Stochastic Analysis and Stochastic PDEs

(Abstracts of talks received up to April 16, 2012)

16-20 April 2012

Mathematics Institute, The University of Warwick

- S. Aida. Tohuku University

  Title. Tunneling for spatially cut-off $P(\phi)_2$-Hamiltonians

  We study the asymptotic behavior of low-lying spectrum of $P(\phi)_2$-Hamiltonian $-L + V_\lambda$ under semi-classical limit $\lambda \to \infty$. This Hamiltonian is related with the quantized scalar field with space-time dimension 2 and the potential function V_λ is defined by a polynomial function and the Wick product. We consider two cases:

  (i) the space is $\mathbb{R}$ and the one particle Hamiltonian is $\sqrt{m^2 - \Delta}$

  (ii) the space is a finite interval $[-l/2, l/2]$ and the one particle Hamiltonian is $\sqrt{m^2 - \Delta}$, where Δ is the Laplacian with periodic boundary condition.

  In the case of (i), we consider spatially cut-off $P(\phi)_2$-Hamiltonian. In both cases, we show that the semi-classical limit of the lowest eigenvalue (first eigenvalue) of the Hamiltonian is determined under suitable assumptions on the potential functions. Next, we consider the case where the potential function is double-well type. In this case, we prove that the gap of the first and the second eigenvalues goes to 0 exponentially fast under semi-classical limit. We give an upper bound on the convergence rate by an approximate Agmon distance between zero points of the potential function in the case of (i) and (ii). Moreover, in the case of (ii), we show that the approximate Agmon distance is equal to the infinite dimensional analogue of the Agmon distance in the case of Schrödinger operators in finite dimensional spaces. The infinite dimensional analogue of the Agmon distance is a Riemannian distance on the Sobolev space $H^{1/2}$ and the distance is defined by a Riemannian metric which is conformal to L^2-metric.

- Adam Andersson (Chalmers, Sweden)

  Title: Weak error of finite element approximations of a nonlinear stochastic heat equation

Abstract: We consider spatial finite element approximations of a nonlinear stochastic heat equation. The noise is of colored multiplicative type for which white noise is a special case. The rate of weak convergence is proved to be essentially twice that of strong convergence. The proof relies on the Malliavin calculus, following a method introduced by Debussche for proving a weak error estimate for time discretization of the heat equation with white noise in one space dimension. The possibility of having colored noise enables us to work in higher dimensions, in contrast to the white noise case, for which a sufficiently regular solution exists only in one space dimension. This is joint work with Stig Larsson.

- D. Applebaum. Sheffield U.

  Title: Second quantised representation of Mehler semigroups associated with Banach space valued Levy processes.

  Abstract: The solutions of linear SPDEs driven by Banach space valued additive Levy noise are generalised Ornstein-Uhlenbeck processes. These are Markov processes and their transition semigroups are sometimes called Mehler semigroups. If the driving noise is a Brownian motion then Anna Chojnowska-Michalik and Ben Goldys have shown that these semigroups can be represented by means of second quantisation within a suitable chaotic decomposition. The result has recently been extended to the Levy case (for Hilbert space valued noise) by Szymon Peszat using a point process construction. In this talk I will present an alternate approach to this construction based on the use of exponential martingales.

  This talk is based on joint work with Jan van Neerven (Delft)

- M. Arnaudon (Poitiers)

  Title: Generalized Navier-Stokes flows.

  Abstract: "We introduce a notion of generalized Navier-Stokes flows on manifolds, that extends to the viscous case the one defined by Brenier. Their kinetic energy extends the kinetic energy for classical Brownian flows, defined as the $L2$ norm of their drift. We prove that there exists a generalized flow which realizes the infimum of kinetic energies among all generalized flows with prescribed initial and final configuration. We construct generalized flows with prescribed drift and kinetic energy smaller than the $L2$ norm of the drift."

- F. Baudoin, Purdue University.

  Title: Functional Inequalities for subelliptic diffusion operators via curvature bounds

  Abstract: In this talk I shall review some recent results that were obtained by the authors in joint works with M. Bonnefont, N. Garofalo and B. Kim. Let L be a symmetric and subelliptic diffusion operator defined on a manifold M. By using the curvature dimension inequality proposed by Baudoin-Garofalo we will discuss the following properties of L that are usually addressed in a Riemannian framework by using Ricci lower bounds: -Boundedness of the Riesz transform -Existence of log-Sobolev inequalities -Existence of isoperimetric and Gaussian isoperimetric inequalities

- Dominique Bakry (Toulouse)

  Title: Diffusions and orthogonal polynomials.

