Tunneling for spatially cut-off $P(\phi)_2$- Hamiltonian

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Introduction

(Spatially cut-off) $P(\phi)_2$-Hamiltonian $-L + V_\lambda$ is an ∞-dimensional Schrödinger operator defined on $L^2(\mathcal{S}'(I), \mu)$, where $I = [-l/2, l/2]$ or $I = \mathbb{R}$ and $\lambda = \frac{1}{\hbar}$.

I explain my recent results:

- Determination of the semi-classical limit of $E_1(\lambda)$ as $\lambda \to \infty$
- An estimate on the asymptotic behavior of the gap of spectrum $E_2(\lambda) - E_1(\lambda)$ as $\lambda \to \infty$.

Plan of Talk

1. $P(\phi)_2$-Hamiltonian
2. Results for Schrödinger operator $-\Delta + \lambda U(\cdot/\lambda)$
3. Main Result 1 : $\lim_{\lambda\to\infty} E_1(\lambda)$
4. Main Result 2 :

$$\limsup_{\lambda\to\infty} \frac{\log\left(E_2(\lambda) - E_1(\lambda)\right)}{\lambda} \le -d_U^W(-h_0, h_0)$$

$$\left(= -d_U^{Ag}(-h_0, h_0) \text{ if } I = [-l/2, l/2]\right)$$

5. Properties of Agmon distance d_U^{Ag}.

$P(\phi)_2$-Hamiltonian

Let $I = [-l/2, l/2]$ or $I = \mathbb{R}$ and $m > 0$. Let $H^s(I, dx)$ be the Sobolev space with the norm:

$$\|\varphi\|_{H^s(I,dx)} = \|(m^2 - \Delta)^{s/2}\varphi\|_{L^2(I,dx)}.$$

Let $H = H^{1/2}(I, dx)$. Let μ be the Gaussian measure whose covariance operator is $(m^2 - \Delta)^{-1/2}$ on $L^2(I, dx)$. Let us consider a Hilbert space W:

(1) When $I = [-l/2, l/2]$, $W = H^{-\varepsilon}(I, dx)$, where ε is any positive number.

(2) When $\boldsymbol{I} = \mathbb{R}$,

$$\boldsymbol{W} = \Big\{ \boldsymbol{w} \in \boldsymbol{\mathcal{S}}'(\mathbb{R}) \mid \|\boldsymbol{w}\|_{\boldsymbol{W}}^{2} = \int_{\mathbb{R}} |(1 + |\boldsymbol{x}|^{2} - \boldsymbol{\Delta})^{-1} \boldsymbol{w}(\boldsymbol{x})|^{2} d\boldsymbol{x} < \infty \Big\}.$$

Then $(\boldsymbol{W}, \boldsymbol{H}, \boldsymbol{\mu})$ is an abstract Wiener space in the sense of Gross. Define a self-adjoint operator $\boldsymbol{A}$ on $\boldsymbol{H}$ by

$$\begin{aligned} \boldsymbol{A}\boldsymbol{h} &= (\boldsymbol{m}^{2} - \boldsymbol{\Delta})^{1/4} \boldsymbol{h}, \\ \mathrm{D}(\boldsymbol{A}) &= \boldsymbol{H}^{1} \subset \boldsymbol{H}. \end{aligned}$$

Definition 1 (Free Hamiltonian)

Let $\mathcal{E}_A$ be the Dirichlet form defined by

$$\mathcal{E}_A(f,f) = \int_W \|ADf(w)\|_H^2 d\mu(w) \quad f \in \mathrm{D}(\mathcal{E}_A),$$

where

$$\mathrm{D}(\mathcal{E}_A) = \Big\{ f \mid Df(w) \in \mathrm{D}(A) \text{ and } \int_W \|ADf(w)\|_H^2 d\mu(w) < \infty \Big\},$$

D : H*-derivative,*

$-L$: *the non-negative generator of* $\mathcal{E}_A$.

