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On finite difference approximations for degenerate filtering

István Gyöngy

Maxwell Institute, School of Mathematics, Edinburgh University

based on joint work with N.V. Krylov

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1. **Introduction: schemes for degenerate PDEs**

$$du_t(x) = (\mathcal{L}u_t(x) + f_t(x))\,dt, \qquad (t,x) \in [0,T] \times \mathbb{R}^d =: H_T$$

$$u_0(x) = \psi(x), \qquad x \in \mathbb{R}^d,$$

where $\mathcal{L} = a^{\alpha\beta} D_\alpha D_\beta$, $\alpha, \beta \in \{0,1,...,d\}$, $D_i = \frac{\partial}{\partial x_i}$ for $i \neq 0$, $D_0 = I$. Assume

$$a^{ij} z^i z^j \geq 0 \qquad z = (z^1, z^2, ..., z^d) \in \mathbb{R}^d.$$

For fixed $h \neq 0$ replace $\mathcal{L}$ by $L^h = \sum_{\lambda,\mu \in \Lambda} \bar{a}^{\lambda\mu} \delta_{h,\lambda} \delta_{h,\mu}$, where Λ is a finite set of vectors, and

$$\delta_{h,\lambda}\varphi(x) = \frac{\varphi(x + h\lambda) - \varphi(x)}{h}, \quad \text{for } \lambda \neq 0,$$

$$\delta_{h,\lambda} = Identity \quad \text{for } \lambda = 0$$

Approximate the Cauchy problem by

$$du_t^h(x) = (L^h u_t^h(x) + f_t(x))\,dt, \qquad (t,x) \in [0,T] \times \mathbb{G}_h,$$

$$u_t^h(x) = \psi(x), \qquad x \in \mathbb{G}_h,$$

$$\mathbb{G}_h = \{h\lambda_1 + h\lambda_2 + ... + h\lambda_n : \lambda_i \in \Lambda \cup (-\Lambda)\}$$

Tasks:

(1) Estimate $\sup_{(t,x)\in[0,T]\times\mathbb{G}_h} |u_t^h(x) - u_t(x)|$

(2) Investigate Richardson extrapolation, i.e., if

$$u^h = u + \sum_{i=1}^{k} \frac{h^i}{i!} u^{(i)} + h^{k+1} r_k^h$$

with $u^{(1)}$,...,$u^{(k)}$ independent of h,

$\sup_{[0,T]\times\mathbb{G}_h} |r^h| \leq K$ with constant K independent of h.

(1) and (2) have been studied thoroughly in the literature in the strongly parabolic case, i.,e, when

$$\sum_{i,j=1,d} a^{ij} z^i z^j \geq \kappa |z|^2 \quad \text{with a constant } \kappa > 0$$

There are only a few papers in the degenerate case. The difficulty is to estimate the (discrete) gradient of u^h independently of h. Gradient estimates and hence rate of convergence estimates for finite difference schemes (in space and time) are obtained in

H. Dong-N.V. Krylov, *On the rate of convergence of finite-difference approximations for degenerate linear parabolic equations with C^1 and C^2 coefficients,* (2005).

First order and higher order derivative estimates for finite difference schemes in the space variables and results on Richardson extrapolation are obtained in

N.V. Krylov-I.G, *First derivative estimates for finite difference schemes,* (2009)

N.V. Krylov-I.G, *Higher order derivative estimates for finite-difference schemes,* (2009)

N.V. Krylov-I.G, *Accelerated finite difference schemes for second order degenerate elliptic and parabolic problems in the whole space*, (2011).

In all the above papers the finite difference schemes are *monotone* schemes and the maximum principle plays a crucial role. A different approach is used to get results on Richardson extrapolation for *non-monotone* finite difference schemes of stochastic PDEs (under uniform stochastic parabolicity condition) in

N.V. Krylov-I.G, *Accelerated finite difference schemes for stochastic parabolic partial differential equations in the whole space* (2010).

