A SUPPORT THEOREM FOR STOCHASTIC WAVES IN DIMENSION THREE

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Introduction

Objective

To prove a characterization of the topological support of the law of the solution of a stochastic wave equation in spatial dimension $d = 3$.

Definition For a random vector $X \to \mathbb{M}$, the topological support is the smallest closed $F \subset \mathbb{M}$ such that $\left(P \circ X^{-1}\right)(F) > 0$.

- What type of solution? Random field solution
- What topology? Hölder
- What method? Approximations

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Main result

Approximation in probability and in Hölder norm of a stochastic wave equation by smoothing the driving noise.
(Wong–Zakai's type Theorem).

References

- For the method: Aida–Kusuoka–Stroock, 1993; Millet–S.-S., 1994; Bally–Millet–S.-S., 1995; Gyöngy–Nualart–S.-S, 1997; Millet–S.-S, 2000...
- For the background on the wave equation: Dalang 1999; Dalang–S.-S, 2009; Dalang–Quer-Sardanyons, 2011; ...

Plan of the work

- Vanishing initial conditions (joint work with F. Delgado)
- Non null initial conditions (work in progress with F. Delgado)

Why we draw such a distinction?

This question is related to

- stationarity of the solution,
- choice of the stochastic integral in the formulation of (1).

Discussion on The Model

Stochastic wave equation in spatial dimension $d = 3$

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(t,x) = \sigma\big(u(t,x)\big)\dot{M}(t,x) + b\big(u(t,x)\big), \\ u(0,x) = u_0(x), \quad \frac{\partial}{\partial t} u(0,x) = v_0(x), \end{cases}$$

$t \in [0, T]$, $x \in \mathbb{R}^3$.

Interpretation in mild form

$$\begin{aligned} u(t,x) &= [G(t) * v_0](x) + \frac{\partial}{\partial t}([G(t) * u_0](x)) \\ &\quad + \int_0^t \int_{\mathbb{R}^3} G(t-s, x-y)\sigma(u(s,y))M(ds,dy) \\ &\quad + \int_0^t [G(t-s,\cdot) * b(u(s,\cdot))](x)ds, \end{aligned} \tag{1}$$

$G(t) = \frac{1}{4\pi t}\sigma_t(dx)$.

The noise

$\{M(\varphi), \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^4)\}$ Gaussian process

- $E(M(\varphi)) = 0$,
- $E\left(M(\varphi)M(\psi)\right) = \int_0^t ds \int_{\mathbb{R}^3} \mu(d\xi)\mathcal{F}\varphi(s)\overline{\mathcal{F}\psi(s)}(\xi)$,

μ non-negative tempered symmetric measure on $\mathbb{R}^3$.

In non-rigorous terms

$$E(\dot{M}(t,x)\dot{M}(s,y)) = \delta(t-s)f(x-y),$$

$f = \mathcal{F}\mu$.

M as a cylindrical Wiener process

$\mathcal{H}$ is the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ of test functions with the semi-inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} \mu(d\xi) \mathcal{F}\varphi(\xi) \overline{\mathcal{F}\psi(\xi)}.$$

The process $B_t(\varphi) = M(1_{[0,t]}\varphi)$ is a cylindrical Wiener process: Gaussian, zero mean and

$$E(M_t(\varphi) M_s(\psi) = \min(s,t) \langle \varphi, \psi \rangle_{\mathcal{H}}.$$

In particular, for any CONS $(e_j)_{j\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^3)$,

$$(W_t^j = B_t(e_j), t \in [0, T])_{j\in\mathbb{N}}$$

defines a sequence of independent standard Brownian motions.

