Weak error of finite element approximations of a nonlinear stochastic heat equation

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(joint work with Stig Larsson)

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A semi-linear stochastic heat equation

The stochastic evolution equation

$$\begin{cases} \mathrm{d}X(t) + [AX(t) - f(X(t))]\,\mathrm{d}t = g(X(t))\,\mathrm{d}W(t), \qquad t \in (0, T], \\ X(0) = X_0. \end{cases}$$

$\mathrm{D} \subset \mathbf{R}^d$ convex polygonal domain. $H = L_2(\mathrm{D})$, $Q \in \mathcal{L}(H)$ selfadjoint nonnegative operator, $\{W(t)\}_{t \geq 0}$ an H-valued Q-Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$, $U_0 = Q^{1/2}(H)$, $\mathcal{L}_2^0 = \mathrm{HS}(U_0, H)$, $A = -\Delta$, $\mathcal{D}(A) = H_0^1(\mathrm{D}) \cap H^2(\mathrm{D})$.

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- $X_0 \in \dot{H}^\alpha$,
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- $f \in C_b^2(H, H)$,
- $g \in C_b^2(\dot{H}^\alpha, \mathrm{HS}(U_0, \dot{H}^{\beta-1}))$, i.e., $\|A^{\frac{\beta-1}{2}} g(x) Q^{\frac{1}{2}}\|_{\mathrm{HS}} < \infty$, $\forall x \in \dot{H}^\alpha$,

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- $g'' = 0$.

Mild solution

Let $\{E(t)\}_{t\geq 0}$ be the analytic semigroup generated by $-A$.

$\exists!$ solution $X \in C([0,T], L_2(\Omega, \dot{H}^\beta))$ to the mild equation

$$X(t) = E(t)X_0 + \int_0^t E(t-s)f(X(s))\,\mathrm{d}s + \int_0^t E(t-s)g(X(s))\,\mathrm{d}W(s), \quad t \in (0,T].$$

Approximation by the finite element method

A discretized equation:

$$\begin{cases} \mathrm{d}X_h(t) + [A_h X_h(t) - P_h f(X_h(t))]\,\mathrm{d}t = P_h g(X_h(t))\,\mathrm{d}W(t), \qquad t \in (0, T] \\ X_h(0) = P_h X_0. \end{cases}$$

Finite element spaces $\{S_h\}_{h\in(0,1]}$ of continuous piecewise linear functions corresponding to a quasi-uniform family of triangulations of D.

A_h is the discrete Laplacian satisfying

$$\langle A_h \psi, \chi \rangle_H = \langle \nabla\psi, \nabla\chi \rangle_H, \quad \forall \psi, \chi \in S_h.$$

$P_h \colon H \to S_h$ orthogonal projection w.r.t. $\langle \cdot, \cdot \rangle_H$.

Mild solution of discretized equation

Let $\{E_h(t)\}_{t\geq 0}$ be the analytic semigroup generated by $-A_h$.

For every $h \in (0,1]$ $\exists!$ solution $X_h \in C([0,T], L_2(\Omega, S_h))$ to the mild equation

$$X_h(t) = E_h(t)P_hX_0 + \int_0^t E_h(t-s)P_hf(X(s))\,\mathrm{d}s + \int_0^t E_h(t-s)P_hg(X(s))\,\mathrm{d}W(s), \quad t \in (0,T].$$

Malliavin integration by parts formula

For all $F \in \mathbf{D}^{1,2}(H)$ and $\Phi \in L_2(\Omega \times [0,T], \mathcal{L}_2^0)$

$$\begin{aligned}\mathbf{E}\Big\langle F, \int_0^T \Phi(t)\,\mathrm{d}W(t)\Big\rangle &= \mathbf{E}\int_0^T \mathrm{Tr}\{Q\Phi(t)^* D_t F\}\,\mathrm{d}t \\ &= \langle DF, \Phi\rangle_{L_2(\Omega\times[0,T],\mathcal{L}_2^0)}.\end{aligned}$$

In particular for $u \in C^2(H, \mathbf{R})$

$$\mathbf{E}\Big\langle u_x(X_h(t)), \int_0^T \Phi(s)\,\mathrm{d}W(s)\Big\rangle = \mathbf{E}\int_0^T \mathrm{Tr}\{Q\Phi(s)^* u_{xx}(F) D_s X_h(t)\}\,\mathrm{d}s.$$

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$$D_s^u X_h(t) = E_h(t-s)P_h\sigma(X_h(s))u + \int_0^t E_h(t-r)P_h f'(X_h(r)) D_s^u X_h(r)\,\mathrm{d}r$$
$$+ \int_0^t E_h(t-r)P_h\sigma'(X_h(r))\cdot D_s^u X_h\,\mathrm{d}W(r).$$

Main result

Theorem

For every test function $\Phi \in C_b^2(H, \mathbf{R})$ *and* $\epsilon > 0$ *there exist a* $C > 0$ *such that*

$$\mathbf{E}[\Phi(X(T)) - \Phi(X_h(T))] \leq Ch^{2\beta-\epsilon}.$$

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Method by A. Debussche (2011).

Assume that $f = 0$ and $X_0 = 0$.