The results of this paper are extended to fully discretized schemes in

E. Hall, *Accelerated spatial approximations for time discretized stochastic partial differential equations*, (2012).

Results on Richardson extrapolation for degenerate SPDEs are obtained in

I.G. *Accelerated finite difference schemes for degenerate stochastic parabolic partial differential equations in the whole space*, (2011).

2. **Nonlinear Filtering, Zakai equation**

$Z = (X, Y)$ signal-observation

$$dX_t = h(Z_t)\,dt + \sigma(Z_t)\,dW_t + b(Z_t)\,dV_t,$$

$$dY_t = H(Z_t)\,dt + dV_t, \quad X_0 = \xi \in \mathbb{R}^d, \quad Y_0 = \eta \in \mathbb{R}^{d_2},$$

(W, V) is a $d_1 + d_2$-dimensional Wiener process independent of (ξ, η).

- Compute the best estimate of $\varphi(X_t)$ given $\mathcal{Y}_t = (Y_s)_{s\in[0,t]}$

$$E(\varphi(X_t)|\mathcal{Y}_t) = \int_{\mathbb{R}^d} \varphi(x)\, P(t,dx) = \int_{\mathbb{R}^d} \varphi(x)\, p(t,x)\, dx,$$

where

$$P(t,dx) := P(X_t \in dx|\mathcal{Y}_t), \qquad p(t,x) := P(t,dx)/dx.$$

Theorem 2.1 Assume

- $P(\xi \in dx|\eta)/dx \in W_2^1(\mathbb{R}^d)$

- $|D_x^k(\sigma, b, h, H)| \leq K \qquad \text{for } k \leq 4$

Then p_t exists and can be computed as

$$p_t(x) = \frac{u_t(x)}{\int u_t(x)\, dx},$$

where u is the solution of the Zakai equation

$$du_t(x) = \mathcal{L}u_t(x)\,dt + \mathcal{M}^r u_t(x)\,dY_t^r,$$

$$u_0(x) = p_0(x) = P(\xi \in dx|\eta)/dx.$$

Here $\mathcal{L} = L^*$, $\mathcal{M}^r = M^{r*}$ are the adjoint of

$$L := a_t^{ij}(x)D_iD_j + h_t^i(x)D_i, \qquad M^r := H_t^r(x) + b_t^{ir}(x)D_i$$

$$a_t(x) := \frac{1}{2}(\sigma_t\sigma_t^*(x) + b_tb_t^*(x), \qquad h_t(x) := h(x, Y_t)$$

$\sigma_t(x) := \sigma(x, Y_t)$, $H_t(x) := H(x, Y_t)$.

3. **Parabolic SPDEs**

$$du_t = (\mathcal{L}_t u_t + f_t)\,dt + (\mathcal{M}_t^\rho u_t + g_t^\rho)\,dw_t^\rho, \quad (1)$$

for $(t,x) \in [0,T] \times \mathbb{R}^d =: H_T$,

$$u_0(x) = \psi(x), \quad x \in \mathbb{R}^d, \quad (2)$$

where w^r are independent $\mathcal{F}_t$-Wiener processes,

$\mathcal{L}_t = a_t^{\alpha\beta} D_\alpha D_\beta, \quad \mathcal{M}_t^\rho = b_t^{\alpha\rho} D_\alpha, \ \ \alpha, \beta \in \{0, 1, ..., d\}$

$a_t^{\alpha\beta} = a_t^{\alpha\beta}(\omega, x) \in \mathbb{R}$, $b_t^\alpha = (b_t^{\alpha\rho}(\omega, x))_{\rho=1}^\infty \in l_2$,

$f_t = f_t(x, \omega) \in \mathbb{R}$, $g_t = (g_t^\rho(\omega, x)) \in l_2$,

$\psi = \psi(\omega, x) \in \mathbb{R}$.

The theory of (1)-(2) and their numerical analysis are well-developed under the condition of
Strong Stochastic Parabolicity:
There is a constant $\lambda > 0$ such that

$$\sum_{i,j=1}^{d} (2a^{ij} - b^{i\rho} b^{j\rho}) z^i z^j \geq \lambda |z|^2 \tag{3}$$

for all $(\omega, t, x) \in \Omega \times H_T$ and $z \in \mathbb{R}^d$.

