Tunneling for spatially cut-off $P(\phi)_2$- Hamiltonian

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Introduction

(Spatially cut-off) $P(\phi)_2$-Hamiltonian $-L + V_\lambda$ is an $\infty$-dimensional Schrödinger operator defined on $L^2(S'(I), \mu)$, where $I = [-l/2, l/2]$ or $I = \mathbb{R}$ and $\lambda = \frac{1}{\hbar}$.

I explain my recent results:

• Determination of the semi-classical limit of $E_1(\lambda)$ as $\lambda \to \infty$

• An estimate on the asymptotic behavior of the gap of spectrum $E_2(\lambda) - E_1(\lambda)$ as $\lambda \to \infty$. 

Plan of Talk

1. $P(\phi)_2$-Hamiltonian

2. Results for Schrödinger operator $-\Delta + \lambda U(\cdot/\lambda)$

3. Main Result 1: $\lim_{\lambda \to \infty} E_1(\lambda)$

4. Main Result 2:

$$\limsup_{\lambda \to \infty} \frac{\log (E_2(\lambda) - E_1(\lambda))}{\lambda} \leq -d_W^U (-h_0, h_0)$$

(= $-d_{U}^{Ag} (-h_0, h_0)$ if $I = [-l/2, l/2]$)

5. Properties of Agmon distance $d_{U}^{Ag}$. 
$P(\phi)_2$-Hamiltonian

Let $I = [-l/2, l/2]$ or $I = \mathbb{R}$ and $m > 0$. Let $H^s(I, dx)$ be the Sobolev space with the norm:

$$\|\varphi\|_{H^s(I, dx)} = \|(m^2 - \Delta)^{s/2}\varphi\|_{L^2(I, dx)}.$$

Let $H = H^{1/2}(I, dx)$. Let $\mu$ be the Gaussian measure whose covariance operator is $(m^2 - \Delta)^{-1/2}$ on $L^2(I, dx)$. Let us consider a Hilbert space $W$:

(1) When $I = [-l/2, l/2]$, $W = H^{-\varepsilon}(I, dx)$, where $\varepsilon$ is any positive number.
(2) When $I = \mathbb{R}$,

$$W = \left\{ w \in S'(\mathbb{R}) \mid \right.$$  

$$\|w\|_W^2 = \int_{\mathbb{R}} |(1 + |x|^2 - \Delta)^{-1}w(x)|^2 dx < \infty \left. \right\}.$$

Then $(W, H, \mu)$ is an abstract Wiener space in the sense of Gross. Define a self-adjoint operator $A$ on $H$ by

$$Ah = (m^2 - \Delta)^{1/4}h,$$

$$D(A) = H^1 \subset H.$$
Definition 1 (Free Hamiltonian)

Let $\mathcal{E}_A$ be the Dirichlet form defined by

$$\mathcal{E}_A(f, f) = \int_W \|ADf(w)\|^2_H d\mu(w) \quad f \in D(\mathcal{E}_A),$$

where

$$D(\mathcal{E}_A) = \left\{ f \mid Df(w) \in D(A) \text{ and } \int_W \|ADf(w)\|^2_H d\mu(w) < \infty \right\},$$

$D : H$-derivative,

$L :$ the non-negative generator of $\mathcal{E}_A.$
Definition 2 Let \( P(x) = \sum_{k=0}^{2M} a_k x^k \) with \( a_{2M} > 0 \).

