Spiking the hard edge

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Beta Tracy-Widom, no spikes

Many years ago J. Ramírez, B. Virág extended the Tracy-Widom laws to the limiting distributions of the largest points in for the standard beta ensembles.

In particular, consider the law on n points $\lambda_1, \dots, \lambda_n$ with density proportional to

$$\prod_{j<k} |\lambda_j - \lambda_k|^\beta \prod_{k=1}^{n} w(\lambda_k)$$

for $w(\lambda) = e^{-\beta\lambda^2/4}$ (beta-Hermite) or $w(\lambda) = \lambda^{\frac{\beta}{2}(m-n)+1} e^{-\beta\lambda/2}$ (beta-Laguerre). Then the limit distribution is described by:

$$TW_\beta = \sup_{f \in L} \frac{2}{\sqrt{\beta}} \int_0^\infty f^2(x)\, db(x) - \int_0^\infty [(f'(x))^2 + x f^2(x)]\, dx.$$

Here L are those functions with $\int_0^\infty f^2 = 1, \int_0^\infty [(f')^2 + x f^2] < \infty$ and $f(0) = 0$.

Riccati correspondence

This variational formula has the interpretation that $-TW_\beta$ is the ground state eigenvalue for

$$\mathcal{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x)$$

on $\mathbb{R}_+$ with Dirichlet conditions at the origin. In this way the above established a conjecture of Edelman-Sutton.

For fixed values of the spectral parameter λ, following the zeros of

$$d\psi'(x) = (x - \lambda)\psi(x)dx + \frac{2}{\sqrt{\beta}}\psi(x)db(x) \text{ with } \psi(0) = 0, \psi'(0) = 1$$

counts the eigenvalues of $\mathcal{H}_\beta$ below λ. Now set $p(x) = \psi'(x)/\psi(x)$, which starts at $+\infty$, solves

$$dp(x) = \frac{2}{\sqrt{\beta}} db(x) + (-\lambda + x - p^2(x))dx$$

and $F_\beta(\lambda) = \mathbb{P}_\infty(p(\cdot, -\lambda) \text{ never explodes})$. For $\beta = 1, 2, 4$ these have Painlevé expressions.

Tridiagonals

For the Laguerre setting, put, for any $\beta > 0$,

$$L_\beta = \frac{1}{\sqrt{\beta}} \begin{bmatrix} \chi_{m\beta} & \chi_{(n-1)\beta} & & & \\ & \chi_{(m-1)\beta} & \chi_{(n-2)\beta} & & \\ & & \ddots & \ddots & \\ & & & \chi_{(m-n+2)\beta} & \chi_\beta \\ & & & & \chi_{(m-n+1)\beta} \end{bmatrix}$$

where each χ_r is independent.

Then, the eigenvalues of $L_\beta L_\beta^\dagger$ follow the law of the $w(\lambda) = \lambda^{\frac{\beta}{2}(m-n)+1} e^{-\beta\lambda}$ beta ensemble.

This fact is due to Dumitriu and Edelman. At $\beta = 1, 2$ can be arrived at by the well known Householder transformations from the "full" sample covariance matrices $S = XX^\dagger$ for X an $n \times m$ independent Gaussian matrix.

Spiked models, Wishart setup

Classically, one takes the matrix $S = XX^\dagger$ with $X = [X_1 \dots X_m]$, each $X_k \in R^n$ drawn from a centered population, in the regime $m \gg n$.

In this case $\frac{1}{m}S$ is a consistent estimator for the *population covariance matrix*, $\mathbf{\Sigma} = \mathbb{E}X_1{X_1}^\dagger$.

In the RMT regime $m = O(n)$, we have at this point discussed the limit of λ_{max} when $\Sigma = I$.

Johnstone asked the question: When can λ_{max} detect between $\mathbf{\Sigma} = I$ and an arbitrary $\mathbf{\Sigma}$?

To cut the problem down to size introduce the "spiked" model in which

$$\mathbf{\Sigma} = \mathrm{diag}(\mathrm{c}_1, \mathrm{c}_2, \dots, \mathrm{c}_\mathrm{r}, 1, 1, \dots, 1)$$

and r is fixed as $n \uparrow \infty$. How does the law of λ_{max} depend on $c's$?

The phase transition

In 2005 Baik, Ben Arous, and Peché found the following phenomena at $\beta = 2$, which I only describe in for a single "spike".