In general the Zakai equation in nonlinear filtering satisfies (3) only with $\lambda = 0$:

Assumption P. (**Stochastic parabolicity**) For all $(\omega, t, x) \in \Omega \times H_T$ and $z \in \mathbb{R}^d$

$$\sum_{i,j=1}^{d} (2a^{ij} - b^{i\rho} b^{j\rho}) z^i z^j \geq 0.$$

In the case of the Zakai equation:

$$2a^{ij} - b^{i\rho}b^{j\rho} = \sigma^{ik}\sigma^{kj}$$

$$(2a^{ij} - b^{i\rho}b^{j\rho})z^i z^j = \sigma^{ik}\sigma^{kj}z^i z^j = \sum_k |\sigma^{ik}z^i|^2 \geq 0.$$

We will use the following result on solvability of (1)-(2) in H^m, for integers $m \geq 0$, where H^m denotes the usual Hilbert-Sobolev space of functions on $\mathbb{R}^d$ with norm $|\cdot|_m$, defined by

$$|\phi|_m^2 = \sum_{|\gamma|\leq m} \int |D^\gamma \varphi(x)|^2 \, dx.$$

Assumption R. (i) a^{ij} and their derivatives in x up to order $\max(m,2)$; $a^{0\alpha}$, $a^{0\alpha}$, b^{α} and their derivatives in x up to order m are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable functions and in magnitude bounded by a constant K.
(ii) $\psi \in H^m$ is $\mathcal{F}_0$-measurable, f is an H^m-valued, g^ρ are H_2^{m+1}-valued predictable processes, such that

$$\mathcal{K}_m^2 := \int_0^T (|f_t|_m^2 + \sum_\rho |g_t^\rho|_{m+1}^2)\, dt + |u_0|_m^2 < \infty.$$

Remark 1. If Assumption R(ii) holds with $m > d/2$ then we have a continuous function of x which equals to u_0 dx-a.e., and we have continuous functions of x which coincide with f_t and g_t dx-a.e. Thus when Assumption R holds with $m > d/2$, we always assume that ψ, f_t and g_t are continuous in x for all t.

Definition. An H^1-valued weakly continuous process $u = (u_t)_{t\in[0,T]}$ is a solution to (1)-(2) if almost surely for all $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$(u_t,\varphi) = (u_0,\varphi)+\int_0^t (-a_s^{ij} D_j u_s, D_i\varphi)+(a_s^\alpha D_\alpha u_s,\varphi)+(f_s,\varphi)\,ds$$

$$+\int_0^t (b_s^{\alpha\rho} D_\alpha u_s + g_s^\rho,\varphi)\,dw_s^\rho, \qquad t\in[0,T],$$

where $a^j := -D_i a^{ij} + a^{0j} + a^{j0}$ and $a^0 := a^{00}$.

Theorem 3.1. Let Assumptions P-R hold with $m \geq 1$. Then (1)-(2) has a unique solution u. Moreover, u is H^m-valued weakly continuous process, it is a continuous process with values in H^{m-1}, and for $q > 0$

$$E \sup_{t \leq T} \|u_t\|_m^q \leq N E \mathcal{K}_m^q, \quad \text{with } N = N(m, d, q, K).$$

Remark 2. We will assume that $m - 1 > d/2$. Then by Sobolev embedding the solution $u_t(x)$ is a continuous function of (t, x).

4. **Finite difference schemes**

For $\alpha = i \in \{1, ..., d\}$ and $h \in \mathbb{R} \setminus \{0\}$ define

$$\delta^h_\alpha u(x) = \frac{1}{2h}(u(x + he_i) - u(x - he_i)),$$

and for $\alpha = 0$ let δ^h_α be the unit operator.