Let \( g \in C_0^\infty(I) \) with \( g(x) \geq 0 \) for all \( x \) and define

\[
V(h) = \int_I P(h(x)) g(x) \, dx \quad h \in H
\]

\[
U(h) = \frac{1}{4} \|Ah\|_H^2 + V(h) \quad \text{for } h \in D(A).
\]

Remark 3 \( V \) is well-defined on \( H \) and we can rewrite

\[
U(h) = \frac{1}{4} \int_I (h'(x)^2 + m^2 h(x)^2) \, dx
\]

\[
+ \int_I P(h(x)) g(x) \, dx \quad h \in H^1.
\]
Definition 4 (1) Let $\lambda > 0$. For the polynomial $P = P(x) = \sum_{k=0}^{2M} a_k x^k$ with $a_{2M} > 0$, define

$$
\int_I : P \left( \frac{w(x)}{\sqrt{\lambda}} \right) : g(x) \, dx
$$

$$
= \sum_{k=0}^{2M} a_k \int_I : \left( \frac{w(x)}{\sqrt{\lambda}} \right)^k : g(x) \, dx.
$$

We write

$$
: V \left( \frac{w}{\sqrt{\lambda}} \right) : = \int_I : P \left( \frac{w(x)}{\sqrt{\lambda}} \right) : g(x) \, dx
$$
and

\[ V_\lambda(w) = \lambda : V \left( \frac{w}{\sqrt{\lambda}} \right) : . \]

(2) It is known that \((-L + V_\lambda, \mathcal{F}C^\infty_A(W))\) is essentially self-adjoint, where \(\mathcal{F}C^\infty_A(W)\) denotes the set of smooth cylindrical functions.

We use the same notation \(-L + V_\lambda\) for the self-adjoint extension.

It is known that \(-L + V_\lambda\) is bounded from below and the lowest eigenvalue \(E_1(\lambda)\) is simple.
Some known results

• (Hoegh-Krohn and Simon 1972)
  \[ \sigma_{\text{ess}}(-L + V_\lambda) \cap [E_1(\lambda), E_1(\lambda) + m) = \emptyset. \]

• (Simon 1972) Example of spatially cut-off \( P(\phi)_2 \)-Hamiltonian for which there exist an eigenvalue which is in a continuous spectrum.

• (Dereziński and Gérard, 2000) \(-L + V_\lambda\) does not have singular continuous spectrum.

• (A.Arai, 1996) Calculation of \( \lim_{\lambda \to \infty} \text{tr} e^{t(L-V_\lambda)/\lambda} \) for certain \( P(\phi) \)-type models (not including \( P(\phi)_2 \)-model)
Schrödinger operators on $\mathbb{R}^N$

Assume

(i) $U \in C^\infty(\mathbb{R}^N)$, $U(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $\lim \inf_{|x| \to \infty} U(x) > 0$.

(ii) $\{ x \mid U(x) = 0 \} = \{ x_1, \ldots, x_n \}$.

(iii) $Q_i = \frac{1}{2} D^2 U(x_i) > 0$ for all $i$.

Then the lowest eigenvalue $E_1(\lambda)$ of $-\Delta + \lambda U(\cdot / \sqrt{\lambda})$ is simple and

$$
\lim_{\lambda \to \infty} E_1(\lambda) = \min_{1 \leq i \leq n} \text{tr} \sqrt{Q_i}.
$$
In addition to (i), (ii), (iii), we assume the symmetry of $U$:

(iv) $U(x) = U(-x)$,

(v) $\{x \mid U(x) = 0\} = \{-x_0, x_0\}$ \quad (x_0 \neq 0).

Then we have (due to Harrell, Jona-Lasinio, Martinelli and Scoppola, Simon, Helffer and Sjöstrand,...)

$$\lim_{\lambda \to \infty} \frac{\log(E_2(\lambda) - E_1(\lambda))}{\lambda} = -d_{Ag}^U(-x_0, x_0),$$

where $E_2(\lambda)$ is the second eigenvalue and $d_{Ag}^U(-x_0, x_0)$ is the Agmon distance between $-x_0$ and $x_0$ such that
\[
d_{U}^{Ag}(-x_{0}, x_{0}) = \inf \left\{ \int_{-T}^{T} \sqrt{U(x(t))} |\dot{x}(t)| \, dt \mid x \text{ is a smooth curve on } \mathbb{R}^{N} \right\}
\]

with \( x(-T) = -x_{0}, x(T) = x_{0} \) \{.

The definition is independent of \( T > 0 \).

