Occupation probabilities for coalescing (annihilating) Brownian motions on $\mathbb{R}$

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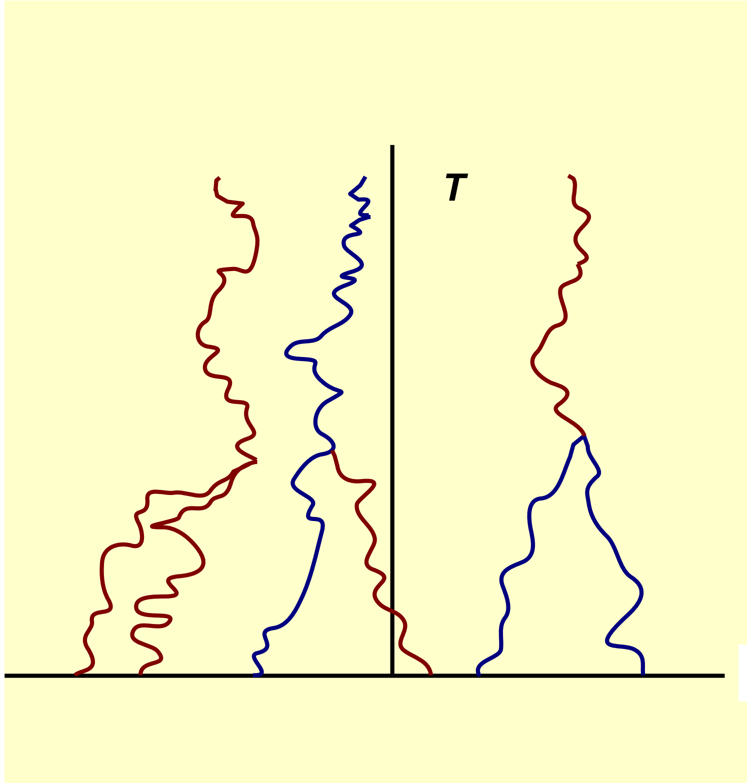
Plan

- **Quantities**
  - Coalescing (annihilating) Brownian motions (CABM)
  - Review of known results
  - The main result for occupation probabilities
- **Structures**
  - Pfaffian representation for occupation probabilities
  - Duality and the maximal entrance law
  - Pfaffian point processes and CABM
- **Conclusions**

CABM's



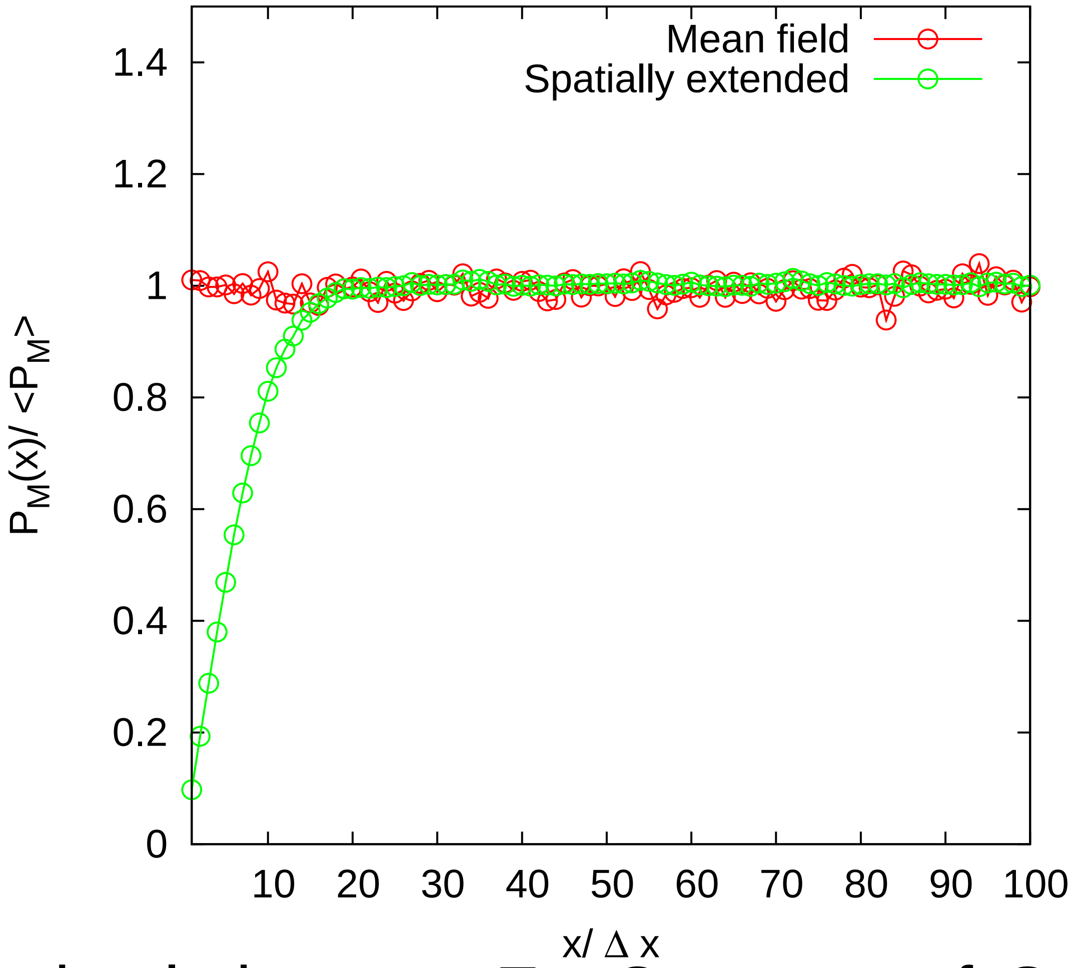
- Thinning relation:
  (X_t^i) under $P^A_{\Theta(\Omega)} \stackrel{D}{=}$
  $\Theta(X_t^i)$ underP^C_Ω
- **Proof** :Colouring