We approximate u by solving

$$du^h_t = (L^h_t u^h_t + f_t)\, dt + (M^{h,\rho}_t u^h_t + g^\rho_t)\, dw^\rho_t, \tag{4}$$

$$u^h_0 = \psi, \tag{5}$$

for $t \in [0, T]$, $x \in \mathbb{G}_h := |h|\mathbb{Z}^d$, where

$$L^h_t = a^{\alpha\beta}_t \delta^h_\alpha \delta^h_\beta \quad M^{h,\rho}_t = b^{\alpha\rho}_t \delta^h_\alpha.$$

Let $l_2(\mathbb{G}_h)$ be the set of real-valued functions ϕ on $(\mathbb{G}_h)$ such that $|\phi|^2_{l_2(\mathbb{G}_h)} := |h|^d \sum_{x\in\mathbb{G}_h} |\phi(x)|^2 < \infty$.

Remark. Equation (4) is a system of SDEs for $\{u_t(x) : x \in \mathbb{G}_h\}$. Therefore if, for instance, (a.s.)

$$u_0^h \in l_2(\mathbb{G}_h), \qquad \int_0^T |f_t|^2_{l_2(\mathbb{G}_h)} + \sum_\rho |g_t^\rho|^2_{l_2(\mathbb{G}_h)}\, dt < \infty,$$

and Assumption 2 (i) holds, then equation (4) has a unique $l_2(\mathbb{G}_h)$-valued continuous solution.

If $r > d/2$ then Sobolev's embedding of H^r into C_b implies $H^r \subset l_2(\mathbb{G}_h)$. Therefore if

$$\mathcal{K}_r^2 = \|\psi\|_r^2 + \int_0^T \|f_s\|_r^2 + \sum_\rho \|g_s^\rho\|_r^2\, ds < \infty \quad (a.s.),$$

for some $r > d/2$, then (4)-(5) has a unique $l_2(\mathbb{G}_h)$-valued continuous solution $(u_t^h)_{t\in[0,T]}$.

We want to estimate u^h independently of h. Take an integer $l \geq 0$.

Assumption 1.(i) There is an $\mathbb{R}^{d \times d_1}$-valued function $\sigma = \sigma_t^{ik}(x)$ on $\Omega \times H_T$ such that

$$\tilde{a}^{ij} := 2a^{ij} - b^{i\rho}b^{j\rho} = \sigma^{ik}\sigma^{jk}, \quad i,j = 1,...,d; \; k = 1,...,d_1$$

(ii) σ is $l+1$ times continuously differentiable in x,

$$|D^j\sigma| \leq K \quad j = 1,...,l+1.$$

Assumption 2.(i) $a^{\alpha 0}$, $a^{0\alpha}$ are l-times, b^α are $l+1$-times continuously differentiable in x, for $\alpha = 0, 1, ..., d$,

$$|D^j a^{\alpha,0}| + |D^j a^{0,\alpha}| \le K, \quad |D^k b^\alpha|_{l_2} \le K,$$

for $j = 0, ..., l$ and $k = 1, ..., l+1$.
(ii) $\psi \in H^l$, f is an H^l-valued predictable process, and g^r are H^{l+1}-valued predictable processes,

$$\mathcal{K}_l^2 = |\psi|_l^2 + \int_0^T |f_t|_l^2 + |g_t|_{l+1}^2 \, dt < \infty$$

Theorem 4.1. Let Assumptions 1-2 hold with $l > d/2$. Then for $q > 0$

$$E \sup_{t \in [0,T]} \sup_{x \in \mathbb{G}_h} |u_t^h(x)|^q \le N E \mathcal{K}_l^q$$

with constant $N = N(T, d, q, K)$.

Theorem 4.2. Let $l > d/2$. Let Assumption 1 hold with $l > d/2$ and let Assumption R hold with $m > 4 + l$. Assume $E\mathcal{K}_m^q < \infty$ for some $q > 0$. Then

$$E \sup_{t\in[0,T]} \sup_{x\in\mathbb{G}_h} |u_t^h(x) - u_t(x)|^q \le Nh^{2q} E\mathcal{K}_m^q$$

with $N = N(T, l, m, d, q, K)$.