The Agmon distance \( d_{U}^{Ag}(-x_{0}, x_{0}) \) is equal to the following action integral which is introduced by Carmona and Simon (1981). The minimizing path of the following variational problem is called an instanton.
\[ d_{CS}^{U}(-x_0, x_0) = \inf \left\{ \int_{-T}^{T} \left( \frac{1}{4} |x'(t)|^2 + U(x(t)) \right) \, dt \right\} \]
\[ \text{subject to } x \text{ is a smooth curve on } \mathbb{R}^N \text{ with} \]
\[ x(-T) = -x_0, \ x(T) = x_0, \ T > 0 \} . \]

The elementary inequality \( ab \leq \frac{a^2}{2} + \frac{b^2}{2} \) implies
\[ \int_{-T}^{T} \sqrt{U(x(t))} |x'(t)| \, dt \leq \int_{-T}^{T} \left( \frac{1}{4} |x'(t)|^2 + U(x(t)) \right) \, dt \]
and \( d_{Ag}^{U}(-x_0, x_0) \leq d_{CS}^{U}(-x_0, x_0) \).
\(-L + V_\lambda\) as an \(\infty\)-dimensional Schrödinger operator

\(-L + V_\lambda\) is informally unitarily equivalent to the \(\infty\)-dimensional Schrödinger operator on \(L^2(L^2(I, dx), dw)\):

\[-\Delta_{L^2(I)} + \lambda : U(w/\sqrt{\lambda}) : -\frac{1}{2} \text{tr}(m^2 - \Delta)^{1/2},\]

where

\[\cdot U(w) : = \frac{1}{4} \int_I w'(x)^2 dx \]

\[+ \int_I \left( \frac{m^2}{4} w(x)^2 + : P(w(x)) : g(x) \right) dx.\]
In fact, $P(\phi)_2$-Hamiltonian is related with the quantization of the classical field (nonlinear Klein-Gordon equation):

$$\frac{\partial^2 w}{\partial t^2}(t, x) = -2(\nabla U)(w(t, x)), (t, x) \in \mathbb{R} \times I$$

$$U(w) = \frac{1}{4} \int_I (w'(x)^2 + m^2 w(x)^2) \, dx$$

$$+ \int_I P(w(x))g(x) \, dx$$

$$2(\nabla U)(w(t, x)) = -\frac{\partial^2 w}{\partial x^2}(t, x) + m^2 w(t, x)$$

$$+ 2P'(w(t, x))g(x).$$
Main Result 1

Assumption 5

(A1) $U(h) \geq 0$ for all $h \in H^1$ and

$$\mathcal{Z} = \{ h \in H^1 \mid U(h) = 0 \} = \{ h_1, \ldots, h_n \}$$

is a finite set.

(A2) The Hessian $D^2 U(h_i) \ (1 \leq i \leq n)$ is strictly positive. The derivative $D$ stands for the $H$-derivative.
Remark 6

\[ D^2 U(h_i) = \frac{1}{2}A^2 + D^2 V(h_i) \]

is an unbounded operator on \( H \).

\[ \inf \sigma(D^2 U(h_i)) > 0 \iff \inf \sigma(m^2 - \Delta + 4v_i) > 0 \]

where

\[ v_i(x) = \frac{1}{2}P''(h_i(x))g(x). \]
Theorem 7 \hspace{1cm} \textit{Assume (A1) and (A2) hold.} \\

Let $E_1(\lambda) = \inf \sigma(-L + V_\lambda)$. Then \\

$$
\lim_{\lambda \to \infty} E_1(\lambda) = \min_{1 \leq i \leq n} E_i,
$$

where \\

$$
E_i = \inf \sigma(-L + Q_{v_i}),
$$

$$
Q_{v_i} = \int_I : w(x)^2 : v_i(x) \, dx,
$$

$$
v_i(x) = \frac{1}{2} P''(h_i(x))g(x).
$$
Main Result 2 (Tunneling estimate)

Let

\[ E_2(\lambda) = \inf \{ \sigma(-L + V_\lambda) \setminus \{ E_1(\lambda) \} \} . \]

We prove that \( E_2(\lambda) - E_1(\lambda) \) is exponentially small when \( \lambda \to \infty \) under a certain assumption on \( P \).

