Random Periodic Solutions of SDEs and SPDEs

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Talk at EPSRC Symposium Workshop - Stochastic Analysis and Stochastic PDEs
University of Warwick

17 April 2012

Idea of regarding SDEs and SPDEs as random dynamical systems

Consider a stochastic differential equation on the state space H

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t \\ X_0 &= x \end{aligned} \quad (1)$$

Now set

$$\Phi(t, \omega)x = X_t^x(\omega), \omega \in \Omega, x \in H,$$

and $\theta : [0, T] \times \Omega \to \Omega$ by

$$(\theta_t \omega)(s) = W(t + s) - W(t).$$

We can also define $X_{t,s}^x$ the solution of (1) with $X_{s,s}^x = x$.

Then we can see that for any $t, s \geq 0$, then for all ω in a full measure set $\Omega_{t,s,x} \subset \Omega$,

$$X^x_{t+s,s}(\omega) = X^x_{t,0}(\theta_s \omega)$$

and

$$X^x_{t+s,0}(\omega) = X^{X^x_{s,0}(\omega)}_{s+t,s}(\omega) = X^{X^x_{s,0}(\omega)}_{t,0}(\theta_s \omega).$$

This suggests that for any t, s, x

$$\Phi(t+s,\omega)x = \Phi(s,\theta_t\omega) \circ \Phi(t,\omega)x, a.s.$$

The exceptional set of measure 0 may depend on t, s, x.

There is a modification of Φ can be chosen so that almost surely we have $x \to \Phi(s, \omega, x)$ is a diffeomorphism and

$$\Phi(t+s, \omega, x) = \Phi(t, \theta(s)\omega) \circ \Phi(s, \omega)x \;\; for\ all\ s, t \geq 0, \;\; x \in R^d.$$

There is a gap which needs to be filled by perfection. The key is to establish the continuity of in x through Kolmogorov's continuity theorem in the finite dimensional case.
SDEs: Elworthy 1978; Carverhill and Elworthy 1983; Kunita 1981, 1990; Arnold 1998, $\cdots$
BAD news is that SPDEs is an infinite dimensional space and Kolmogorov continuity theorem does not work.

Mohammed, Zhang and Zhao, Memoirs of AMS 2008,
Garrido-Atienza, Lu and Schamulfuss, JDE (2010)

established RDS for a large class of SPDEs.

Driving flow

- Model for noise: Driving system

$(\Omega, \mathcal{F}, P)$ – a probability space.
$\{\theta_t\}_{t\in R}$ be a measurable flow on Ω:

$$\theta : R \times \Omega \to \Omega$$

such that
(i) θ is measurable,
(ii) $\theta_0 = id_\Omega$, $\theta_s \circ \theta_t = \theta_{s+t}$ for all $s, t \in R$.

Driving flow: P-preserving measurable flow.

Definition of RDS

Definition 1

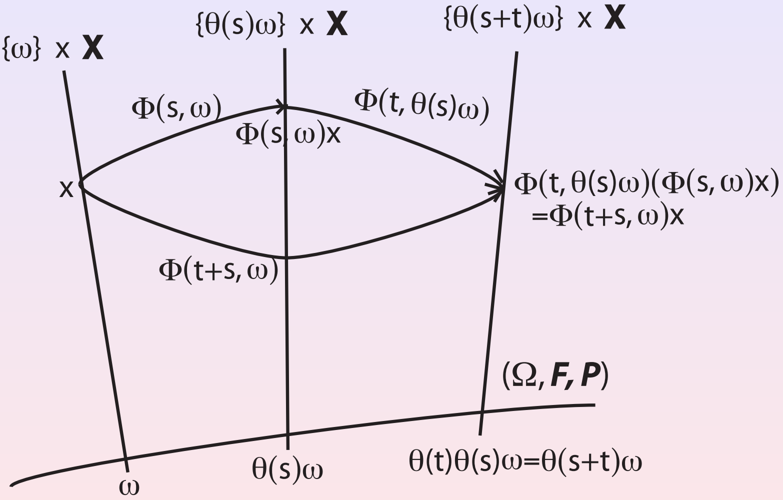
A measurable random dynamical system on the measurable space $(X, \mathcal{B}(X))$ *over a driving flow* $(\Omega, \mathcal{F}, P, \theta_t)$ *is a mapping:*

$$\Phi : R \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \Phi(t, \omega, x),$$

with the following properties:
(i) Measurability: Φ *is* $(\mathcal{B}(R) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(X))$*-measurable.*
(ii) Cocycle property:

$$\Phi(0, \omega) = id_X \ \textit{for all } \omega \in \Omega,$$

$$\Phi(t + s, \omega) = \Phi(t, \theta_s \omega) \circ \Phi(s, \omega) \ \textit{for all } s, t \in R, \ \omega \in \Omega \, .$$


{ω} x X
{θ(s)ω} x X
{θ(s+t)ω} x X
Φ(s, ω)
Φ(t, θ(s)ω)
Φ(s, ω)x
x
Φ(t, θ(s)ω)(Φ(s, ω)x)
=Φ(t+s, ω)x
Φ(t+s, ω)
(Ω, F, P)
ω
θ(s)ω
θ(t)θ(s)ω=θ(s+t)ω