Dalang's integral as an i.d. Itô integral

Theorem (Dalang–Quer-Sardanyons, 2011)
Let $g \in \mathcal{P}_0$ (integrands admissible for the Dalang's integral). Then $g \in L^2(\Omega \times [0, T]; \mathcal{H})$ and

$$\int_0^t \int_{\mathbb{R}^3} g(s, y) M(ds, dy) = \sum_{j \in \mathbb{N}} \int_0^t \langle g(s, \cdot), e_j \rangle_{\mathcal{H}} W^j(ds).$$

Example
Let $\{Z(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\}$ be predictable, with spatially homogeneous covariance and

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^3} E(|Z(t, x)|^2) < \infty.$$

Then

$$\{g(t, x) := G(t, dx) Z(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\} \in \mathcal{P}_0$$

The stochastic wave equation

$$
\begin{aligned}
u(t,x) = {} & [G(t) * v_0](x) + \frac{\partial}{\partial t}\left([G(t) * u_0](x)\right) \\
& + \sum_{j \in \mathbb{N}} \int_0^t \langle G(t-s, x-\cdot)\sigma(u(s,\cdot)), e_j \rangle_{\mathcal{H}} W_j(ds) \\
& + \int_0^t G(t-s,\cdot) * b(u(s,\cdot))(x)\, ds, \qquad (2)
\end{aligned}
$$

$t \in [0, T]$, $x \in \mathbb{R}^3$.

We are interested in random field solutions $\{u(t,x), (t,x) \in [0, T] \times \mathbb{R}^3\}$.

Background: Dalang, EJP 1999

Hypotheses:

- u_0, v_0 vanish,
- $\sigma, b : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous,
- $\Gamma(dx) = |x|^{-\beta} dx$, $\beta \in]0, 2[$.

Theorem There exists a unique random field solution to (2). This is an adapted process $\{u(t,x), (t,x) \in [0, T] \times \mathbb{R}^3\}$ satisfying (2) for any $(t,x) \in [0, T] \times \mathbb{R}^3$.
The solution is L^2-continuous and bounded in L^p:

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} E(|u(t,x)|^p) < \infty.$$

Support Theorem

Sample path properties of the wave equation

Notation

- For $t_0 \in [0,T]$, $K \subset \mathbb{R}^3$ compact, $\rho \in]0,1[$,

$$\|g\|_{\rho,t_0,K} := \sup_{(t,x)\in[t_0,T]\times K} |g(t,x)| + \sup_{\substack{(t,x),(\bar t,\bar x)\in[t_0,T]\times K \\ t\neq \bar t, x\neq \bar x}} \frac{|g(t,x)-g(\bar t,\bar x)|}{(|t-\bar t|+|x-\bar x|)^\rho},$$

- $\mathcal{C}^\rho([t_0,T]\times K)$ is the space of real functions g such that $\|g\|_{\rho,t_0,K} < \infty$.

Theorem (Dalang–S.-S., 2009)

Almost surely, the sample paths of the random field solution of (2) *belong to the space* $\mathcal{C}^\rho([t_0,T]\times K)$ *with* $\rho \in \left]0, \frac{2-\beta}{2}\right[$.

Support theorem (null initial conditions)

For $t \in]0, T]$, set $\mathcal{H}_t := L^2([0,t];\mathcal{H})$. Let

$$\Phi^h(t,x) = \Big\langle G(t-\cdot, x-\cdot)\sigma(\Phi^h), h\Big\rangle_{\mathcal{H}_t} + \int_0^t ds[G(t-s,\cdot) * b(\Phi^h(s,\cdot))](x),$$

$h \in \mathcal{H}_T$,

Theorem (Delgado–S.-S., 2011)

Let $u = \{u(t,x), (t,x) \in [t_0, T] \times K\}$, $t_0 > 0$, be the random field solution to (2). Fix $\rho \in \left]0, \frac{2-\beta}{2}\right[$. Then the topological support of the law of u in the space $\mathcal{C}^\rho([t_0,T] \times K)$ is the closure in $\mathcal{C}^\rho([t_0,T] \times K)$ of the set of functions $\{\Phi^h, h \in \mathcal{H}_T\}$.

A method to prove the support theorem

Part I

Assume that there exist:

- $\xi_1 : \mathcal{H}_T \to \mathcal{C}^\rho([t_0, T] \times K)$,
- $w^n : \Omega \to \mathcal{H}_T$,

such that for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left\{\|u - \xi_1(w^n)\|_{\rho,t_0,K} > \epsilon\right\} = 0.$$

Then $\operatorname{supp}(\mathbb{P} \circ u^{-1}) \subset \overline{\xi_1\left(\mathcal{H}_T\right)}$.