To state our estimate, we introduce infinite dimensional analogue of Agmon distance in quantum mechanics.
Let us fix $T > 0$ and take $h, k \in H (= H^{1/2}(I))$.

Let

$$H^1_{T,h,k}(I)$$

$$= \left\{ u = u(t, x) \ ((t, x) \in (-T, T) \times I) \right\}$$

$$u \in H^1((-T, T) \times I),$$

$$u(-T, \cdot) = h, \ u(T, \cdot) = k \text{ in the sense of trace} \right\}$$

Note $H^1_{T,h,k}(I) \neq \emptyset$. 
Let $U$ be a non-negative potential function which we introduced. Here we do not assume $(A1)$, $(A2)$.

For any $u \in H_{T,h,k}^1(I)$, the following properties hold:

(i) $u(t, \cdot) \in H^1(I)$ for almost every $t$ and $t(\in [-T,T]) \mapsto U(u(t, \cdot))$ is a Lebesgue measurable function,

(ii) $t(\in (-T,T)) \mapsto u(t, \cdot) \in L^2(I)$ is an absolutely continuous function and its derivative is in $L^2((-T,T) \rightarrow L^2(I))$,

(iii) $\int_{-T}^{T} \sqrt{U(u(t, \cdot))} \| \partial_t u(t, \cdot) \|_{L^2(I)} dt < \infty$. 

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The fact (iii) follows from the following argument. Let \( u \in H^1((-T, T) \times I) \) and define

\[
I_{T,P}(u) = \frac{1}{4} \iint_{(-T,T)\times I} \left( \left| \frac{\partial u}{\partial t}(t, x) \right|^2 + \left| \frac{\partial u}{\partial x}(t, x) \right|^2 \right) \, dt \, dx + \iint_{(-T,T)\times I} \left( \frac{m^2}{4} u(t, x)^2 + P(u(t, x))g(x) \right) \, dt \, dx.
\]

By Sobolev’s theorem, \( I_{T,P}(u) < \infty \) for any \( u \in H^1((-T, T) \times I) \).
Hence

\[ \int_{-T}^{T} \sqrt{U(u(t, \cdot))} \| \partial_t u(t, \cdot) \|_{L^2} \]

\[ \leq \int_{-T}^{T} U(u(t, \cdot)) dt + \frac{1}{4} \int_{-T}^{T} \| \partial_t u(t, \cdot) \|_{L^2}^2 dt \]

\[ = I_{T,P}(u) < \infty. \]

Now we define an infinite dimensional analogue of Agmon distance.
Definition 8 Let $0 < T < \infty$. We define the Agmon distance between $h, k \in H (= H^{1/2}(I))$ by

$$d_{U}^{Ag}(h, k) = \inf \left\{ \int_{-T}^{T} \sqrt{U(u(t, \cdot))} \| \partial_{t}u(t, \cdot) \|_{L^2} dt \mid u \in H^{1}_{T, h, k}(I) \right\}.$$

The definition of $d_{U}^{Ag}$ does not depend on $T$. To prove tunneling estimates, we need another quantity $d_{U}^{W}$. 
Definition 9  Let $u$ be a non-negative bounded continuous function on $W$. Let $\tilde{H}$ be the all mappings $c : [-1, 1] \to L^2(I, dx)$ such that

(i) $c$ is an absolutely continuous path on $L^2$ and
$$\int_{-1}^{1} \|c'(t)\|_{L^2}^2 dt < \infty.$$