  Diffusion semigroups are described through their generators, which are in general in Rn or an open set in it second order differential operators of the form
  $$L(f)(x) = \sum_{i,j} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_i b_i(x)\frac{\partial f}{\partial x_i}$$
  The easiest cases are when one is able to diagonalize this operator in an basis of orthogonal polynomials, since then one is able to have a quite explicit description of the associated law of the underlying process. In dimension 1, there are not many examples of such a situation. It reduces to the family of Jacobi, Laguerre and Hermite polynomials. In higher dimension, many examples come from Lie group actions of homogeneous spaces, or generalizations of them, through root systems or other algebraic constructions. We shall give a complete characterization of the problem : on which open sets in Rn one may expect to find a probability measure for which the associated orthogonal polynomials are eigenvectors of diffusion operators. We shall give a complete description of all the models in dimension 2, where we are able to completely solve this problem. There are exactly 11 compact sets (up to affine transformations), and 7 non compact ones, on which there exist such a measure. We shall also describe all the associated measures and operators.

- Benjamin Gess (Bielefeld)

  Title: Stochastic dynamics induced by porous media equations with space-time linear multiplicative noise.

  Abstract: We prove the generation of a continuous random dynamical system on L^1 for porous media equations on a bounded domain perturbed by space-time linear multiplicative noise. By proving regularizing properties of the flow and uniform upper bounds we obtain the existence of a random attractor, which is compact and attracting with respect to the L^∞ norm. Our results apply to general noise including fractional Brownian motion for all Hurst parameters and we obtain a pathwise Wong-Zakai result for driving noise given by a continuous semimartingale.

  Arxiv-preprint: http://arxiv.org/abs/1108.2413

- A.B. Cruzeiro (Lisbon) **Title: Stochastic calculus of variations on the diffeomorphisms group**

  On the diffeomorphisms group endowed with a L^2 metric the corresponding Brownian motion is not well defined. We propose a stochastic calculus of variations for some truncated diffusions on this space.

- Z. Brzezniak (York)

  Title: Strong and weak solutions to the stochastic Landau-Lifshitz equation and their asymptotic behaviour

  Abstract: I will speak about the existence of weak solutions (and uniqueness in the case of strong solutions) to the stochastic Landau-Lifshitz equations on multi (one)-dimensional spatial domains. I will also describe the corresponding Large Deviations Principle and its applications to the theory of ferromagnetic wires.

- D. Crisan (Imperial College)

  Title: Robust Filtering: Correlated Noise and Multidimensional Observation

  Abstract: In the late seventies, Clark pointed out that it would be natural for π_t, the probability measure that is the solution of the stochastic filtering problem, to depend continuously on the observed data $Y = \{Y_s, s \in [0,t]\}$. Indeed, if the signal and the observation noise are independent one can show that, for any suitably chosen test function f, there exists a continuous map θ_t^f, defined on the space of continuous paths endowed with the uniform convergence topology such that $\pi_t(f) = \theta_t^f(Y)$, almost surely. This type of *robust* representation is also valid when the signal and the observation noise, provided the observation process is scalar. For a general correlated noise and multidimensional observations such a representation does not exist. By using the theory of rough paths we provide a solution to this deficiency: The observation process Y is "lifted" to the process $\mathbf{Y}$ that consists of Y and its corresponding Lévy area process and we show that there exists a continuous map θ_t^f, defined on a suitably chosen space of Hölder continuous paths such that $\pi_t(f) = \theta_t^f(\mathbf{Y})$, almost surely.

  This is joint work with J. Diehl, P. K. Friz and H. Oberhauser.

- A. debusschee (Cachan, Bretange)

  Title. Diffusion limit for stochastic kinetic equations

  Abstract: In this talk, we consider a kinetic equation containing random terms. The kinetic equation contains a small parameter and it is well known that, after scaling, when this parameter goes to zero the limit problem is a diffusion equation in the PDE sense, ie a parabolic equation of second order. A smooth noise is added, accounting for external perturbation. It scales also with the small parameter. We show that the limit equation is now a stochastic parabolic equation where the noise is multiplicative and in the Stratonovitch form. The method is to generalize the "perturbed test function" method. This is a joint work with J. Vovelle.

