ensemble

Local limit

- The eigenvalues form a point process Λ_n on $\mathbb{R}_+$
- To study local behavior of eigenvalues, rescale and translate the eigenvalues to zoom in on a particular region of the spectrum (typically edges and bulk) and then let $n \to \infty$.



Does $a_n(\Lambda_n - b_n) \Rightarrow ?$ as $n \to \infty$

The Laguerre spectrum

Let $\lambda_1, \dots, \lambda_n$ have joint density distribution $\mathbb{P}_\beta$. Suppose that $m/n \to \gamma \in [1, \infty)$. Then,

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i/n} \to \mu \text{ weakly, as } n \to \infty.$$

$$\frac{d\mu}{dx} = \tilde{\sigma}^{\gamma}(x) = \frac{\sqrt{(x-a^2)(b^2-x)}}{2\pi x} 1_{[a^2, b^2]}(x), a = \gamma^{1/2} - 1, b = \gamma^{1/2} + 1.$$

Marchenko-Pastur distribution.

Laguerre spectrum: $[(\sqrt{m} - \sqrt{n})^2, (\sqrt{m} + \sqrt{n})^2]$.

The bulk limit of the β-Laguerre ensemble

Theorem (Bulk limit of the Laguerre ensemble, J.,Valkó, 2011)

Fix $\beta > 0$, assume that $m/n \to \gamma \in [1, \infty)$ and let $c \in (a^2, b^2)$ for $a = \gamma^{1/2} - 1, b = \gamma^{1/2} + 1$. Let Λ_n^L denote the point process with joint eigenvalue $\mathbb{P}_\beta$. Then

$$2\pi\tilde{\sigma}^\gamma(c)\left(\Lambda_n^L - cn\right) \Rightarrow Sine_\beta.$$

- In the edges case, the limit point process is described by the mean of stochastic operators.
- Valkó and Virág, in 2007, showed that, for the β-Hermite ensemble, the bulk scaling can be described by a discrete point process: the Sine_β process or Brownian carousel.
- We prove that the bulk scaling of the β-Laguerre ensemble converges to the Sine_β process which completes the picture about the point process scaling limits of the Laguerre ensemble.

Description of the local limits

Model solvable for $\beta = 1, 2, 4$

$\beta = 2$: Determinantal processes.

- Bulk: Sine_2 determinantal process with kernel

$$K(x, y) = \frac{sin\left(\pi(x - y)\right)}{\pi(x - y)}$$

Problem: For general $\beta > 0$, there is no form of the general β correlations that seem amenable to a description of the asymptotics.

Matrix model for the Laguerre β-ensemble, $\beta > 0$

It was then a big deal when Dimitriu and Edelman,in 2002 found that if $A_{n,m}$ is the following $n \times n$ bidiagonal matrix:

$$A_{n,m} = \frac{1}{\sqrt{\beta}} \begin{bmatrix} \tilde{\chi}_{\beta(m-1)} & & & & \\ \chi_{\beta(n-1)} & \tilde{\chi}_{\beta(m-2)} & & & \\ & \ddots & \ddots & & \\ & & \chi_{\beta \cdot 2} & \tilde{\chi}_{\beta(m-n+1)} & \\ & & & \chi_{\beta} & \tilde{\chi}_{\beta(m-n)} \end{bmatrix}$$

where the entries are independent, and each χ_r stands for the square root of an independent χ^2 with parameter r. In other words,

$$\mathbb{P}(\chi_r \in dx) = \frac{1}{2^{r-2/2}\Gamma(r/2)} x^{r-1} e^{-x^2/2} dx.$$

Then, the eigenvalues of the tridiagonal matrix $L_\beta^n = A_{n,m} A_{n,m}^T$ are distributed according to $\mathbb{P}_\beta$.