Definition 2 *Let* $P(x) = \sum_{k=0}^{2M} a_k x^k$ *with* $a_{2M} > 0$*.*

Let $g \in C_0^\infty(I)$ *with* $g(x) \geq 0$ *for all* x *and define*

$$
\begin{aligned}
V(h) &= \int_I P(h(x))g(x)dx \quad h \in H \\
U(h) &= \frac{1}{4}\|Ah\|_H^2 + V(h) \quad \text{for} \quad h \in \mathrm{D}(A).
\end{aligned}
$$

Remark 3 V *is well-defined on* H *and we can rewrite*

$$
\begin{aligned}
U(h) &= \frac{1}{4}\int_I \left(h'(x)^2 + m^2 h(x)^2\right) dx \\
&\quad + \int_I P(h(x))g(x)dx \qquad h \in H^1.
\end{aligned}
$$

Definition 4 (1) *Let $\lambda > 0$. For the polynomial $P = P(x) = \sum_{k=0}^{2M} a_k x^k$ with $a_{2M} > 0$, define*

$$\int_I :P\left(\frac{w(x)}{\sqrt{\lambda}}\right): g(x)dx$$
$$= \sum_{k=0}^{2M} a_k \int_I :\left(\frac{w(x)}{\sqrt{\lambda}}\right)^k: g(x)dx.$$

We write

$$:V\left(\frac{w}{\sqrt{\lambda}}\right): \ = \ \int_I :P\left(\frac{w(x)}{\sqrt{\lambda}}\right): g(x)dx$$

and

$$V_\lambda(w) = \lambda : V\left(\frac{w}{\sqrt{\lambda}}\right) : .$$

(2) *It is known that* $(-L + V_\lambda, \mathfrak{F}C_A^\infty(W))$ *is essentially self-adjoint, where* $\mathfrak{F}C_A^\infty(W)$ *denotes the set of smooth cylindrical functions.*

We use the same notaion $-L + V_\lambda$ *for the self-adjoint extension.*

It is known that $-L + V_\lambda$ *is bounded from below and the lowest eigenvalue* $E_1(\lambda)$ *is simple.*

Some known results

- (Hoegh-Krohn and Simon 1972)
  $\boldsymbol{\sigma_{ess}(-L + V_\lambda) \cap [E_1(\lambda), E_1(\lambda) + m) = \emptyset}$.

- (Simon 1972) Example of spatially cut-off $\boldsymbol{P(\phi)_2}$-Hamiltonian for which there exist an eigenvalue which is in a continuous spectrum.

- (Dereziński and Gérard, 2000) $\boldsymbol{-L + V_\lambda}$ does not have singular continuous spectrum.

- (A.Arai, 1996) Calculation of $\boldsymbol{\lim_{\lambda \to \infty} \mathrm{tr}\, e^{t(L - V_\lambda)/\lambda}}$ for certain $\boldsymbol{P(\phi)}$-type models (not including $\boldsymbol{P(\phi)_2}$-model)

Schrödinger operators on $\mathbb{R}^N$

Assume

(i) $U \in C^\infty(\mathbb{R}^N)$, $U(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $\liminf_{|x|\to\infty} U(x) > 0$.

(ii) $\{x \mid U(x) = 0\} = \{x_1, \ldots, x_n\}$.

(iii) $Q_i = \frac{1}{2}D^2U(x_i) > 0$ for all i.

Then the lowest eigenvalue $E_1(\lambda)$ of $-\Delta + \lambda U(\cdot/\sqrt{\lambda})$ is simple and

$$\lim_{\lambda\to\infty} E_1(\lambda) = \min_{1\leq i\leq n} \operatorname{tr}\sqrt{Q_i}.$$

In addition to (i), (ii), (iii), we assume the symmetry of $\boldsymbol{U}$:

(iv) $\boldsymbol{U(x) = U(-x)}$,

(v) $\boldsymbol{\{x \mid U(x) = 0\} = \{-x_0, x_0\}} \quad \boldsymbol{(x_0 \neq 0)}$.

Then we have (due to Harrell, Jona-Lasinio, Martinelli and Scoppola, Simon, Helffer and Sjöstrand,...)

$$\boldsymbol{\lim_{\lambda \to \infty} \frac{\log(E_2(\lambda) - E_1(\lambda))}{\lambda} = -d_U^{Ag}(-x_0, x_0)},$$

where $\boldsymbol{E_2(\lambda)}$ is the second eigenvalue and $\boldsymbol{d_U^{Ag}(-x_0, x_0)}$ is the Agmon distance between $\boldsymbol{-x_0}$ and $\boldsymbol{x_0}$ such that

$$
\begin{aligned}
d_U^{Ag}(-x_0, x_0) \;=\; \inf\Big\{ \int_{-T}^{T} \sqrt{U(x(t))}\,|\dot{x}(t)|\,dt \\
\Big|\; x \text{ is a smooth curve on } \mathbb{R}^N \\
\text{with } x(-T) = -x_0,\; x(T) = x_0 \Big\}.
\end{aligned}
$$

The definition is independent of $T > 0$.