- Particles perform independent Brownian motions on $\mathbb{R}$ until they meet
- At the moment of collision particles instantly coagulate. The aggregate follows a Brownian path with the same diffusion rate
- The main object of interest: $\rho_n(t, x_1, \ldots, x_n)dx_1 \ldots dx_n$ - the probability of finding n particles in $dx_1, \ldots, dx_n$ at time t

Mean field analysis

- 'Chemist': Aggregation rate is proportional to the number of pairs of particles within the interaction range
- Hence the rate equation: $\partial_t \rho_1(t) = -\lambda(r, D)\rho_1(t)^2$
- Solution: $\rho_1(t) \sim \frac{1}{t}$
- Implicit mean field assumption: $\rho_n(t) \sim \rho_1(t)^n$
- Consequently, $\rho_n \sim \frac{1}{t^n}$

The breakdown of mean field theory: anti-correlations



(Monte-Carlo simulations on $\mathbb{Z}$. Courtesy of Colm Connaughton)

Known rigorous results for $t \to \infty$

$d = 1$

- $\rho_1(t) \sim \frac{1}{\sqrt{t}}$ (Bramson-Griffeath, **Z**, 1980; Bramson-Lebowitz, 1988)
- Exact expression for $\rho_n(t)$ for all t's using empty interval method in the form of the sum of products of duffusive Greens' functions with alternating signs. (ben-Avraham, 1998)

$d \geq 2$

- For $d = 2$, $\rho_1(t) \sim \frac{ln(t)}{t}$ (Bramson-Lebowitz, 1988)
- For $d > 2$, $\rho_1(t) \sim \frac{1}{t}$ (Bramson-Lebowitz, 1988; van den Berg and Kesten, finite reaction rates, 2002)

Predictions from (non-rigorous) renormalization group analysis

$$\rho_n(t) \sim \begin{cases} t^{-\frac{n}{2}-\frac{n(n-1)}{4}} & d=1 \\ \left(\frac{\ln t}{t}\right)^n (\ln t)^{-\frac{n(n-1)}{2}} & d=2 \\ t^{-n} & d>2 \end{cases}$$

- Note the non-linear dependence of the scaling exponent on n in $d=1$

Multi-scaling of occupation probabilities

Theorem 1 *Under the maximal entrance law for C(A)BM's,*

$$\sup_{|x_i|<<t^{1/2}} \left| \rho_t^{(2n)}(x_1, x_2, \ldots, x_{2n}) - c_n t^{-n} \left| \Delta_{2n}\left(\frac{x}{\sqrt{t}}\right) \right| \right| \to 0 \textit{ as } t \to \infty,$$

$\Delta_{2n}(x) = \prod_{1 \leq i < j \leq 2n} (x_i - x_j)$, $c_n^{ABM} = \frac{1}{4^n} c_n^{CBM}$

- IC's: Maximal entrance law initial conditions (Arratia, 1981): 'one particle per site' at $t = 0$ - as in Brownian web
- Construction: Poisson(λ) initial distribution with $\lambda \to \infty$