Let $u(t,x) = \mathbf{E}\Phi(X(t,x))$, $t \in [0,T]$ and $x \in H$ where $X(0,x) = x$.
Satisfies the Kolmogorov equation

$$u_t(t,x) + Lu(t,x) = 0, \quad t \in [0,T), \quad x \in H,$$
$$u(0,x) = \Phi(x), \quad x \in H.$$

Here

$$Lu(t,x) = \langle Ax, u_x(t,x)\rangle - \frac{1}{2}\operatorname{Tr}\{g(x)Qg^*(x)u_{xx}(t,x)\}.$$

By Itô's formula and the Kolmogorov equation

$$
\begin{aligned}
&\mathbf{E}[\Phi(X(T)) - \Phi(X_h(T))] \\
&\quad = \mathbf{E}[u(T, X_0) - u(0, X_h(T))] \\
&\quad = \mathbf{E}\Big[\int_0^T u_t(T-s, X_h(s)) + L_h u(T-s, X_h(s))\,\mathrm{d}s\Big] \\
&\quad = \mathbf{E}\Big[\int_0^T (L_h - L)u(T-s, X_h(s))\,\mathrm{d}s\Big] \\
&\quad = \mathbf{E}\Big[\int_0^T \langle (A_h - A)X_h(s), u_x(T-s, X_h(s))\rangle\,\mathrm{d}s \\
&\qquad -\frac{1}{2}\int_0^T \mathrm{Tr}\{(P_h g(X_h(s))Qg^*(X_h(s))P_h \\
&\qquad\qquad -g(X_h(s))Qg^*(X_h(s)))u_{xx}(t, X_h(s))\}\,\mathrm{d}s\Big] \\
&\quad = I + J
\end{aligned}
$$

$$
\begin{aligned}
I &\le \left| \mathbf{E} \int_0^T \left\langle A_h P_h (I - R_h) u_x(T - s, X_h(s)), X_h(s) \right\rangle \mathrm{d}s \right| \qquad (R_h = A_h^{-1} P_h A) \\
&= \left| \mathbf{E} \int_0^T \left\langle u_x(T - s, X_h(s)), \right. \right. \\
&\qquad\qquad \left. \left. \int_0^s (A_h P_h (I - R_h))^* E_h(s - r) P_h g(X_h(r)) \, \mathrm{d}W(r) \right\rangle \mathrm{d}s \right| \\
&= \left| \mathbf{E} \int_0^T \int_0^s \mathrm{Tr} \left\{ Q g^*(X_h(r)) P_h E_h(s - r) A_h P_h (I - R_h) \right. \right. \\
&\qquad \left. \left. u_{xx}(T - s, X_h(s)) D_r X_h(s) \right\} \mathrm{d}r \, \mathrm{d}s \right| \\
&\le \mathbf{E} \int_0^T \int_0^s \| Q^{\frac{1}{2}} g^*(X_h(s)) A^{\frac{\beta - 1}{2}} \|_{\mathrm{HS}} \| A^{\frac{1 - \beta}{2}} P_h A_h^{\frac{\beta - 1}{2}} \|_{\mathcal{L}(H)} \\
&\qquad \times \| A_h^{1 - \epsilon} E_h(s - r) P_h \|_{\mathcal{L}(H)} \| A_h^{\frac{1 - \beta + 2\epsilon}{2}} P_h (I - R_h) A^{-\frac{1 + \beta - \epsilon}{2}} \|_{\mathcal{L}(H)} \\
&\qquad \times \| A^{\frac{1 + \beta - \epsilon}{2}} u_{xx}(T - s, X_h(s)) A^{\frac{1 - \beta}{2}} \|_{\mathcal{L}(H)} \\
&\qquad \times \| A^{\frac{\beta - 1}{2}} D_r X_h(s) Q^{\frac{1}{2}} \|_{\mathrm{HS}} \, \mathrm{d}r \, \mathrm{d}s \\
&\le C h^{2\beta - 3\epsilon} \int_0^T \int_0^s (T - s)^{-\frac{2 - \epsilon}{2}} (s - r)^{-1 + \epsilon} \, \mathrm{d}r \, \mathrm{d}s.
\end{aligned}
$$

Estimates

Let $0 \leq s \leq r \leq 1$. Then

$$\|A_h^{\frac{s}{2}} P_h (I - R_h) A^{-\frac{r}{2}}\|_{\mathcal{L}(H)} \leq C h^{r-s}, \quad h \in (0,1].$$

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Let $\gamma \geq 0$. Then

$$\|A^{\gamma} E(t)\|_{\mathcal{L}(H)} + \|A_h^{\gamma} E_h(t) P_h\|_{\mathcal{L}(H)} \leq C t^{-\gamma}, \quad \forall t > 0.$$

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Let $\lambda \in [0, \frac{1+\beta}{2})$. Then

$$\|A^{\lambda} u_x(t,x)\|_{\mathcal{L}(H)} \le C t^{-\lambda} |\Phi|_{C_{\mathrm{b}}^1}, \quad \forall t \in (0,T], \quad \forall x \in H.$$

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Let $\lambda, \rho \in [0, \frac{1+\beta}{2})$ be such that $\lambda + \rho < 1$. Then

$$\|A^{\rho} u_{xx}(t,x) A^{\lambda}\|_{\mathcal{L}(H)} \le C t^{-(\rho+\lambda)} |G|_{C_{\mathrm{b}}^2}, \quad \forall t \in (0,T], \quad \forall x \in H.$$

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Let $\gamma \in [0, \beta)$. Then

$$\mathbf{E}[\|A_h^{\frac{\beta-1+\gamma}{2}} D_s X_h(t) Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2] \leq C(t-s)^{-\gamma}, \quad \forall t \in [0,T], \quad \forall s \in [0,t).$$

Thank you for your attention!