The Agmon distance $d_U^{Ag}(-x_0, x_0)$ is equal to the following action integral which is introduced by Carmona and Simon (1981). The minimizing path of the following variational problem is called an instanton.

$$d_U^{CS}(-x_0, x_0) = \inf\Big\{ \int_{-T}^{T} \left(\frac{1}{4}|x'(t)|^2 + U(x(t)) \right) dt$$

$$\Big|\ x \text{ is a smooth curve on } \mathbb{R}^N \text{ with}$$

$$x(-T) = -x_0,\ x(T) = x_0,\ T > 0 \Big\}.$$

The elementary inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ implies

$$\int_{-T}^{T} \sqrt{U(x(t))}|x'(t)|dt \leq \int_{-T}^{T} \left(\frac{1}{4}|x'(t)|^2 + U(x(t)) \right) dt$$

and $d_U^{Ag}(-x_0, x_0) \leq d_U^{CS}(-x_0, x_0)$.

$-L + V_\lambda$ as an ∞-dimensional Schrödinger operator

$-L + V_\lambda$ is informally unitarily equivalent to the ∞-dimensional Schrödinger operator on $L^2(L^2(I, dx), dw)$:

$$-\Delta_{L^2(I)} + \lambda : U(w/\sqrt{\lambda}) : -\frac{1}{2}\mathrm{tr}(m^2 - \Delta)^{1/2},$$

where

$$\begin{aligned} : U(w) : \; = \; & \frac{1}{4}\int_I w'(x)^2 dx \\ & + \int_I \left(\frac{m^2}{4} w(x)^2 + : P(w(x)) : g(x) \right) dx. \end{aligned}$$

In fact, $P(\phi)_2$-Hamiltonian is related with the quantization of the classical field (nonlinear Klein-Gordon equation):

$$\begin{aligned}
\frac{\partial^2 w}{\partial t^2}(t,x) &= -2(\nabla U)(w(t,x)), (t,x) \in \mathbb{R} \times I \\
U(w) &= \frac{1}{4}\int_I \left(w'(x)^2 + m^2 w(x)^2\right) dx \\
&\quad + \int_I P(w(x))g(x)dx \\
2(\nabla U)(w(t,x)) &= -\frac{\partial^2 w}{\partial x^2}(t,x) + m^2 w(t,x) \\
&\quad +2P'(w(t,x))g(x).
\end{aligned}$$

Main Result 1

Assumption 5

(A1) $U(h) \geq 0$ *for all* $h \in H^1$ *and*

$$\mathcal{Z} = \{h \in H^1 \mid U(h) = 0\} = \{h_1, \ldots, h_n\}$$

is a finite set.

(A2) *The Hessian* $D^2U(h_i)$ $(1 \leq i \leq n)$ *is strictly positive. The derivative* D *stands for the* H*-derivative.*

Remark 6

$$D^2U(h_i) = \frac{1}{2}A^2 + D^2V(h_i)$$

is an unbounded operator on H.

$$\inf \sigma(D^2U(h_i)) > 0 \iff \inf \sigma(m^2 - \Delta + 4v_i) > 0$$

where

$$v_i(x) = \frac{1}{2}P''(h_i(x))g(x).$$

Theorem 7 *Assume* **(A1)** *and* **(A2)** *hold.*

Let $\boldsymbol{E_1(\lambda) = \inf \sigma(-L + V_\lambda)}$. *Then*

$$\lim_{\lambda\to\infty} E_1(\lambda) = \min_{1\leq i\leq n} E_i,$$

where

$$E_i = \inf \sigma(-L + Q_{v_i}),$$

$$Q_{v_i} = \int_I : w(x)^2 : v_i(x)dx,$$

$$v_i(x) = \frac{1}{2}P''(h_i(x))g(x).$$

Main Result 2 (Tunneling estimate)

Let

$$E_2(\lambda) = \inf \{\sigma(-L + V_\lambda) \setminus \{E_1(\lambda)\}\}.$$

We prove that $E_2(\lambda) - E_1(\lambda)$ is exponentially small when $\lambda \to \infty$ under a certain assumption on P.

To state our estimate, we introduce infinite dimensional analogue of Agmon distance in quantum mechanics.