Remarks

- This follows from Portmanteau's theorem.
- The closure refers to the Hölder norm $\|\cdot\|_{\rho,t_0,K}$.
- $\xi_1(w^n) := \Phi^{w^n}$.

Part II

Assume that:

- there exists a mapping $\xi_2 : \mathcal{H}_T \to \mathcal{C}^\rho([t_0, T] \times K)$,
- for any $h \in \mathcal{H}_T$, there exists a sequence $T_n^h : \Omega \to \Omega$ such that $\mathbb{P} \circ (T_n^h)^{-1} \ll \mathbb{P}$,
- the following convergence holds

$$\lim_{n \to \infty} \mathbb{P} \left\{ \|u(T_n^h) - \xi_2(h)\|_{\rho, t_0, K} > \epsilon \right\} = 0.$$

Then $\mathrm{supp}(\mathbb{P} \circ u^{-1}) \supset \overline{\xi_2 \left(\mathcal{H}_T\right)}$.

This follows from Girsanov's theorem.

Next: Choices for w^n, ξ_1, ξ_2, T_n^h.

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Choice for w^n

Let $\Delta_i = \left[\frac{iT}{2^n}, \frac{(i+1)T}{2^n}\right[$. For $1 \leq j \leq n$, let

$$\dot{W}_j^n(t) = \begin{cases} \sum_{i=0}^{2^n-1} 2^{n\theta_1} T^{-1} W_j(\Delta_i) 1_{\Delta_{i+1}}(t), & t \in [2^{-n}T, T], \\ \\ 0, & t \in [0, 2^{-n}T[, \end{cases}$$

$\theta_1 \in]0, \infty[$.

For $j > n$, put $\dot{W}_j^n = 0$. Set

$$w^n(t,x) = \sum_{j \in \mathbb{N}} \dot{W}_j^n(t) e_j(x).$$

Remark:

$M(ds) = \sum_{j \in \mathbb{N}} W_j(ds) \sim w^n(s)ds.$

Choice for ξ_1, ξ_2

$\xi_1, \xi_2 : L^2\left([0,T];\mathcal{H}\right) \to \mathcal{C}^\rho([t_0,T]\times K)$

$\xi_1(h) = \xi_2(h) = \Phi^h$.

Choice for T_n^h

$T_n^h(\omega) = \omega - w^n + h$.

For the rigorous setting: abstract Wiener space associated with $\{W^j, j \in \mathbb{N}\}$.

Approximation result

$$
\begin{aligned}
X(t,x) = &\int_0^t \int_{\mathbb{R}^3} G(t-s,x-y)(A+B)(X(s,y))M(ds,dy)\\
&+ \langle G(t-\cdot,x-*)D(X(\cdot,*)),h\rangle_{\mathcal{H}_t}\\
&+ \int_0^t \int_{\mathbb{R}^3} G(t-s,x-y)b(X(s,y))dsdy,
\end{aligned}
$$

$$
\begin{aligned}
X_n(t,x) = &\int_0^t \int_{\mathbb{R}^3} G(t-s,x-y)A(X_n(s,y))M(ds,dy)\\
&+ \langle G(t-\cdot,x-*)B(X_n(\cdot,*)),w^n\rangle_{\mathcal{H}_t}\\
&+ \langle G(t-\cdot,x-*)D(X_n(\cdot,*)),h\rangle_{\mathcal{H}_t}\\
&+ \int_0^t \int_{\mathbb{R}^3} G(t-s,x-y)b(X_n(s,y))dsdy,
\end{aligned}
$$

With an appropriate choice of the coefficients A, B, D, b:

1. $A = D = 0$, $B := \sigma$;
2. $A = -B = D = \sigma$,

the two convergences follow from the next

Theorem

The coefficients are Lipschitz. Suppose also that

$$\theta_1 \in \left[0, \frac{6-\beta}{4}\right[.$$

Fix $t_0 > 0$ and a compact set $K \subset \mathbb{R}^3$. Then for any $\rho \in \left]0, \frac{2-\beta}{2}\right[$, $\lambda > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\|X_n - X\|_{\rho,t_0,K} > \lambda\right) = 0.$$

Local $L^p(\Omega)$ convergence

Prove that for a sequence $L_n(T) \uparrow \Omega$,

$$\lim_{n\to\infty} \mathbb{E}\left(\|X_n - X\|^p_{\rho,t_0,K} 1_{L_n(T)}\right) = 0.$$

(Similar idea as in Millet– S.-S (2000) for 2-d wave equation).

