Localization of solutions to stochastic porous media equations: finite speed of propagation

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Let $\mathcal{O}$ be a bounded and open domain of $\mathbb{R}^d$, $d = 1, 2, 3$, with smooth boundary $\partial\mathcal{O}$. Consider the stochastic porous media equation

$$\begin{aligned} & dX - \Delta(|X|^{m-1}X)dt = \sigma(X)dW_t, && t \geq 0, \\ & X = 0 && \text{on } \partial\mathcal{O}, \\ & X(0) = x && \text{in } \mathcal{O}, \end{aligned} \tag{SPME}$$

where $m \geq 1, \Delta$ is the Dirichlet Laplacian and W_t is a Wiener process in $L^2(\mathcal{O})$ of the form

$$W_t = \sum_{k=1}^{N} \beta_k(t) e_k$$

for some $N \in \mathbb{N}$, where

$\{\beta_k\}_{k=1}^N$ are independent Brownian motions on a filtered probability space $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$, while $\{e_k\}_{k\in N}$ is an orthonormal system in $L^2(\mathcal{O})$ and

$$\sigma(X)W_t = \sum_{k=1}^{N} \mu_k X e_k \beta_k(t),$$

where $\mu_k,\ 1 \leq k \leq N$, are nonnegative numbers.
We assume that $e_k \in C^2(\overline{\mathcal{O}})$ and

$$\sum_{k=1}^{N} \mu_k^2 e_k^2(x) \geq \rho > 0, \quad \forall x \in \overline{\mathcal{O}}. \tag{P}$$

Let $H := H^{-1}(\mathcal{O})$ be the dual of $H_0^1(\mathcal{O})$ with norm $|\cdot|_{-1}$.

Definition 1.1

An H-valued continuous $\mathcal{F}_t$-adapted process $X = X(t,\xi)$ is called a strong solution to (SPME) *on* $(0,T)\times\mathcal{O}$ *if*

$$X \in L^2(\Omega, C([0,T];H)) \cap L^\infty(0,T;L^2(\Omega\times\mathcal{O})),$$
$$|X|^{m-1}X \in L^2(0,T;L^2(\Omega,H_0^1(\mathcal{O}))),$$
$$X(t) = x + \int_0^t \Delta(|X(s)|^{m-1}X(s))ds + \int_0^t \sigma(X(s))dW_s,\ t\in[0,T]. \qquad \text{(SPME')}$$

Here we use the standard notation $L^p(E;B)$, $p\in[0,\infty]$, for a measure space $(E,\mathcal{E},\mu)$ and a Banach space B, i.e., $L^p(E;B)$ denotes the space of all B-valued measurable maps $f : E \to B$ such that $|f|_B^p$ is μ-integrable.

Our main result says that if $1 < m \leq 5$, which is the case of slow diffusion under stochastic perturbation, then the process $X = X(t, \cdot)$ has the property of "**finite speed propagation of disturbances**" in the following sense (see [Antontsev/Shmarev, Nonlinear Analysis 2005]):
If $x = 0$ in $B_r(\xi_0) = \{\xi \in \mathbb{R}^d;\ |\xi - \xi_0| < r\} \subset \mathcal{O}$, then there is a function $r(\,\cdot\,, \omega) : [0, T] \to (0, r_0)$, decreasing in t, such that $X(t, \xi, \omega) = 0$ in $B_{r(t,\omega)}(\xi_0)$ for $0 \leq t \leq t(\omega)$, for $\mathbb{P}$-a.e. $\omega \in \Omega$.
In this sense, we speak about finite speed of propagation of $X(t)$.

Remark

In the case $0 < m < 1$ *(fast diffusion) and if* $d = 1$ *also for* $m = 0$ *the solution* $X = X(t, x)$ *has the finite extinction property with positive probability (see [Barbu/DaPrato/R., CRAS 2009, CMP 2009 resp.]) which also can be seen as a localization property for the solutions of* (SPME).

Our technique of proof is based on the results in [Barbu/R., JDE 2011], which allow to transform the problem to a deterministic partial differential equation (PDE) with random coefficients. This latter PDE, however, is not of porous media or any other known type, so that the necessary estimates become more complicated.