Let us fix $T > 0$ and take $h, k \in H (= H^{1/2}(I))$.

Let

$$
\begin{aligned}
& H^1_{T,h,k}(I) \\
& \quad = \Big\{ u = u(t,x) \ ((t,x) \in (-T,T) \times I) \ \Big| \\
& \quad u \in H^1((-T,T) \times I), \\
& \quad\ u(-T,\cdot) = h,\ u(T,\cdot) = k \text{ in the sense of trace} \Big\}
\end{aligned}
$$

Note $H^1_{T,h,k}(I) \neq \emptyset$.

Let U be a non-negative potential function which we introduced. Here we do not assume **(A1), (A2)**.

For any $u \in H^1_{T,h,k}(I)$, the following properties hold:

(i) $u(t,\cdot) \in H^1(I)$ for almost every t and $t(\in [-T,T]) \mapsto U(u(t,\cdot))$ is a Lebesgue measurable function,

(ii) $t(\in (-T,T)) \to u(t,\cdot) \in L^2(I)$ is an absolutely continuous function and its derivative is in $L^2((-T,T) \to L^2(I))$,

(iii) $\int_{-T}^{T} \sqrt{U(u(t,\cdot))} \|\partial_t u(t,\cdot)\|_{L^2(I)} dt < \infty$.

The fact (iii) follows from the following argument. Let $u \in H^1((-T,T)\times I)$ and define

$$
\begin{aligned}
&I_{T,P}(u) \\
&\quad = \frac{1}{4}\iint_{(-T,T)\times I}\left(\left|\frac{\partial u}{\partial t}(t,x)\right|^2 + \left|\frac{\partial u}{\partial x}(t,x)\right|^2\right) dtdx \\
&\quad + \iint_{(-T,T)\times I}\left(\frac{m^2}{4}u(t,x)^2 + P(u(t,x))g(x)\right) dtdx.
\end{aligned}
$$

By Sobolev's theorem, $I_{T,P}(u) < \infty$ for any $u \in H^1((-T,T)\times I)$.

Hence

$$\int_{-T}^{T} \sqrt{U(u(t,\cdot))}\|\partial_t u(t,)\|_{L^2}$$

$$\leq \int_{-T}^{T} U(u(t,\cdot))dt + \frac{1}{4}\int_{-T}^{T} \|\partial_t u(t,\cdot)\|_{L^2}^2 dt$$

$$= I_{T,P}(u) < \infty.$$

Now we define an infinite dimensional analogue of Agmon distance.

Definition 8 *Let* $0 < T < \infty$. *We define the Agmon distance between* $h, k \in H (= H^{1/2}(I))$ *by*

$$d_U^{Ag}(h,k) = \inf\left\{ \int_{-T}^{T} \sqrt{U(u(t,\cdot))}\|\partial_t u(t,\cdot)\|_{L^2} dt \;\middle|\; u \in H^1_{T,h,k}(I) \right\}.$$

The definition of d_U^{Ag} does not depend on T.

To prove tunneling estimates, we need another quantity d_U^W.

Definition 9 *Let* u *be a non-negative bounded continuous function on* W. *Let* $\bar{H}$ *be the all mappings* $c : [-1, 1] \to L^2(I, dx)$ *such that*

(i) c *is an absolutely continuous path on* L^2 *and*

$$\int_{-1}^{1} \|c'(t)\|_{L^2}^2 dt < \infty.$$

(ii) $c(t) \in H$ *for almost every* $t \in [-1, 1]$ *and* $c(\cdot) \in L^2((-1, 1) \to H)$

For $w_1, w_2 \in W$, *define*

$$\rho_u^W(w_1, w_2) = \inf\left\{ \int_{-1}^{1} \sqrt{u(w_1 + c(t))} \|c'(t)\|_{L^2} dt \;\middle|\; c \in \bar{H}, \; c(-1) = 0 \text{ and } c(1) = w_2 - w_1 \right\}.$$

- If $h \in H$, then $\rho_u^W(w, w+h) < \infty$ for any $w \in W$.
- $H^1_{1,h,k}(I) \subset \bar{H}$.