Theorem 4.3. Let $l > d/2$. Let Assumptions 1 hold with $l > d/2$ and let Assumption R hold with $m > 4 + d/2$. Then for each $\varepsilon > 0$ there is a finite r. v. ξ_ε such that almost surely

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{G}_h} |u_t^h(x) - u_t(x)| \le \xi_\varepsilon h^{2-\varepsilon}$$

for all $h > 0$.

5. **Accelerated finite difference schemes**

Let $k \geq 0$ be a fixed integer.
Aim: existence of $u_t^{(j)}(x)$, $(t,x) \in H_T$, $j = 0, ..., k$, independent of h, s.t. $u^{(0)}$ is the solution of (1)-(2), and for each $h \neq 0$ almost surely

$$u_t^h(x) = \sum_{j=0}^{k} \frac{h^j}{j!} u_t^{(j)}(x) + h^{k+1} r_t^h(x), \quad x \in \mathbb{G}_h,\ t \in [0,T], \tag{6}$$

where R^h is an $l_2(\mathbb{G}_h)$-valued continuous process, s. t.

$$E \sup_{t \in [0,T]} \sup_{x \in \mathbb{G}_h} |r_t^h(x)|^q \leq N E \mathcal{K}_m^q \tag{7}$$

for some $q > 0$ with a constant N independent of h.

Theorem 5.1. Let $l > d/2$. Let Assumption 1 hold with $l > d/2$, let Assumption R hold with

$$m > 2k + 3 + l \tag{8}$$

for some integer $k \geq 0$ and let $E\mathcal{K}_m^q < \infty$ for some $q > 0$. Then

(i) expansion (6) and estimate (7) hold with a constant $N = N(T, d, m, q, k, K)$,

(ii) $u^{(j)} = 0$ for odd j,

(iii) if k is odd then instead of (8) we need only

$$m > 2k + 2 + l.$$

Define

$$\tilde{u}^h = \sum_{j=0}^{\tilde{k}} \lambda_j u^{2^{-j}h}, \tag{9}$$

where

$$(\lambda_0, \lambda_1, ..., \lambda_{\tilde{k}}) := (1, 0, 0, ..., 0)V^{-1}, \quad \tilde{k} = [\tfrac{k}{2}],$$

and V^{-1} is the inverse of

$$V^{ij} := 4^{-(i-1)(j-1)}, \quad i, j = 1, ..., \tilde{k} + 1.$$

Theorem 5.2. Under the conditions of Theorem 3.1

$$E \sup_{t \le T} \sup_{x \in \mathbb{G}_h} |\tilde{u}^h_t(x) - u_t(x)|^q \le N|h|^{q(k+1)} E\mathcal{K}^q_m, \tag{10}$$

with $N = N(T, d, m, k, q, K)$.

Theorem 5.3. Let $l > d/2$. Let Assumption 1 hold with $l > d/2$, let Assumption R hold with $m \geq 2k+3+l$ if k is even, and with $m \geq 2k+2+l$ if k is odd. Then for every $\varepsilon > 0$ there is η_ε such that almost surely

$$\sup_{t \leq T} \sup_{x \in \mathbb{G}_h} |\tilde{u}^h_t(x) - u_t(x)| \leq \eta_\varepsilon |h|^{k+1-\varepsilon} \tag{11}$$

for all $h > 0$.

Example 1. Assume $d = 2$ and the conditions of Theorem 5 with $l = 2, m = 10$. Then

$$\tilde{u}^h := \tfrac{4}{3} u^{h/2} - \tfrac{1}{3} u^h$$

satisfies

$$E \sup_{t \leq T} \sup_{x \in \mathbb{G}_h} |u_t(x) - \tilde{u}^h_t(x)|^q \leq N h^{4q}.$$

Results for the Zakai equation

Assumption 1 is a very unpleasant condition to satisfy. Even if $\tilde{a} = \tilde{a}(x)$, $x \in \mathbb{R}^d$ is a smooth function with values in the set of nonnegative matrices, its square root is only Lipschitz continuous in general. In the case of the Zakai equation, however, we have

$$\tilde{a}^{ij} = 2a^{ij} - b^{ir}b^{jr} = 2\frac{1}{2}(\sigma\sigma^* + bb^*)^{ij} - (bb^*)^{ij} = \sigma^{ik}\sigma^{jk}.$$

Thus to satisfy Assumption 1 it is sufficient to require that σ has bounded derivatives in x up to a sufficiently high order.