The main results

Theorem I

Assume that $x \in L^{m+1}(\mathcal{O})$. Then (SPME) *has a unique strong solution X. If $x \geq 0$ a.e. in $\mathcal{O}$, then $X \geq 0$ a.e. in $\Omega\times(0,T)\times\mathcal{O}$ and*

$$E\int_0^T ds \int_{\mathcal{O}} \left|\nabla(|X|^{m-1}X)\right|^2 d\xi + \sup_{t\in[0,T]} E\int_{\mathcal{O}} |X(t,\xi)|^{m+1} d\xi \leq C\int_{\mathcal{O}} |x|^{m+1} d\xi.$$

Remark

Existence and uniqueness, as well as nonnegativity of solutions to (SPME) has been discussed in several papers (see e.g. [Barbu/DaPrato/R., Indiana Univ. Math. J. 2007], [Ren/R./Wang, JDE 2007]). But the notion of solution was different. More precisely, solutions were not required to satisfy (SPME'), but only that

$$t \mapsto \int_0^t |X(s)|^{m-1} X(s) ds$$

is a continuous process in $H_0^1(\mathcal{O})$, and that (SPME') holds with the Laplacian in front of the ds-integral. We refer to [R./Wang, JDE 2008] for a detailed discussion. In the present paper, we need the stronger notion of solution as in Definition 1. For recent results on existence of such "strong" solutions for general SPDE of gradient type, including our situation as a special case, see [Gess, arXiv 2011].

Below, we are only concerned with small $T > 0$, so we may assume that $T \leq 1$. Furthermore, for a function $g : [0, 1] \to \mathbb{R}$, we define its α-Hölder norm, $\alpha \in (0, 1)$, by

$$|g|_\alpha := \sup_{\substack{s,t\in[0,1]\\ s\neq t}} \frac{|g(t) - g(s)|}{|t - s|^\alpha}.$$

Fix $\alpha \in (0, \frac{1}{2})$ and define the "Hölder ball of radius R"

$$\Omega^\alpha_{H,R} = \{\omega \in \Omega \mid |\beta_k(\omega)|_\alpha \leq R,\ 1 \leq k \leq N\}.$$

Then, $\Omega^\alpha_{H,R} \nearrow \Omega$ as $R \to \infty$ $\mathbb{P}$-a.s.
Now, we are ready to formulate our main result.

Theorem II

Assume that $d = 1, 2, 3$ *and* $1 < m \leq 5$, *and that* $x \in L^\infty(\mathcal{O})$, $x \geq 0$, *is such that*

$$\mathrm{support}\{x\} \subset B^c_{r_0}(\xi_0).$$

Let for $R > 0$

$$\delta(R) := \left(\frac{1}{m+1} \left(\frac{\rho}{2}\right)^{1/2} c_1^{-1} \left(\sum_{k=1}^N |\nabla e_k|_\infty \mu_k \right)^{-1} \times \exp\left[\frac{1}{2}\,(1-m) \left(\frac{1}{2}\, c_2 + \sum_{k=1}^N |e_k|_\infty \mu_k \right) \right] \right) \wedge 1,$$

where c_1, c_2 *(depending on* R*) are as in the Key Lemma below and* ρ *as in* (P).

The main results

Theorem II (continued)

Define for $T \in (0,1]$

$$\Omega_T^{\delta(R)} := \left\{ \sup_{t\in[0,T]} |\beta_k(t)| \leq \delta(R) \text{ for all } 1 \leq k \leq N \right\}.$$

Then, for $\omega \in \Omega_T^{\delta(R)} \cap \Omega_{H,R}^{\alpha}$*, there is a decreasing function* $r(\cdot,\omega) : [0,T] \to (0, r_0]$*, and* $t(\omega) \in (0,T]$ *such that for all* $0 \leq t \leq t(\omega)$*,*

$$\begin{aligned} &X(t,\omega) = 0 \text{ on } B_{r(t,\omega)}(\xi_0) \supset B_{r(t(\omega),\omega)}(\xi_0), \quad \text{and} \\ &X(t,\omega) \not\equiv 0 \text{ on } B^c_{r(t,\omega)} \subset B^c_{r(t(\omega),\omega)}(\xi_0). \end{aligned} \tag{L}$$

Since $\Omega_T^{\delta(R)} \nearrow \Omega$ *as* $T \to 0$ *up to a* $\mathbb{P}$*-zero set, and hence*

$$\mathbb{P}\left(\bigcup_{M\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \Omega_{1/N}^{\delta(M)} \cap \Omega_{H,M}^{\alpha} \right) = 1,$$

it follows that we have **finite speed of propagation of disturbances** (*"localization"*) *for* $(X_t)_{t\geq 0}$ $\mathbb{P}$*-a.s..*

The main results

As follows from the proof, $r(t, \cdot)$, $t \in [0, T]$, is an $\{\mathcal{F}_t\}$-adapted process.
The conditions $m \leq 5$ and $x \in L^\infty(\mathcal{O})$ might seem unnatural, but they are technical assumptions required by the work [Barbu/R., JDE 2011] on which the present proof essentially relies.