Choice of the localization

$$L_n(t) = \left\{ \sup_{1\le j\le n} \sup_{0\le i\le [2^n t T^{-1} - 1]^+} 2^{n\theta_1} |W_j(\Delta_i)| \le \alpha 2^{n\theta_2} n^{\frac{1}{2}} \right\}$$

Property

$$\|w^n 1_{L_n(t')} 1_{[t,t']}\|_{\mathcal{H}_T} \le C n 2^{n\theta_2} |t' - t|^{\frac{1}{2}}.$$

Lemma *For* $\alpha > (2\ln 2)^{\frac{1}{2}}$ *and* $\theta_2 + \theta_1 + \frac{1}{2} \ge 0$,

$$\lim_{n\to\infty} \mathbb{P}(L_n(T)^c) = 0.$$

Ingredients

For any $\theta_1 \in]0,\infty[$, $\theta_2 \in \left]0, \frac{4-\beta}{4}\right[$,

- Local $L^p(\Omega)$ estimates of increments

$$\sup_{n\geq 1} \left\| \left[X_n(t,x) - X_n(\bar{t},\bar{x})\right] 1_{L_n(\bar{t})} \right\|_p \leq C\Big(\left|\bar{t}-t\right| + \left|\bar{x}-x\right|\Big)^{\rho},$$

$\rho \in \left]0, \frac{2-\beta}{2}\right[$.

- Pointwise convergence

$$\lim_{n\to\infty} \|(X_n(t,x) - X(t,x))1_{L_n(t)}\|_p = 0, \ \ p \in [1,\infty).$$

To obtain the convergence in probability, $\theta_2 - \theta_1 + \frac{1}{2} \geq 0$, thus

$$\theta_1 \in \left]0, \frac{6-\beta}{4}\right[.$$

A few technical details

Increments in space

Notation

$$\varphi_{n,p}(t,x,\bar{x}) = \mathbb{E}\left(\left|X_n(t,x) - X_n(t,\bar{x})\right|^p 1_{L_n(t)}\right),$$

$t \in [t_0, T], x, \bar{x} \in K, p \in [1, \infty[$.

Proposition (a simplified version)

$$\varphi_{n,p}(t,x,\bar{x}) \le C \left[f_n + |x-\bar{x}|^{\frac{\alpha_2 p}{2}} + \int_0^t ds (\varphi_{n,p}(s,x,\bar{x})) \right.$$
$$\left. + |x-\bar{x}|^{\alpha_1 \frac{p}{2}} \int_0^t ds \left[\varphi_{n,p}(s,x,\bar{x})\right]^{1/2} \right],$$

with $\lim_{n\to\infty} f_n = 0$, $\alpha_1 \in [0, (2-\beta)\wedge 1)[$, $\alpha_2 \in]0, (2-\beta)[$.

Lemma (Gronwall's type) *u, b and k are nonnegative continuous functions in $J = [\alpha, \beta]$; $\bar{p} \geq 0$, $\bar{p} \neq 1$, $a > 0$. Suppose that*

$$u(t) \leq a + \int_\alpha^t b(s)u(s)ds + \int_\alpha^t k(s)u^{\bar{p}}(s)ds, \qquad t \in J.$$

Then

$$u(t) \leq \exp\left(\int_\alpha^\beta b(s)ds\right) \left[a^{\bar{q}} + \bar{q}\int_\alpha^\beta k(s)\exp\left(-\bar{q}\int_\alpha^s b(\tau)d\tau\right)ds\right]^{\frac{1}{\bar{q}}},$$

for $t \in [\alpha, \beta_1)$, where $\bar{q} = 1 - \bar{p}$ and β_1 is choosen so that the expression beween $[\ldots]$ is positive in the subinterval $[\alpha, \beta_1)$ ($\beta_1 = \beta$ if $\bar{q} > 0$).