(ii) $c(t) \in H$ for almost every $t \in [-1, 1]$ and $c(\cdot) \in L^2((-1, 1) \to H)$
For \( w_1, w_2 \in W \), define

\[
\rho^W_u (w_1, w_2) = \inf \left\{ \int_{-1}^{1} \sqrt{u(w_1 + c(t))}\|c'(t)\|_{L^2} dt \mid c \in \bar{H}, c(-1) = 0 \text{ and } c(1) = w_2 - w_1 \right\}.
\]

- If \( h \in H \), then \( \rho^W_u (w, w + h) < \infty \) for any \( w \in W \).
- \( H^1_{1,h,k}(I) \subset \bar{H} \).
Definition 10  Let $\mathcal{F}_U^W$ be the set of non-negative bounded globally Lipschitz continuous functions $u$ on $W$ which satisfy the following conditions.

(1)  $0 \leq u(h) \leq U(h)$ for all $h \in H^1$, 
\[
\{ h \in H^1 \mid U(h) - u(h) = 0 \} = \mathcal{Z},
\]
\[
D^2 (U - u) (h_i) > 0, \quad \text{for all } h_i \in \mathcal{Z},
\]
where $\mathcal{Z}$ is the zero point set of $U$.

(2)  There exists $\varepsilon > 0$ such that

$$u(w) = \varepsilon \| w - h_i \|_W^2 \quad \text{in a n.b.d. of } h_i, \; 1 \leq i \leq n.$$
Let $h, k \in H$ and we write

$$B_\varepsilon(h) = \{ w \in W \mid \|w - h\|_W \leq \varepsilon \}.$$ 

**Definition 11** For $u \in F_W^U$, set

$$\rho^W_{-u}(h, k) = \lim_{\varepsilon \to 0} \inf_{w \in B_\varepsilon(h), \eta \in B_\varepsilon(k)} \rho^W_u(w, \eta)$$

and define

$$d^W_U(h, k) = \sup_{u \in F^W_U} \rho^W_{-u}(h, k).$$

**Lemma 12**

$$d^W_U(h, k) \leq d^{Ag}_U(h, k) < \infty \quad \text{for all } h, k \in H.$$
Assumption 13 (Double-well potential function)

Let $P = P(x)$ be the polynomial function which defines $U$.

We consider the following assumption.

(A3) For all $x$, $P(x) = P(-x)$ and $\mathcal{Z} = \{h_0, -h_0\}$, where $h_0 \neq 0$.

The following is our second main theorem.
Theorem 14  Assume that $U$ satisfies (A1),(A2),(A3).

(1)  $d^W_U(h_0, -h_0) > 0$ and

$$\limsup_{\lambda \to \infty} \frac{\log(E_2(\lambda) - E_1(\lambda))}{\lambda} \leq -d^W_U(h_0, -h_0).$$

(2) Let $I = [-l/2, l/2]$. Then

$$d^W_U(-h_0, h_0) = d^{Ag}_U(-h_0, h_0).$$
Properties of Agmon distance

(1) Properties of Agmon distance

Proposition 15

(1) Assume $U$ is non-negative. Then $d_{U}^{Ag}$ is a continuous distance function on $H$.

(2) Let $U(h) = \frac{1}{4} \|Ah\|_{H}^{2}$ (that is $P = 0$). Then

$$d_{U}^{Ag}(0, h) = \frac{1}{4} \|h\|_{H}^{2} \quad \text{for any } h \in H.$$

(3) Let $I = [-l/2, l/2]$ and assume $(A1), (A2)$. Then

$$d_{U}^{Ag}(h, k) = d_{U}^{W}(h, k) \quad \text{for any } h, k \in H^{1}.$$
(2) Instanton

Let us consider a non-linear elliptic boundary value problem

\[ \frac{\partial^2 u}{\partial t^2}(t, x) + \frac{\partial^2 u}{\partial x^2}(t, x) = m^2 u(t, x) + 2P'(u(t, x))g(x) \]

\[ \lim_{t \to -\infty} u(t, x) = -h_0(x), \quad \lim_{t \to \infty} u(t, x) = h_0(x). \]

\[ (t, x) \in \mathbb{R} \times I \]