- S. Fang (U. Bourgogne)

  Title: Diffusion processes with the singular drift on complete Riemannian manifolds

Abstract: On a complete Riemannian manifold, we will construct the diffusion process associated to the infinitesimal generator $(1/2)\Delta_M + Z$ with Z in a Sobolev space. A sufficient condition is given to insure the strong completeness of the Brownian flow.

- I. Gyongy (Edinburgh)

  Finite difference schemes in degenerate filtering

  Abstract: We present sharp estimates for the accuracy of finite difference schemes in space variables for the Zakai equation of nonlinear filtering, when the equation can degenerate. We show that the accuracy of the schemes can be made as high as one wishes by Richardson extrapolation if the coefficients of the diffusion process, describing the signal and observation model, are sufficiently smooth. The talk is based on joint results with N.V. Krylov.

- M. Hairer (Warwick)

  Title: Diffusion Monte-Carlo with a twist

  We present a modification of the standard diffusion Monte-Carlo method. The modified method has two main advantages over the "standard" DMC method:

  1. Its variance is always smaller.
  2. In the simple case of a reweighted random walk, it has a continuous limit (in an appropriate state space).

  We also give a description of the continuous limit in question, which is a kind of branching process which produces offspring at infinite rate. This is joint work with Jonathan Weare.

- Eric Hall (Edinburgh)

  Title: Accelerated spatial approximations for time discretized stochastic partial differential equations

  Abstract: We investigate numerical solutions of the Cauchy problem for linear second order parabolic stochastic partial differential equations (SPDE) defined on the whole space. Such SPDE arise in applications of the nonlinear filtering theory of partially observable diffusion processes where solutions are desired in real-time. Thus there is a keen interest in developing accurate numerical schemes for their solutions. We consider finite difference approximations in uniform grids in time and space and give sufficient conditions for accelerating the strong convergence with respect to the spatial approximation to higher order accuracy by Richardson's method, an extrapolation technique. This is done by first proving the existence of an expansion in powers of the spatial discretization parameter for the solution to our space-time scheme, provided appropriate regularity conditions are satisfied. Hence we apply Richardson's method and show that suitable mixtures of the spatial difference approximations at different mesh sizes converge to the time discretized solution with arbitrarily high accuracy. This work extends the results of Gyngy and Krylov [SIAM J. Math. Anal., 42 (2010), pp. 2275–2296] to schemes that discretize in time as well as space.

- Erika Hausenblas (Leoben)

  Title: Numerical Approximation of Stochastic Evolution equations in UMD Banach spaces

  In a joint work with Cox we have shown a perturbation result with respect to a stochastic evolution equation. To be more precise, we consider the effect of perturbations of A on the solution to the following quasi-linear parabolic stochastic partial differential equation:

  $$\begin{cases} dU(t) = AU(t)\,dt + F(t,U(t))\,dt + G(t,U(t))\,dW_H(t), \quad t > 0; \\ U(0) = x_0. \end{cases} \tag{SDE}$$

  Here A is the generator of an analytic C_0-semigroup on a UMD Banach space X with type τ, $G : [0,T]\times X \to \mathcal{L}(H, X^A_{\theta_G})$ and $F : [0,T]\times X \to X^A_{\theta_F}$ for some $\theta_G > -\frac{1}{2}$, $\theta_F > -\frac{3}{2} + \frac{1}{\tau}$. We assume F and G to satisfy certain global Lipschitz and linear growth conditions. The spaces $X^A_{\theta_F}$ and $X^A_{\theta_G}$ are certain interpolation, resp. extrapolation spaces.

  Let A_0 denote the perturbed operator and U_0 the solution to (SDE) with A substituted by A_0. We provide estimates for $\|U - U_0\|_{L^p(\Omega;C([0,T];X))}$ in terms of $D_\delta(A, A_0) := \|R(\lambda : A) - R(\lambda : A_0)\|_{\mathcal{L}(X^A_{\delta-1},X)}$. Here $\delta \in [0,1]$ is assumed to satisfy $0 \leq \delta < \min\{\frac{3}{2} - \frac{1}{\tau} + \theta_F,\, \frac{1}{2} - \frac{1}{p} + \theta_G\}$.