The Sine_β process

Let Z be a 2-dimensional Brownian motion and consider the one-parametrised system of SDE for $\lambda \in \mathbb{R}$.

$$d\alpha_\lambda = \lambda \frac{\beta}{4} \exp\left(-\frac{\beta}{4} t\right) dt + \Re\left(\left(e^{-i\alpha_\lambda} - 1\right) dZ\right)$$

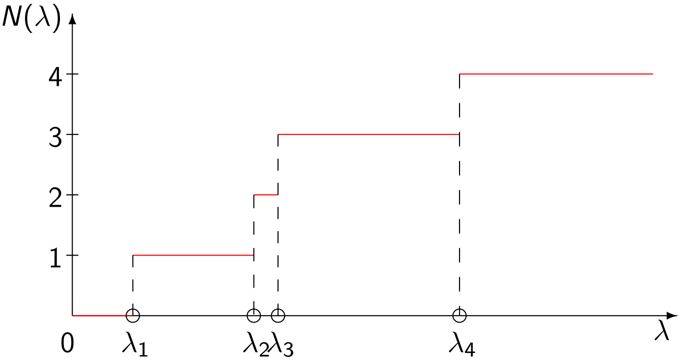
with $\alpha_\lambda(0) = 0$. Then,

$$N(\lambda) = \frac{1}{2\pi} \lim_{t\to\infty} \alpha_\lambda(t)$$

exists, and is integer valued a.s. We define the Sine_β point process as the discontinity points of $N(\lambda)$.

$$N(\lambda) = \sharp\{ \text{ points of } \text{Sine}_\beta \in (0, \lambda]\}$$

The Sine_β process



The Sine_β process

For a fixed $\lambda > 0$,

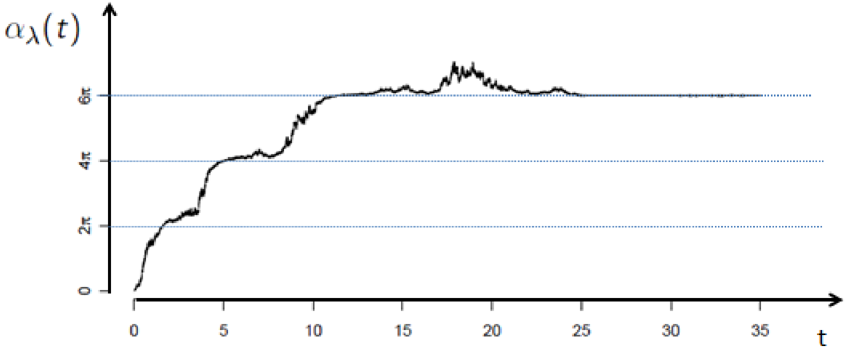
$$\begin{aligned}\Re\left(\left(e^{-i\alpha_\lambda}-1\right)dZ\right) &= \Re\left(e^{-i\frac{\alpha_\lambda}{2}}\left(e^{-i\frac{\alpha_\lambda}{2}}-e^{i\frac{\alpha_\lambda}{2}}\right)dZ\right)\\ &= 2\sin\left(\frac{\alpha_\lambda}{2}\right)\Im\left(e^{-i\frac{\alpha_\lambda}{2}}dZ\right)\\ &= 2\sin\left(\frac{\alpha_\lambda}{2}\right)dW\end{aligned}$$

where W is a Brownian motion. Thus,

$$d\alpha_\lambda = \lambda\frac{\beta}{4}\exp\left(-\frac{\beta}{4}t\right)dt + 2\sin\left(\frac{\alpha_\lambda}{2}\right)dW$$

with $\alpha_\lambda(0) = 0$.

A simulation of $\alpha_\lambda(t)$



- If it hits an integer multiple of 2π it will stay above it.
- The limit as $t \to \infty$ exists and is an integer multiple of 2π.

Ideas of the proof

Tridiagonal representation of the β-Laguerre ensemble: Let A be the following $n \times n$ bidiagonal matrix:

$$A = \frac{1}{\sqrt{\beta}} \begin{bmatrix} a_1 & & & & \\ b_1 & a_2 & & & \\ & \ddots & \ddots & & \\ & & b_{n-2} & a_{n-1} & \\ & & & b_{n-1} & a_n \end{bmatrix}, a_k = \tilde{\chi}_{\beta(n-k)}, b_k = \chi_{\beta(n-k)}$$

Then the eigenvalues of AA^T are distributed according to $\mathbb{P}_\beta$.

- We want to understand the bulk scaling limit of the eigenvalues of AA^T.
- Problem: not enough independence– we will use another matrix.