Sketch of Proof for Theorem II

For the proof we shall assume $\xi_0 = 0 \in \mathcal{O}$ and set $B_r = B_r(0)$. We consider the transformation from [Barbu/DaPrato/R., CMP 2009]

$$y(t) = e^{\mu(t)} X(t), \ t \geq 0,$$

where $\mu(t,\xi) = -\sum_{k=1}^{N} \mu_k e_k(\xi)\beta_k(t), \ t \geq 0, \ \xi \in \mathcal{O}$.

Then

$$\begin{aligned} &\frac{dy}{dt} - e^{\mu}\Delta(y^m e^{-m\mu}) + \frac{1}{2}\widetilde{\mu} y = 0, \ t > 0, \ \mathbb{P}\text{-a.s.}, \\ &y(0) = x, \\ &y^m \in H_0^1(\mathcal{O}), \ \forall t > 0, \ \mathbb{P}\text{-a.s.}, \end{aligned} \qquad \text{(TPME)}$$

where

$$\widetilde{\mu} := \sum_{k=1}^{N} \mu_k^2 e_k^2.$$

By Theorem I, we have $\mathbb{P}$-a.s.

$$y \geq 0, \ y^m(t) \in H_0^1(\mathcal{O}) \cap L^{\frac{m+1}{m}}(\mathcal{O}), \ \text{a.e. } t \geq 0.$$

Sketch of Proof for Theorem II

Furthermore, by [Barbu/R., JDE 2011]:

Key Lemma

Assume that $1 \le d \le 3$ *and* $m \in]1,5]$. *Then, if* $x \in L^\infty(\mathcal{O})$, *the solution* y *to* (TPME) *satisfies* $\mathbb{P}$*-a.s. for every* $T > 0$

$$y \in L^\infty((0,T) \times \mathcal{O}) \cap C([0,T];H),$$

$$y^m \in L^2(0,T;H_0^1(\mathcal{O})), \ \frac{dy}{dt} \in L^2(0,T;H).$$

Moreover, for every $T \in (0,1]$, $\alpha \in (0, \frac{1}{2})$, $R > 0$, *there exist constants* $c_1, c_2 > 0$ *depending on* α, R, $\mathcal{O}$, $|x|_\infty$, $\max_{1 \le k \le N}(|e_k|_\infty, |\nabla e_k|_\infty, |\Delta e_k|_\infty)$, *but not on* T *such that* $\mathbb{P}$*-a.s. on* $\Omega^\alpha_{H,R}$,

$$\|y\|_{L^\infty((0,T)\times\mathcal{O})} \le c_1 \exp\left[c_2 \max_{1\le k\le N} \sup_{t\in[0,T]} |\beta_k(t)|\right].$$

Sketch of Proof for Theorem II

Consider and the localizing function $\psi_\varepsilon(\xi) := \varrho_\xi(|\xi|)$, $\xi \in \mathcal{O}$, where ϱ_ε is smooth with $1_{[0,r+\varepsilon]} \leq \varrho_\varepsilon \leq 1_{[0,r+2\varepsilon]}$. Then multiply (TPME) by ψ_ε and integrate over $\mathcal{O}$ to obtain:

$$\frac{1}{m+1}\int_{\mathcal{O}}(y(t,\xi))^{m+1}\psi_\varepsilon(\xi)d\xi + \int_0^t ds\int_{\mathcal{O}}\nabla(ye^{-\mu})^m\cdot\nabla(e^\mu y^m\psi_\varepsilon)d\xi$$
$$+\frac{1}{2}\int_0^t ds\int_{\mathcal{O}}\widetilde{\mu}y^{m+1}\psi_\varepsilon d\xi = \frac{1}{m+1}\int_{\mathcal{O}}x^{m+1}\psi_\varepsilon d\xi.$$