  With the help of this result, we prove almost sure uniform convergence rates for space approximations of semi-linear stochastic evolution equations with multiplicative noise in Banach spaces. The space approximations we consider are spectral Galerkin and finite elements, but can also be applied to wavelets in Besov spaces. A more theoretical application is the Yosida approximation, to which the result can be applied as well.

- H. Kunita.

  Title: Non-degenerate SDE with jumps and its hypoelliptic property

  We discuss SDE with jumps on $\mathbf{R}^d$:

  $$\xi_t = X(\xi_{t-}, dt),$$

  where $X(x,t)$ is a Lévy process with spatial parameter x, written by

  $$X(x,t) = \int_0^t b(x,s)ds + \int_0^t \sigma(x,s)dW(s) + \int_0^t \int_{\mathbf{R}^m} g(x,s,z)\tilde{N}(dsdz).$$

Here, $W(t)$ is a standard m-dimensional Brownian motion and $\tilde{N}(dsdz)$ is a compensated Poisson random measure. Coefficients b, σ, g of the equation are assumed to be smooth. Then the solution ξ_t is a jump diffusion. Its generator is given by

$$\begin{aligned} A(s)\varphi &= \frac{1}{2}\sum_{i,j} a^{ij}(x,s)\varphi_{x_i x_j} + \sum_i b^i(x,s)\varphi_{x_i} \\ &\quad + \int (\varphi(x+g(x,s,z)) - \varphi(x) - \sum_i g^i(x,s,z)\varphi_{x_i}(x))\nu(dz), \end{aligned}$$

where ν is the Lévy measue of $\tilde{N}(dsdz)$.

We will introduce *nondegenerate* SDE's in the sense of Malliavin and Picard. Though its definition is complicated, it includes the following SDE's. Let $V_j(x,t), j = 1, ..., m$ be time-dependent vector fields defined from diffusion coefficient matrix $(\sigma^{ij}(x,t))$ by $V_j(x,t) = \sigma^{\cdot j}(x,t)$. Next, let B be the infinitesimal covariance matrix of the Lévy measure ν and let (τ^{kj}) be a square root of B. Using jump coefficient $g(x,t,z)$, we define vector fields by

$$\tilde{V}_j(x,t) = \sum_k (\partial_{z_k} g(x,t,z)|_{z=0})\tau^{kj}, \quad j = 1, ..., m.$$

If $\{V_j(x,t), \tilde{V}_j(x,t), j = 1, ..., m\}$ spans $\mathbf{R}^d$ for any x, t, the SDE is nondegenerate. More generally, if the above vector fields and a drift vector field $V_0(x,t)$ satisfy Hörmander's condition, the SDE is nondegenerate.

We show that the solution of nondegenerate SDE has the *hypoelliptic property* described below. Let $\xi_{s,t}(x), t > s$ be the solution of the SDE starting from x at time s and let $c(x,s)$ be a bounded C^∞-function. The Feynman-Kac operator $P^c_{s,t}\varphi(x)$ given by

$$P^c_{s,t}\varphi(x) = E\Big[\exp\Big\{\int_s^t c(\xi_{s,r}(x), r)dr\Big\}\varphi(\xi_{s,t}(x))\Big]$$

can be extended from a smooth function φ to a tempered distribution Φ. The extended function $u(x,s) = P^c_{s,t}\Phi(x)$ is a smooth function of (x,s) and satisfies Kolmogorov's backward equation.

$$\Big(\frac{\partial}{\partial s} + A(s) + c(x,s)\Big)u(x,s) = 0.$$

Further, $p(s,x;t,y) := P^c_{s,t}\delta_y(x)$ is the fundamental solution for the Cauchy problem associated with the above equation.

- Terry Lyons. (Oxford)

  Title. The expected signature of Brownian motion on exit from a domain

Abstract: In joint work with Ni Hao, we explain how the expected signature of a measure on rough paths captures the law of that process, and show how to calculate this in the case of a markov measure on rough paths stopped on exit from a domain.

- Mario Maurelli (Scuola Normale Superiore, Pisa)

  Title: DEs and linear SPDEs with rough coefficients arising from fluid dynamics

  Abstract. In this talk I will consider the problem of existence and Wiener uniqueness for stochastic continuity equations (SCEs) and similar equations, in the case of rough coefficients. The SCE represents the evolution of a mass driven by the associated SDE and therefore is useful to capture regularization-by-noise phenomena and splitting/coalescence behaviour of the mass.