If $c < \mathfrak{c}$: $\mathbb{P}\Big(\sigma_n(\lambda_{\max} - \mu_n) \le t\Big) \to F_2(t)$.

If $c > \mathfrak{c}$: $\mathbb{P}\Big(\sigma_n'(\lambda_{\max} - \mu_n') \le t\Big) \to \int_{-\infty}^t e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$.

If $c = \mathfrak{c} - wn^{-1/3}$: $\mathbb{P}\Big(\sigma_n(\lambda_{\max} - \mu_n) \le t\Big) \to F(t,w) = F_2(t)f(t,w)$, all described in terms of Painlevé II.

This is a steepest descent analysis on a fairly explicit integral kernel of the underlying determinantal process.

Soon after D. Wong obtained a result for $\beta = 4$, and there is also work of Mo at $\beta = 1$.

Spikes for general beta

The main insight is that if you spike at say $\beta = 1, 2$ the bidiagonalization procedure goes through with the only change in the top entry: $\chi_{m\beta}$ becomes $\sqrt{c}\chi_{m\beta}$.

Bloemendal-Virág (2011) proved (among other things) that $\lambda_{max}(L_{\beta,c}L_{\beta,c}^{\dagger})$ under critical spiking ($c = \mathfrak{c} - wn^{-1/3}$) has a scaled limit distribution given by

$$TW_{\beta,w} = \sup_{f \in L'} \frac{2}{\sqrt{\beta}} \int_0^\infty f^2(x)db(x) - \int_0^\infty [(f'(x))^2 + xf^2(x)]dx$$

where L' is again those functions with $\int_0^\infty [(f')^2 + xf^2] < \infty$, but subject to

$$f'(0) = wf(0)$$

at the origin.

So all is familiar, but with Robin rather than Dirichlet conditions. This extra variable now makes the PDE tied to the Riccati correspondence more meaningful (one can get back the Painlevé formulas for instance).

Bringing in the hard edge

In the basic $XX^\dagger$ Gaussian Wishart ensemble the behavior of the limiting density of states can be very different in the vicinity of λ_{min} vs. λ_{max}.

With $\frac{m}{n} \to \gamma$, that object is given by

$$\frac{1}{n}\sum_{k=1}^{n}\delta_{\lambda_k}(\lambda) \to \frac{\sqrt{(\lambda-\ell_-)(\ell_+-\lambda)}}{2\pi\lambda}\,d\lambda$$

where $\ell_\pm = (1 \pm \sqrt{\gamma})^2$.

When $\gamma > 1$ both edges are "soft", and we have Tracy-Widom fluctuations.

When $\gamma = 1$, then $a_- = 0$ and the eigenvalues now feel the "hard edge" of the origin.

In fact, if $m = n + a$ as $n \uparrow \infty$ there is a one-parameter family of limit laws for λ_{min} indexed by a (first described in terms of Painlevé by Tracy-Widom).

As $a \to \infty$ recover the soft-edge Tracy-Widom laws. (And true for all β, as should be.)

Spiking (down) the hard edge

Return now to the spiked tridiagonal model, tuned to see the hard edge: for any $a > -1$

$$L_{\beta,c} = \frac{1}{\sqrt{\beta}} \begin{bmatrix} \sqrt{c}\chi_{(n+a)\beta} & \chi_{(n-1)\beta} & & & \\ & \chi_{(n+a-1)\beta} & \chi_{(n-2)\beta} & & \\ & & \ddots & \ddots & \\ & & & \chi_{(a+2)\beta} & \chi_{\beta} \\ & & & & \chi_{(a+1)\beta} \end{bmatrix}$$

We want to understand the limit law for $\lambda_{min}(L_{\beta,c}L_{\beta,c}^{\dagger})$, tuning $c \downarrow 0$ as $n \uparrow \infty$.

With $c = 1$, the charactering limit operator was already found by J. Ramírez and myself (2009).

The "critical" scaling turns is $c = \frac{\kappa}{n}$.

At $\beta = 2$ this problem was studied at the level of correlation functions (via multiple-orthogonal polynomials) by Desrosiers-Forrester.