Ideas of the proof

Consider the $2n \times 2n$ tridiagonal matrix:

$$B = \frac{1}{\sqrt{\beta}} \begin{bmatrix} 0 & a_1 & & & \\ a_1 & 0 & b_1 & & \\ & b_1 & \ddots & \ddots & \\ & & \ddots & \ddots & a_n \\ & & & a_n & 0 \end{bmatrix}$$

- If $[u_1, v_1, u_2, \ldots]^T$ is an eigenvector for B with eigenvalue λ, then, $[u_1, u_2, \ldots]^T$ is an eigenvector for $A^T A$ with eigenvalue λ^2 and $[v_1, v_2, \ldots]^T$ is an eigenvector for AA^T with eigenvalue λ^2.
- ADVANTAGE: independence of the entries modulo symmetry.

Ideas of the proof

Tridiagonal matrix representation L_n^β for the β-Laguerre ensemble. The proof relies on being able to track the eigenvalues of a tridiagonal matrix. Consider the $n \times n$ tridiagonal matrix

$$M = \begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & c_{n-1} & a_n \end{bmatrix}, b_i > 0, c_i > 0.$$

If $[u_1, u_2, \ldots, u_n]^T$ is an eigenvector corresponding to Λ, then the ratio $r_{\ell,\Lambda} = u_{\ell+1}/u_\ell$ satisfies the single term recursive recursion:

$$r_{0,\Lambda} = \infty, r_{\ell,\Lambda} = \frac{1}{b_\ell}\left(-\frac{c_{\ell-1}}{r_{\ell-1,\Lambda}} + \Lambda - a_\ell\right).$$

FACT: Λ is an eigenvalue if and only if $r_{n,\Lambda} = 0$.

We can turn $r_{\ell,\Lambda}$ into an angle

$$r_{\ell,\Lambda} \in \mathbb{R} \cup \{\infty\} \longleftrightarrow z_\ell = e^{i\varphi_{\ell,\Lambda}}, \varphi_{\ell,\Lambda}\text{: phase function.}$$

$(\varphi_{\ell,\Lambda})_{\ell \leq n}$ has the following properties:

- For each fixed $0 \leq \ell \leq n$, $\varphi_{\ell,\Lambda}$ is continuous monotone increasing function of Λ.
- $\varphi_{n,\Lambda} = 0 \mod 2\pi$ identifies the eigenvalues. More precisely,

$$\sharp\{(\varphi_{n,0}, \varphi_{n,\Lambda}] \cap 2\pi\mathbb{Z}\} = \sharp \text{ eigenvalues in } (0, \Lambda].$$

- $\varphi_{\ell,\Lambda}$ is a Markov chain with respect to the filtration $\mathcal{F}_\ell = \sigma(\varphi_{k,\Lambda} : k \leq \ell)$.

Rescale according to theorem:

$$\Lambda = \frac{\lambda}{2\pi\sigma(c)} + cn,$$

and consider $\varphi_{\ell,\lambda}$.

Problem: The increments $\varphi_{\ell+1,\lambda} - \varphi_{\ell,\lambda}$ are not infinitesimal in the limit.

However, a simple transformation (monotone and 2π invariant) allows to reguralise it. Call $\tilde{\varphi}_{\ell,\lambda}$ the reguralised version.

- The regularised phase function $\tilde{\varphi}_{\ell,\lambda}$ converges, as $n \to \infty$ to the solution to a stochastic differential equation (version of the Kurtz theorem). More precisely, for $t < 1$,

$$\tilde{\varphi}_{\lfloor nt \rfloor,\lambda} \Rightarrow \tilde{\varphi}_\lambda(t) \text{ in distribution as } n \to \infty.$$

- Particularly, $\alpha_{\lfloor nt \rfloor,\lambda} = \tilde{\varphi}_{\lfloor nt \rfloor,\lambda} - \tilde{\varphi}_{\lfloor nt \rfloor,0}$ converges to a time changed version of the stochastic Sine equation.
- Letting $t \to 1$, we obtain that

$$\sharp \{(\tilde{\varphi}_{n,0}, \tilde{\varphi}_{n,\lambda}] \cap 2\pi\mathbb{Z}\} \Rightarrow \lim_{t\to 1} \frac{\alpha_\lambda(t)}{2\pi} = N(\lambda).$$

Thank you !