  I will focus on Le Jan's theory, which uses Wiener chaos decomposition and selects a unique Wiener generalized solution to a SCE ([LR]). This is suitable for models arising in fluid dynamics, in cases of non strong uniqueness; it can also be applied to restore uniqueness, starting from an ill-posed linear PDE ([M]). If time permits, examples and further research directions will be mentioned.

  Collaboration with F. Flandoli (Università di Pisa).

  [LR] Y. Le Jan, O. Raimond, Integration of Brownian vector fields, Ann. Probab. 30 (2002), 826-873.

  [M] M. Maurelli, Wiener chaos and uniqueness for stochastic transport equation, C. R. Acad. Sci. Paris 349 (2011), 669-672.

- Leonid Mytnik (Technion)

  Title: Multifractal analysis of superprocesses with stable branching in dimension one

  Abstracts: It has been well-known for a long time that the super-Brownian motion with $1+\beta$-stable branching mechanism has densities at any fixed time, provided that the spatial dimension d is small enough ($d < 2/\beta$). Then, in 2003, it was shown that in the case of $\beta < 1$ there is a dichotomy for the corresponding density functions: there are either continuous if $d = 1$, or locally unbounded in dimensions $d \in (1, 2/\beta)$. Recently we determined the spectrum of singularities of the continuous densities in dimension $d = 1$. This is a joint work with V. Wachtel.

- M. Neklyudov (Tübingen Universität)

  Title: The role of noise in finite ensembles of nanomagnetic particles

  Abstract. The dynamics of finitely many nanomagnetic particles is described by the stochastic Landau-Lifshitz-Gilbert equation. We show that the system relaxes exponentially fast to the unique invariant measure which is described by a Boltzmann distribution. Furthermore, we provide Arrhenius type law for the rate of the convergence to the distribution. Then, we discuss two implicit discretizations to approximate transition functions both, at finite and infinite times: the first scheme is shown to inherit the geometric 'unit-length' property of single spins, as well as the Lyapunov structure, and is shown to be geometrically ergodic; moreover, iterates converge strongly with rate for finite times. The second scheme is computationally more efficient since it is linear; it is shown to converge weakly at optimal rate for all finite times. We use a general result of Shardlow and Stuart to then conclude convergence to the invariant measure of the limiting problem for both discretizations. Computational examples will be reported to illustrate the theory. This is a joint work with A. Prohl.

- Etienne Pardoux (Marseille) **Title : Feller's branching diffusion with logistic growth : genealogies, and a path-valued** Markov process indexed by the initial population size.

  Abstract : We study various questions related to models of population with competition. One such model is Feller's branching diffusion with logistic growth. We describe the genealogy in such a population, and discuss some properties of the forest of genealogical trees for large initial population size, which are induced by the competition. We also discuss the path-valued Markov process which describes the time evolution of the population, as a function of its initial size. The corresponding process solves a Poissonian SDE driven by the Poisson process of excursions (in the sense of Pitma-Yor) of Feller's diffusion.

- Michael Rockner (Bielefeld)

  Title: Localization of Solutions to Stochastic Porous Media Equations: Finite Speed of Propagation

  Abstract: We present a localization result for stochastic porous media equations with linear multiplicative noise. More precisely, we prove that the solution process P-a.s. has the property of"finite speed propagation of disturbances" in the sense of [Antontsev/Shmarev, Nonlinear Analysis 2005]. Joint work with Viorel Barbu.

- M.Romito (Florence)

  Title: Densities for the 3D Navier-Stokes equations with Gaussian noise

  abstract: We present three different methods to prove existence of a density with respect to the Lebesgue measures for the law of any finite dimensional projection of solutions of 3D Navier-Stokes. While the first two methods provide only a qualitative result, the third method ensures a bit of regularity in Besov spaces, without restrictions on the decay of the coefficients of the (although non– degenerate) driving noise. If time permits it, the hypoelliptic case will be also discussed. This is a joint work with A. Debussche (ENS Cachan Bretagne).