D. Bainov, P. Simenov: Integral Inequalities and Applications.

Where $(\cdot)^{\frac{1}{2}}$ does come from?

$$\mathbb{E}\left(|X_n(t,x) - X_n(t,\bar{x})|^p 1_{L_n(t)}\right) \leq C \sum_{i=1}^{4} R_n^i(t,x,\bar{x}),$$

$$R_n^1(t,x,\bar{x}) =$$

$$\mathbb{E}\left(\left|\int_0^t \int_{\mathbb{R}^3} [G(t-s,x-y) - G(t-s,\bar{x}-y)] Z_n(s,y) M(ds,dy)\right|^p\right),$$

$$Z_n(s,y) = A(X_n(s,y)) 1_{L_n(s)}.$$

Apply Burkholder's inequality and Plancherel's identity:

$$R_n^1(t,x,\bar{x}) = \mathbb{E}\left(\left|\int_0^t \int_{\mathbb{R}^3} \left[G(t-s,x-y) - G(t-s,\bar{x}-y)\right] \times Z_n(s,y)M(ds,dy)\right|^p\right)$$

$$\le C\mathbb{E}\left(\left|\int_0^t ds \left\| \left[G(t-s,x-*) - G(t-s,\bar{x}-*)\right] Z_n(s,*)\right\|_{\mathcal{H}}^2\right|\right)^{p/2}$$

$$\overset{(*)}{=} C\mathbb{E}\left(\int_0^t ds \int_{\mathbb{R}^3\times\mathbb{R}^3} [G(t-s,x-du) - G(t-s,\bar{x}-du)] f(u-v) \times [G(t-s,x-dv)) - G(t-s,\bar{x}-dv)] Z_n(s,u) Z_n(s,v)\right)^{p/2}$$

$$= \int_0^t ds \int_{\mathbb{R}^3\times\mathbb{R}^3} [f\Delta Z_n \Delta Z_n + Z_n \Delta Z_n \Delta f + Z_n Z_n \Delta^2 f],$$

$(*)$ $f(x) = |x|^{-\beta}$, $\beta \in]0,2[$.

$$
\begin{aligned}
C\varphi_{n,p}(t,x,\bar{x}) \leq f_n & \quad \text{(correction stochastic integrals)} \\
+ |x-\bar{x}|^{\frac{\alpha_2 p}{2}} & \quad (Z_n Z_n \Delta^2 f) \\
+ \int_0^t ds(\varphi_{n,p}(s,x,\bar{x})) & \quad (f \Delta Z_n \Delta Z_n) \\
+ |x-\bar{x}|^{\alpha_1 \frac{p}{2}} \int_0^t ds \Big[\varphi_{n,p}(s,x,\bar{x})\Big]^{1/2}. & \quad (Z_n \Delta Z_n \Delta f)
\end{aligned}
$$

Stationarity

Comparison with $d = 2$

- Different approach to $G(\bar{t} - s, x - dy) - G(t - s, \bar{x} - dy)$ (method from Dalang–S.-S., 2009).
- The approximation of

$$\sum_{j \geq 1} \int \cdots W_j(ds) \quad \text{by} \quad \sum_{j \geq 1} \int \cdots W_j^n(s) ds$$

  is much more difficult.
  - smoother approximations of the noise (parameter θ_1),
  - combination of the two processes: approximation and localization.

References

[1] V. Bally, A. Millet, M. Sanz-Solé, *Approximation and support theorem in Hölder norm for parabolic stochastic partial differential equations*. Ann.Probab., **23**, 178-222, (1995).

[2] R.C. Dalang, M. Sanz-Solé, *Hölder-Sobolev regularity of the solution to the stochastic wave equation in dimension three*. Memoirs of the American Mathematical Society. No. **931**, Vol. 199, (2009).

[3] R.C. Dalang, L. Quer-Sardanyons. *Stochastic integrals for spde's: A comparison*. Expositiones Mathematicae **29**, 67-109, (2011).

[4] A. Millet, M. Sanz-Solé, *The support of an hyperbolic stochastic partial differential equation*. Probab. Theory Related Fields, **98**, 361-387, (1994).

[5] A. Millet, M. Sanz-Solé, *Approximation and support theorem for a wave equation in two space dimensions*. Bernoulli **6**, 5, 887–915, (2000).

Many Thanks!