Pfaffians and interacting particle systems

Theorem 2 *Consider a system of ABM's with* $2n$ *particles at* $t = 0$, $x_1 < x_2 < \dots x_{2n}$. *Then the product moment* $m_t^{(2n)}(x_1, \dots, x_{2n}) = \mathbb{E}^A_{(x_1,\dots,x_n)}\left(\prod_{i \in I_t} g(X_t^i)\right)$ *is given by the Pfaffian of an* $2n \times 2n$ *antisymmetric matrix:*

$$m_t^{(2n)}(x_1, \dots, x_{2n}) = Pf\left((-1)^{\chi(j>i)} m_t^{(2)}(x_i, x_j)\right)$$

Proof. $m_t^{(2n)}(x_1, \dots, x_{2n})$ *solves heat equation on the cell* $x_1 < x_2 < \dots x_{2n} \subset \mathbf{R}^{2n}$. *BC's:* $m_t^{(2n)} \mid_{x_i = x_{i+1}} = m_t^{(2n-2)}$. *IC's:* $m_0^{(2n)} = \prod_{k=1}^{2n} g(x_k)$. *Pfaffian solves the equation and satisfies IC's, BC's. The theorem follows by uniqueness*

Coalescing Brownian motions and Pfaffians

- CBM-ABM duality (Arratia) for the maximal entrance law:

$$P^C_\infty[N_t[a_1, a_2] = 0 \dots N_t[a_{2n-1}, a_{2n}] = 0] = P^A_{(a_i)}(\tau < t)$$

- The right hand side is the Pfaffian of $P^{(A)}_{a_i,a_j}(\tau < t)$ (Brownian hitting prob)
  - Proof: set $g \equiv 0$ in Thm 2
- **Conclusion**: empty interval probabilities are Pfaffians

CBM's and Pfaffian point processes.

Theorem 3 *Under the maximal entrance law for coalescing Brownian motions, the particle positions at time* t *form a Pfaffian point process with kernel* $t^{-1/2}K(xt^{-1/2}, yt^{-1/2})$*, where*

$$K(x,y) = \begin{pmatrix} -F''(y-x) & -F'(y-x) \\ F'(y-x) & sgn(y-x)F(|y-x|) \end{pmatrix}$$

and $F(x) = \pi^{-1/2}\int_x^\infty e^{-z^2/4}dz$*. (Here* $sgn(z) = 1$ *for* $z > 0$*,* $sgn(z) = -1$ *for* $z < 0$ *and* $sgn(0) = 0$*.)*

Proof: Differentiate the pfaffian expression for empty interval probabilities with respect to right end points.

Closing the loop: Theorem 1 follows from the large-t expansion of the pfaffian formulae for ρ_n's

CABM's and random matrices

Corollary 4

$$K_t^{ABM}(x,y) = \frac{1}{\sqrt{2t}} K_{rr}^{Ginibre}\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right),$$

where $K_{rr}^{Ginibre}$ *is the* $N \to \infty$ *limit of the Kernel of the Pfaffian point process characterising the law of real eigenvalues in the real Ginibre*(N) *ensemble,*

$$\mu_N(d\mathbf{M}) = \frac{1}{(2\pi)^{N^2/2}} e^{-\frac{1}{2}Tr(\mathbf{M}^T\mathbf{M})} \lambda_{N\times N}(d\mathbf{M})$$

(Ginibre, Edelman, Sommers, Akemann, Forrester, Sinclair, Borodin, ...)

Conclusions

- Mean field approximation in $d = 1$ is invalidated by strong negative correlations between the particles
- Multi-point probability densities exhibit quadratic multi-scaling
- One-dimensional occupation densities in CBM's are a Pfaffian point process
- The same process describes occupation densities of real eigenvalues in $N \to \infty$ limit of real Ginibre matrix ensemble

Open questions, references

- Is there a relation between CABM's and the $GL(N)$-valued Brownian Motions? ("Ginibre process"). Conjecture presented in $[2]$ incorrect

- Rigorous derivation of logarithmic corrections in $d = 2$?

- **References**:

1. *Multi-Scaling of the n-Point Density Function for Coalescing Brownian Motions* , CMP Vol. **268**, No. **3**, December 2006;
2. *Pfaffian formulae for one dimensional coalescing and annihilating systems*, arXiv Math.PR: 1009.4565; EJP, vol. **16**, Article 76 (2011)