Definition 10 *Let* $\mathcal{F}_U^W$ *be the set of non-negative bounded globally Lipschitz continuous functions* u *on* W *which satisfy the following conditions.*

(1) $$0 \le u(h) \le U(h) \quad \text{for all } h \in H^1,$$

$$\{h \in H^1 \mid U(h) - u(h) = 0\} = \mathcal{Z},$$

$$D^2 (U - u)(h_i) > 0, \quad \text{for all } h_i \in \mathcal{Z},$$

where $\mathcal{Z}$ *is the zero point set of* U.

(2) *There exists* $\varepsilon > 0$ *such that*

$$u(w) = \varepsilon \|w - h_i\|_W^2 \quad \text{in a n.b.d. of } h_i,\ 1 \le i \le n.$$

Let $\boldsymbol{h}, \boldsymbol{k} \in \boldsymbol{H}$ and we write

$$\boldsymbol{B}_{\varepsilon}(\boldsymbol{h}) = \{\boldsymbol{w} \in \boldsymbol{W} \mid \|\boldsymbol{w} - \boldsymbol{h}\|_{\boldsymbol{W}} \leq \varepsilon\}.$$

Definition 11 *For* $\boldsymbol{u} \in \mathcal{F}_{\boldsymbol{U}}^{\boldsymbol{W}}$, *set*

$$\underline{\boldsymbol{\rho}}_{\boldsymbol{u}}^{\boldsymbol{W}}(\boldsymbol{h}, \boldsymbol{k}) = \lim_{\varepsilon \to 0} \inf_{\boldsymbol{w} \in \boldsymbol{B}_{\varepsilon}(\boldsymbol{h}), \boldsymbol{\eta} \in \boldsymbol{B}_{\varepsilon}(\boldsymbol{k})} \boldsymbol{\rho}_{\boldsymbol{u}}^{\boldsymbol{W}}(\boldsymbol{w}, \boldsymbol{\eta})$$

and define

$$\boldsymbol{d}_{\boldsymbol{U}}^{\boldsymbol{W}}(\boldsymbol{h}, \boldsymbol{k}) = \sup_{\boldsymbol{u} \in \mathcal{F}_{\boldsymbol{U}}^{\boldsymbol{W}}} \underline{\boldsymbol{\rho}}_{\boldsymbol{u}}^{\boldsymbol{W}}(\boldsymbol{h}, \boldsymbol{k}).$$

Lemma 12

$$\boldsymbol{d}_{\boldsymbol{U}}^{\boldsymbol{W}}(\boldsymbol{h}, \boldsymbol{k}) \leq \boldsymbol{d}_{\boldsymbol{U}}^{\boldsymbol{Ag}}(\boldsymbol{h}, \boldsymbol{k}) < \infty \qquad \textit{for all } \boldsymbol{h}, \boldsymbol{k} \in \boldsymbol{H}.$$

Assumption 13 (Double-well potential function)

Let $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{x})$ *be the polynomial function which defines* $\boldsymbol{U}$*.*

We consider the following assumption.

(A3) *For all* $\boldsymbol{x}$*,* $\boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{P}(-\boldsymbol{x})$ *and* $\boldsymbol{\mathcal{Z}} = \{\boldsymbol{h_0}, -\boldsymbol{h_0}\}$*, where* $\boldsymbol{h_0} \neq \boldsymbol{0}$*.*

The following is our second main theorem.

Theorem 14 *Assume that* U *satisfies* **(A1),(A2),(A3)**.

(1) $d_U^W(h_0, -h_0) > 0$ *and*

$$\limsup_{\lambda \to \infty} \frac{\log\left(E_2(\lambda) - E_1(\lambda)\right)}{\lambda} \leq -d_U^W(h_0, -h_0).$$

(2) *Let* $I = [-l/2, l/2]$. *Then*

$$d_U^W(-h_0, h_0) = d_U^{Ag}(-h_0, h_0).$$

Properties of Agmon distance

(1) Properties of Agmon distance

Proposition 15

(1) *Assume* U *is non-negative. Then* d_U^{Ag} *is a continuous distance function on* H.