On the other hand, we have

$$\int_{\mathcal{O}}\nabla(ye^{-\mu})^m\cdot\nabla(e^\mu y^m\psi_\varepsilon)d\xi = \int_{\mathcal{O}}|\nabla(ye^{-\mu})^m|^2\psi_\varepsilon e^{(m+1)\mu}d\xi$$
$$+(m+1)\frac{1}{2}\int_{\mathcal{O}}(\nabla(ye^{-\mu})^m\cdot\nabla\mu)e^\mu y^m\psi_\varepsilon d\xi$$
$$+\int_{\mathcal{O}}\left(\nabla(ye^{-\mu})^m\cdot\frac{\xi}{|\xi|}\right)(s,\xi)\rho'_\varepsilon(|\xi|)(e^\mu y^m)(s,\xi)d\xi,$$

Sketch of Proof for Theorem II

Then

$$
\begin{aligned}
&\frac{1}{m+1}\int_{B_{r+\varepsilon}} y^{m+1}(t,\xi)d\xi ds \\
&\quad + \int_0^t ds \int_{B_{r+2\varepsilon}} \psi_\varepsilon e^{(m+1)\mu} |\nabla(ye^{-\mu})^m|^2 d\xi\, ds \quad \underset{\varepsilon\to 0}{\longrightarrow} \quad \phi(r,t) \\
&\quad + \frac{1}{2}\int_0^t ds \int_{B_{r+2\varepsilon}} \psi_\varepsilon \widetilde{\mu} y^{m+1} d\xi\, ds \\
&\quad \le \frac{1}{m+1}\int_{B_{r+2\varepsilon}} \psi_\varepsilon x^{m+1} d\xi \\
&\quad -(m+1)\int_0^t \int_{B_{r+2\varepsilon}} (\nabla(ye^{-\mu})^m \cdot \nabla\mu)\psi_\varepsilon e^\mu y^m d\xi\, ds \\
&\quad -\int_0^t \int_{B_{r+2\varepsilon}\setminus B_{r+\varepsilon}} \left(\nabla(ye^{-\mu})^m \cdot \frac{\xi}{|\xi|}\right)(s,\xi)(e^\mu y^m)(s,\xi)\rho'_\varepsilon(|\xi|)d\xi\, ds.
\end{aligned}
$$

Sketch of Proof for Theorem II

Here define "**energy function**"

$$\phi(t,r) := \int_0^t \int_{B_r} |\nabla(ye^{-\mu})^m|^2 e^{(m+1)\mu} d\xi\, ds, \ \ t \in [0,T], \ \ r \geq 0.$$

In order to prove (L), our aim is to show that ϕ satisfies a differential inequality of the form

$$\frac{\partial \phi}{\partial r}(t,r) \geq Ct^{\theta-1}(\phi(t,r))^{\delta} \text{ on } \Omega^{\alpha}_{H,R} \cap \Omega^{\delta(R)}_{T} \text{ for } t \in [0,T], \ \ r \in [0,r_0],$$

where $0 < \theta < 1$ and $0 < \delta < 1$ and from which (L) will follow (by elementary arguments).

Sketch of Proof for Theorem II

On the other hand, we have as $\varepsilon \to 0$

$$\int_0^t \int_{B_{r+2\varepsilon}\setminus B_{r+\varepsilon}} \left| \left(\nabla (ye^{-\mu})^m \cdot \frac{\xi}{|\xi|} \right) e^{\mu} y^m \rho'_\varepsilon(|\cdot|) \right| d\xi \, ds$$

$$\leq \left(\int_0^t \int_{B_{r+2\varepsilon}\setminus B_{r+\varepsilon}} |\rho'_\varepsilon(|\cdot|)| \, |\nabla (ye^{-\mu})^m|^2 e^{(m+1)\mu} d\xi \, ds \right)^{\frac{1}{2}} \xrightarrow{\varrho'_\varepsilon \sim \frac{1}{\varepsilon}} \left(\frac{\partial \phi}{\partial r}(t,r) \right)^{\frac{1}{2}}$$

$$\times \left(\int_0^t \int_{B_{r+2\varepsilon}\setminus B_{r+\varepsilon}} e^{(1-m)\mu} y^{2m} |\rho'_\varepsilon(|\cdot|)| d\xi \, ds \right)^{\frac{1}{2}} \to \left(\int_0^t ds \int_{\Sigma_r} y^{2m} e^{(1-m)\mu} d\xi \right)^{\frac{1}{2}}.$$