The solution is a candidate of minimizers (instanton) whose action integral attain the value:

\[ \inf_{T > 0, u \in H^1_{T, -h_0, h_0}(I)} I_{T, P}(u), \]
where

\[ I_{T,P}(u) = \frac{1}{4} \int\int_{(-T,T) \times I} \left( \left| \frac{\partial u}{\partial t}(t, x) \right|^2 + \left| \frac{\partial u}{\partial x}(t, x) \right|^2 \right) \, dt \, dx \]

\[ + \int\int_{(-T,T) \times I} \left( \frac{m^2}{4} u(t, x)^2 + P(u(t, x)) g(x) \right) \, dt \, dx. \]

It is very likely that the minimum action integral of the instanton is equal to the Agmon distance \( d_{Ag}^U (-h_0, h_0) \).

I show such a simple example.
(3) Example

Let us consider the case \( I = [-l/2, l/2] \) and \( g = 1 \). Let \( x_0 (\in \mathbb{R}) \neq 0 \) and \( a > 0 \).

We consider the case where

\[
U(h) = \frac{1}{4} \int_I h'(x)^2 \, dx + a \int_I (h(x)^2 - x_0^2)^2 \, dx.
\]

This can be realized by a suitable choice of \( P \).

Note \( \mathcal{Z} = \{x_0, -x_0\} \).

These are zero points also of the potential function

\[
Q(x) = a(x^2 - x_0^2)^2 \quad x \in \mathbb{R}.
\]
Let

\[ d_{1-dim}^{Ag}(-x_0, x_0) = \inf \left\{ \int_{-T}^{T} \sqrt{Q(x(t))}|x'(t)|\,dt \right\} \]

\[ x(-T) = -x_0, \quad x(T) = x_0 \}

This is the Agmon distance which corresponds to

1-dimensional Schrödinger operator \(-\frac{d^2}{dx^2} + Q(x)\) and

\[ d_{1-dim}^{Ag}(-x_0, x_0) = \int_{-x_0}^{x_0} \sqrt{Q(x)}\,dx = \frac{5\sqrt{ax_0^3}}{3}. \]

We can prove the following.
Proposition 16 Assume $2ax_0^2l^2 \leq \pi^2$.

(1) $d^{Ag}_U(−x_0, x_0) = l \, d^{Ag}_{1−dim}(−x_0, x_0)$.

(2) Let $u_0(t) = x_0 \tanh (2\sqrt{ax_0}t)$.

Then $u_0(t)$ is the solution to

$$u''(t) = 2Q'(u(t)) \quad \text{for all } t \in \mathbb{R},$$

$$\lim_{t \to -\infty} u(t) = -x_0, \quad \lim_{t \to \infty} u(t) = x_0$$
and

\[ I_{\infty, P}(u_0) = \left( \frac{1}{4} \int_{-\infty}^{\infty} u_0'(t)^2 dt + \int_{-\infty}^{\infty} Q(u_0(t)) dt \right) l, \]

\[ = d_{1-dim}^{Ag}(-x_0, x_0) l \]

\[ = d_{U}^{Ag}(-h_0, h_0). \]

That is,

- \( u_0 \) is the instanton for both operators: 1-dimensional Schrödinger operator \(- \frac{d^2}{dx^2} + \lambda Q(\cdot/\sqrt{\lambda})\) and \(-L + V_\lambda\).

- The Agmon distance \( d_{U}^{Ag}(-h_0, h_0) \) is equal to the action integral of the instanton in this case.
Open problems

- $d^W_U (-h_0, h_0) = d^A^g_U (-h_0, h_0)$ in the case of $I = \mathbb{R}$?

- Instanton solutions for general cases?

- $\lim_{\lambda \to \infty} \frac{\log (E_2(\lambda) - E_1(\lambda))}{\lambda} = -d^A^g_U (h_0, -h_0)$?
References


• B. Simon, Semiclassical analysis of low lying eigenvalues, II. Tunneling, Annals of Math. 120, (1984), 89–118.