Limit operator(s)

What we show is that $(nL_{\beta,c}L_{\beta,c}^{\dagger})^{-1}$, after embedding into an appropriate L^2 space, converges in norm to the (compact) integral operator:

$$(\mathfrak{G}_{\beta,a,\kappa}f)(x) = \int_0^\infty \int_0^{x\wedge y} \mathfrak{s}(dz) f(y)\mathfrak{m}(dy) + \kappa^{-1}\int_0^\infty f(y)\mathfrak{m}(dy).$$

Here $x \mapsto b(x)$ a Brownian motion,

$$\mathfrak{m}(dx) = \exp\left(-(a+1)x - \frac{2}{\sqrt{\beta}}b(x)\right)dx, \quad \mathfrak{s}(dx) = \exp\left(ax + \frac{2}{\sqrt{\beta}}b(x)\right)dx.$$

This is the resolvent for the diffusion $t \mapsto x_t$ with speed measure $\mathfrak{m}(dx)$, scale function $\int_0^x \mathfrak{s}(dx')$, and killing measure $\kappa\delta_0(x)$.

Take $t \mapsto \bar{x}_t$ with the same speed and scale, but with simple reflection at the origin. With L_t the local time of $\bar{x}_t$ at the origin, x_t equals $\bar{x}_t$ up to time T defined by

$$\mathbb{P}(T > t \,|\, \bar{x}_\cdot) = e^{-\kappa L_t},$$

at which point the path is killed. The unspiked case corresponds to $\kappa = \infty$.

Aside on the "supercritical" regime

This is the analog of the Gaussian limit at the supercritically spiked soft-edge.

Denoting $\Lambda(\beta, a, \kappa)$ the limiting hard edge eigenvalue (one over the largest eigenvalue of $\mathfrak{G}_{\beta,a,\kappa}$):

$$\kappa^{-1}\Lambda(\beta, a, \kappa) \to \frac{1}{\beta}\chi^2_{\beta(a+1)}.$$

as $\kappa \downarrow 0$.

A simple perturbation argument implies that $\kappa\,\mathfrak{G}_{\beta,a,\kappa}$, for small κ, has ground state approaching the constant function with corresponding eigenvalue $\int_0^\infty \mathfrak{m}(dx)$.

The equality in law

$$\int_0^\infty \mathfrak{m}(dx) = \int_0^\infty \exp\left[-(a+1)x - \frac{2}{\sqrt{\beta}} b(x)\right] dx = \frac{1}{\beta}\chi^{-2}_{\beta(a+1)},$$

is due to Dufresne (motivated by a problem in finance).

PDEs and spiked hard to soft transition

Again there is a hitting time description. With the process

$$dp(x) = \frac{2}{\sqrt{\beta}} p(x) db(x) + \Big((a + \tfrac{2}{\beta}) p(x) - p^2(x) - \lambda e^{-x} \Big) dx,$$

it holds

$$\mathbb{P}(\Lambda(\beta, a, \kappa) > \lambda) := \mathbb{P}_\kappa \Big(p(\cdot) \text{ never vanishes} \Big).$$

Can certainly write down a PDE in (λ, κ) for this probability; the connection to Painlevé remains open.

What you can check is that you do indeed recover the spiked soft edge distributions by taking $a \uparrow \infty$:

$$\frac{a^2 - \Lambda(\beta, 2a, a + a^{1/3} w)}{a^{4/3}} \Rightarrow TW_{\beta, \omega}.$$

(The convergence takes place over the entire point processes.)

Spiking and growth models

As is well known, there is an equality in law between last passage with independent exponential weights and the largest eigenvalue of the $\beta = 2$ Laguerre ensemble:

$$\mathbb{P}\Big(\ell_{n,m} \leq \lambda\Big) = \mathbb{P}\Big(\lambda_{max}(n,m) \leq \lambda\Big).$$

Spiking (once) on the random matrix side corresponds to changing the mean of the exponentials on the x-axis in the last passage setup.

The hard edge appears in Hammersley's process: take the unit square with Poisson points of intensity t and make a path from origin to $(1,1)$ by "connecting the dots", keeping the slope positive. Then,

$$\mathbb{P}(\ell_t \leq a) = e^{-t}\langle e^{2\sqrt{t}\sum_{k=1}^{a}\cos(\theta_k)}\rangle_{U(a)} = \mathbb{P}\Big(\Lambda(2,a) \geq 4t\Big).$$

Hammersley's process with boundaries was in fact where the spiked soft edge laws first appeared (in the large t limit). But there are no Painlevé expressions pre-limit, and no apparent connection to any "perturbed" hard edge.