Sketch of Proof for Theorem II

Setting

$$K(t,r) = \frac{1}{2}\int_0^t \int_{B_r} \widetilde{\mu} y^{m+1} ds\, d\xi$$
$$H(t,r) = \sup\left\{ \frac{1}{m+1}\int_{B_r} y^{m+1}(s,\xi)d\xi,\ 0 \le s \le t \right\},$$

and letting $\varepsilon \to 0$ we obtain for $r \in (0, r_0]$,

$$\begin{aligned} &H(t,r) + \phi(t,r) + K(t,r) \\ &\quad \le (m+1)\int_0^t \int_{B_r} |(\nabla(ye^{-\mu})^m \cdot \nabla\mu)e^{\mu} y^m| d\xi\, ds \\ &\quad + \left(\frac{\partial\phi}{\partial r}(t,r)\right)^{\frac{1}{2}} \left(\int_0^t \int_{\Sigma_r} y^{2m} e^{(1-m)\mu} d\xi\, ds\right)^{\frac{1}{2}}, \end{aligned}$$

since $x \equiv 0$ on B_r.

Sketch of Proof for Theorem II

By Cauchy–Schwarz and (P), we have (since $m = \frac{m+1}{2} + \frac{m-1}{2}$)

$$\begin{aligned}
&\int_0^t \int_{B_r} |(\nabla(ye^{-\mu})^m \cdot \nabla\mu)e^{\mu} y^m| d\xi\, ds \\
&\leq \|y^{m-1} e^{(1-m)\mu} |\nabla\mu|^2\|^{1/2}_{L^\infty((0,T)\times\mathcal{O})} \\
&\times \left(\int_0^t ds \int_{B_r} |\nabla(y^m e^{-m\mu})|^2 e^{(m+1)\mu} d\xi\right)^{\frac{1}{2}} \left(\int_0^t ds \int_{B_r} y^{m+1} d\xi\right)^{\frac{1}{2}} \\
&\leq C_T \||\nabla\mu|^2\|^{1/2}_{L^\infty((0,T)\times\mathcal{O})} (\phi(t,r))^{\frac{1}{2}} (K(t,r))^{\frac{1}{2}} \\
&\leq \frac{1}{2(m+1)} (\phi(t,r) + K(t,r)), \\
&\qquad\qquad \forall t \in (0,T],\ r \in (0,r_0], \quad \text{on } \Omega^{\alpha}_{H,R} \cap \Omega^{\delta(R)}_T,
\end{aligned}$$

by the definition of $\delta(R)$ and by the "Key Lemma".

Sketch of Proof for Theorem II

Hence

$$H(t,r)+\phi(t,r)+K(t,r)\leq 2\left(\frac{\partial\phi}{\partial r}(t,r)\right)^{\frac{1}{2}}\left(\int_0^t ds\int_{\Sigma_r} y^{2m}e^{(1-m)\mu}d\xi\right)^{\frac{1}{2}}$$
$$\forall t\in[0,T],\ r\in[0,r_0],\quad \text{on } \Omega^{\alpha}_{H,R}\cap\Omega^{\delta(R)}_T,$$

where the surface integral on the right-hand side can be estimated in terms of $t^{1-\theta}, \phi$ and H to give the desired

$$\frac{\partial\phi}{\partial r}(t,r)\geq ct^{\theta-1}(\phi(t,r))^{\delta} \text{ on } \Omega^{\alpha}_{H,R}\cap\Omega^{\delta(R)}_T \text{ for } t\in[0,T],\ r\in[0,r_0].$$

For this we modify a technique form [Diaz/Veron, TAMS 1985] based on the following interpolation-trace inequality

$$|z|_{L^2(\Sigma_r)}\leq C(|\nabla z|_{L^2(B_r)}+|z|_{L^{\sigma+1}(B_r)})^{\theta}|z|^{1-\theta}_{L^{\sigma+1}(B_r)},$$

for all $\sigma\in[0,1]$ and $\theta=(d(1-\sigma)+\sigma+1)/(d(1-\sigma)+2(\sigma+1))\in\left[\frac{1}{2},1\right)$. We apply this inequality for $z=(y^m e^{-\mu})^m$ and $\sigma=\frac{1}{m}$.