- Marta Sanz-Solé, University of Barcelona

  Title: A support theorem for stochastic waves in dimension three

  Abstract: A characterization of the support in Hölder norm of the law of the solution for a stochastic wave equation with three-dimensional space variable is proved. The result is a consequence of an approximation theorem, in the convergence of probability, for a sequence of evolution equations driven by a family of regularizations of the driving noise.

- M. Scheutzow (TU Berlin)

  Title: Exponential growth rate for a singular linear stochastic delay differential equation

  Abstract: First we briefly review sufficient criteria for the uniqueness of an invariant measure of a stochastic delay differential equation obtained in joint work with Martin Hairer and Jonathan Mattingly. Then, we study the very simple one-dimensional equation $dX(t) = X(t-1)dW(t)$ in more detail and establish the existence of a deterministic exponential growth rate of a suitable norm of the solution via a Furstenberg-Hasminskii-type formula.

- J. Teichmann (ETH, Zurich)

  Title: Finite dimensional realizations for the CNKK-volatility surface model

  abstract: We show that parametrizations of volatility surfaces (and even more involved multivariate objects) by time-dependent Lvy processes (as proposed by Carmona-Nadtochiy-Kallsen-Krhner) lead to quite tractable term structure problems. An interesting SPDE of HJM-type arises and we discuss several solution concepts and solutions of it. In this context we can then ask whether the corresponding term structure equations allow for (regular) finite dimensional realization, which necessarily leads to models driven by an affine factor process. This is another confirmation that affine processes play a particular role in mathematical finance. The analysis is based on a careful geometric analysis of the term structure equations by methods from foliation theory.

- Anton Thalmaier (University of Luxembourg)

  Title: Brownian motion with respect to moving metrics and Perelman's entropy formula

  Abstract: We discuss notions of stochastic differential geometry in the case when the underlying manifold evolves along a geometric flow. Special interest lies in entropy formulas for positive solutions of the heat equation (or conjugate heat equation) under forward Ricci flow.

- A. Truman (Swansea)

  Title: A Stochastic BZ Model and the Satellites of Jupiter and Saturn

  Abstract: We give a stochastic Burgers-Zeldovich Model for a protoring nebula for gas giants such as Neptune and apply it to the satellites of Jupiter and Saturn.

- A. Veretennikov (Leeds)

  Title. On Poisson equations for "ergodic" generators of the second order

  Abstract: For a class of "ergodic" generators which are elliptic operators of the second order and for "small" potentials that may change sign, it is shown that Poisson equation in R^d has a Sobolev solution which allows a Feynman-Kac representation. The work is joint with S.V.Anulova.

- J. Zabczyk, Institute of Mathematics, PAS Warsaw, Poland

  Title: SPDEs with Lévy noise for the bond market

  The talk is devoted to stochastic partial differential equations with Lvy noise describing the evolution of the forward rates of the bond market. In the classical case of the market consisting of default-free bonds only, local and global existence of positive solutions are established. Some recent extensions to equations modeling forward rates in the market with defaultable bonds and with collateral debt obligations (CDO) are discussed as well. Presented results were obtained in collaboration with M. Barski.

- L. Zambotti (Paris VI)

  Title : CSBPs with immigration and Fleming-Viot Processes with Mutation

  Abstract : We study a class of self-similar jump type SDEs driven by Hölder continuous drift and noise coefficients. Using the Lamperti transformation for positive self-similar Markov processes we obtain a necessary and sufficient condition for almost sure extinction in finite time. We then show that in some cases pathwise uniqueness holds in a restricted sense, namely among solutions spending a Lebesgue-negligible amount of time at 0. We then study an associated Fleming-Viot process with mutation.

- Tusheng Zhang (Manchester)

  Title: Mixed Boundary Value Problems of Semilinear Elliptic PDEs and BSDEs with Singular Coefficients

  Abstract: In this work, we prove that there exists a unique weak solution to the mixed boundary value problem for a general class of semilinear second order elliptic partial differential equations with singular coefficients. Our approach is probabilistic. The theory of Dirichlet forms and backward stochastic differential equations with singular coefficients and infinite horizon plays a crucial role.

- H. Zhao

  Title: Random Periodic solution of SDEs and SPDEs.

  Abstract: I will talk about the random periodic solution of sdes and spdes and mathematical tools of infinite horizon integral equations and Wiener-Sobolev compact embedding that we have developed recently. This is based on recent work with C. Feng.