Hence one can get the following results for the Zakai equation.

Theorem 5.4. Assume that the derivatives in x of H, σ, b and h up to order $m > 4 + d/2$ are bounded by a constant K and that $E|p_0|_m^q < \infty$ for some $q > 0$. Then

$$E \sup_{t\in[0,T]} \sup_{x\in\mathbb{G}_h} |u_t^h(x) - u_t(x)|^q \leq Nh^{2q} E|p_0|_m^q$$

with $N = N(T, d, d_1, q, K)$.

Define $\tilde{u}^h = \sum_{j=0}^{\tilde{k}} \lambda_j u^{2^{-j}h}$, where

$$(\lambda_0, \lambda_1, ..., \lambda_{\tilde{k}}) := (1, 0, 0, ..., 0)V^{-1}, \quad \tilde{k} = [\tfrac{k}{2}],$$

and V^{-1} is the inverse of

$$V^{ij} := 4^{-(i-1)(j-1)}, \quad i, j = 1, ..., \tilde{k} + 1.$$

Theorem 5.5. Let $k \geq 0$. Assume that the derivatives in x of (H, σ, b, h) up to order $m \geq 2k+3+d/2$ if k is even and up to order $m \geq 2k+2+d/2$ if k is odd are bounded in magnitude by K. Let $E|p_0|_m^q < \infty$ for some $q > 0$. Then

$$E \sup_{t \leq T} \sup_{x \in \mathbb{G}_h} |\tilde{u}_t^h(x) - u_t(x)|^q \leq N|h|^{q(k+1)} E|p_0|_m^q$$

with $N = N(d, k, m, q, K)$.

Theorem 5.6. Let $k \geq 0$. Assume that the derivatives in x of (H, σ, b, h) up to order $m \geq 2k+3+d/2$ if k is even, and up to order $m \geq 2k+2+d/2$ if k is odd, are bounded in magnitude by K. Let $p_0 \in H^m$ (a.s.). Then for every $\varepsilon > 0$ there is η_ε such that almost surely

$$\sup_{t \leq T} \sup_{x \in \mathbb{G}_h} |\tilde{u}_t^h(x) - u_t(x)| \leq \eta_\varepsilon |h|^{k+1-\varepsilon} \quad \text{for all } h > 0.$$

Example 2.

$$du_t(x) = aD^2u_t(x)\,dt + bDu_t(x)\,dw_t, \qquad t > 0,\ x \in \mathbb{R}$$

$$u_0(x) = \cos x, \qquad x \in \mathbb{R},$$

Let $a = b = 2$.Then $2a - b^2/2 = 0$, i.e., this is a degenerate parabolic SPDE. The unique bounded solution is

$$u_t(x) = \cos(x + 2w_t).$$

The finite difference equation is the following:

$$du_t^h(x) = \frac{u_t^h(x+2h) - 2u_t^h(x) + u_t^h(x-2h)}{2h^2}\,dt$$

$$+\frac{u_t^h(x+h) - u_t^h(x-h)}{h}\,dw_t.$$

Its unique bounded solution with initial $u_0(x) = \cos x$ is

$$u_t^h(x) = \cos(x + 2\phi_h w_t),$$

where

$$\phi_h = \frac{\sin h}{h}.$$

For $t = 1$, $h = 0.1$, and $w_t = 1$ we have

$$u_1(0) = -0.4161468365,$$

$$u_1^h(0) = -0.4131150562, \quad u_1^{h/2}(0) = -0.415389039,$$

$$\tilde{u}_1^h(0) = \tfrac{4}{3}u_1^{h/2}(0) - \tfrac{1}{3}u_1^h(0) = -0.4161470333.$$

Such level of accuracy by $u_1^{\tilde{h}}(0)$ is achieved with $\tilde{h} = 0.0008$, which is more than 60 times smaller than $h/2$.

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