(2) *Let* $U(h) = \frac{1}{4}\|Ah\|_H^2$ *(that is* $P = 0$*). Then*

$$d_U^{Ag}(0, h) = \frac{1}{4}\|h\|_H^2 \qquad \text{for any } h \in H.$$

(3) *Let* $I = [-l/2, l/2]$ *and assume* **(A1), (A2)**. *Then*

$$d_U^{Ag}(h, k) = d_U^W(h, k) \quad \text{for any } h, k \in H^1.$$

(2) Instanton

Let us consider a non-linear ellptic boundary value problem

$$\frac{\partial^2 u}{\partial t^2}(t,x)+\frac{\partial^2 u}{\partial x^2}(t,x) = m^2 u(t,x)+2P'(u(t,x))g(x)$$
$$\lim_{t\to-\infty} u(t,x) = -h_0(x), \quad \lim_{t\to\infty} u(t,x) = h_0(x).$$
$$(t,x) \in \mathbb{R} \times I$$

The solution is a candidate of minimizers (instanton) whose action integral attain the value:

$$\inf_{T>0, u\in H^1_{T,-h_0,h_0}(I)} I_{T,P}(u),$$

where

$$\boldsymbol{I_{T,P}(u)}$$

$$= \frac{1}{4}\iint_{(-T,T)\times I}\left(\left|\frac{\partial u}{\partial t}(t,x)\right|^2 + \left|\frac{\partial u}{\partial x}(t,x)\right|^2\right) dt dx$$

$$+ \iint_{(-T,T)\times I}\left(\frac{m^2}{4}u(t,x)^2 + P(u(t,x))g(x)\right) dt dx.$$

It is very likely that the minimum action integral of the instanton is equal to the Agmon distance $d_U^{Ag}(-h_0, h_0)$.

I show such a simple example.

(3) Example

Let us consider the case $I = [-l/2, l/2]$ and $g = 1$. Let $x_0 (\in \mathbb{R}) \neq 0$ and $a > 0$.

We consider the case where

$$U(h) = \frac{1}{4} \int_I h'(x)^2 dx + a \int_I \left(h(x)^2 - x_0^2 \right)^2 dx.$$

This can be realized by a suitable choice of P.

Note $\mathcal{Z} = \{x_0, -x_0\}$.

These are zero points also of the potential function

$$Q(x) = a(x^2 - x_0^2)^2 \quad x \in \mathbb{R}.$$

Let

$$\boldsymbol{d^{Ag}_{1-dim}(-x_0, x_0) = \inf\Big\{\int_{-T}^{T} \sqrt{Q(x(t))}|x'(t)|dt \;\Big|\; x(-T) = -x_0, \;\; x(T) = x_0\Big\}.}$$

This is the Agmon distance which corresponds to $\mathbf{1}$-dimensional Schrödinger operator $-\frac{d^2}{dx^2} + Q(x)$ and

$$\boldsymbol{d^{Ag}_{1-dim}(-x_0, x_0) = \int_{-x_0}^{x_0} \sqrt{Q(x)}dx = \frac{5\sqrt{a}x_0^3}{3}.}$$

We can prove the following.

Proposition 16 *Assume* $2ax_0^2 l^2 \le \pi^2$.

(1) $d_U^{Ag}(-x_0, x_0) = l\, d_{1-dim}^{Ag}(-x_0, x_0)$.

(2) *Let* $u_0(t) = x_0 \tanh\left(2\sqrt{a}x_0 t\right)$.

Then $u_0(t)$ *is the solution to*

$$u''(t) \ = \ 2Q'(u(t)) \qquad \text{for all } t \in \mathbb{R},$$

$$\lim_{t\to-\infty} u(t) = -x_0, \qquad \lim_{t\to\infty} u(t) = x_0$$

and

$$\begin{aligned} I_{\infty,P}(u_0) &= \left(\frac{1}{4}\int_{-\infty}^{\infty} u_0'(t)^2 dt + \int_{-\infty}^{\infty} Q(u_0(t))dt\right) l, \\ &= d^{Ag}_{1-dim}(-x_0, x_0) l \\ &= d^{Ag}_U(-h_0, h_0). \end{aligned}$$

That is,

• u_0 is the instanton for both operators: 1-dimensional Schrödinger operator $-\frac{d^2}{dx^2} + \lambda Q(\cdot/\sqrt{\lambda})$ and $-L + V_\lambda$.

• The Agmon distance $d^{Ag}_U(-h_0, h_0)$ is equal to the action integral of the instanton in this case.

Open problems

- $d_U^W(-h_0, h_0) = d_U^{Ag}(-h_0, h_0)$ in the case of $I = \mathbb{R}$?

- Instanton solutions for general cases ?

- $\lim_{\lambda\to\infty} \dfrac{\log\left(E_2(\lambda) - E_1(\lambda)\right)}{\lambda} = -d_U^{Ag}(h_0, -h_0).$?

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