RDS versus DS

$$
\begin{array}{rcl}
\text{SDE, SPDE} & \Longleftrightarrow & \text{ODE, PDE} \\
\Updownarrow & & \Updownarrow \\
\text{stochastic dynamical systems} & \Longleftrightarrow & \text{dynamical systems} \\
\downarrow \; long & time & behaviour \; \downarrow \\
\text{equilibrium(random)} & \Longleftrightarrow & \text{equilibrium} \\
\downarrow & & \downarrow \\
\text{random } stationary \;\; periodic & \Longleftrightarrow & fixed \quad periodic \\
solution & & point \quad solution \\
 & and & \\
\text{invariant manifolds(random)} & \Longleftrightarrow & \text{invariant manifolds} \\
in\ a\ neighbourhood & of & an\ equilibrium
\end{array}
$$

A *stationary solution* is a $\mathcal{F}$-measurable random variable $Y^* : \Omega \to S$ such that

$$\Phi(t, \omega, Y^*(\omega)) = Y^*(\theta_t \omega), \quad t \in T \ a.s.$$

References of existence of stationary solution of sdes and spdes: Sinai (1991, 1996), Schmalfuss (2001), E, Khanin, Mazel and Sinai (AM 2000), Duan, Lu and Schmalfuss (AP 2003), Mattingly (CMP, 1999), Caraballo, Kloeden and Schmalfuss (2004), Zhang and Zhao (JFA 2007, JDE 2010, Preprint 2011), Feng, Zhao and Zhou (JDE 2011), Feng, Zhao (JFA 2012).

Random periodic solution–motivating problem

Consider

$$dX(t) \quad = \quad AX(t)dt + b(X_t)dt + \sigma(X_t)dW_t \tag{2}$$

Assume the following deterministic differential equation (when noise turns off):

$$dX(t) \quad = \quad AX(t)dt + b(X_t)dt$$

has a periodic solution $Z(t)$ with period τ, how about (2)? Denote the deterministic dynamical system by $\Phi_t : X \to X$ over time $t \in T$, a periodic solution is a periodic function $Z : I \to X$ with period $\tau \neq 0$ such that

$$\Phi_\tau(Z(t)) = Z(t+\tau) = Z(t) \text{ for all } t \in T. \tag{3}$$

Computable example

If one thinks periodic solution of the SDEs as in the deterministic case, the answer is NO. But this would not be the right notion.

To illustrate the concept, as a simple example, we consider the random dynamical system generated by a perturbation to the following deterministic ordinary differential equation in R^2:

$$\frac{dx(t)}{dt} = x(t) - y(t) - x(t)(x^2(t) + y^2(t)),$$
$$\frac{dy(t)}{dt} = x(t) + y(t) - y(t)(x^2(t) + y^2(t)).$$

It is well-known that above equation has a limit cycle

$$x^2(t) + y^2(t) = 1.$$

Consider a random perturbation

$$dx = (x - y - x(x^2 + y^2))dt + x \circ dW(t),$$
$$dy = (x + y - y(x^2 + y^2))dt + y \circ dW(t).$$

Here $W(t)$ is a one-dimensional Brownian motion. Set

$$\rho^*(\omega) = (2\int_{-\infty}^{0} e^{2s+2W_s(\omega)}ds)^{-\frac{1}{2}}$$

and

$$\phi^\omega(t) = (\rho^*(\omega)\cos(2\pi\alpha + t), \rho^*(\omega)\sin(2\pi\alpha + t)).$$

One can check that

$$\phi^\omega(2\pi + t) = \phi^\omega(t),$$

and

$$\tilde{\Phi}(t,\omega)\phi^\omega(0) = \phi^{\theta_t\omega}(t).$$

From this we can tell that the random dynamical system generated by the stochastic differential equation (4) has a periodic invariant solution. Moreover if $x^2(0) + y^2(0) \neq 0$, then

$$x^2(t, \theta(-t, \omega)) + y^2(t, \theta(-t, \omega)) \to \rho^*(\omega)^2$$

as $t \to \infty$.

Definition 2

(Zhao and Zheng, JDE (2009)) An invariant random periodic solution is an $\mathcal{F}$-measurable periodic function $\phi : \Omega \times I \to X$ of period T such that

$$\phi^\omega(t + T) = \phi^\omega(t) \text{ and } \Phi_t^\omega(\phi^\omega(t_0)) = \phi^{\theta_t \omega}(t + t_0) \text{ for all } t, t_0 \in I. \quad (4)$$

Interactions: periodicity of the deterministic system (rotating due to nonlinearity) and noise (spreading).

Let

$$X(t) = Z(t) + u(t).$$

This can then be transformed to study the τ-periodic solutions of τ-periodic stochastic differential equations.

$$du(t) \quad = \quad Au(t)\,dt + F(t, u(t))\,dt + B_0(t, u(t))dW(t). \tag{5}$$

Work of Feng and Z.

We will study the τ-periodic solutions of τ-periodic stochastic differential equations on R^d:

$$\begin{aligned} du(t) &= Au(t)\,dt + F(t,u(t))\,dt + B_0(t)dW(t), \quad t \geq s, \\ u(s) &= x \in R^d. \end{aligned} \tag{6}$$

Assume F and B_0 satisfy:
Condition (P) *There exists a constant $\tau > 0$ such that for any $t \in R$, $u \in R^d$*

$$F(t,u) = F(t+\tau,u),\ \ B_0(t) = B(t+\tau).$$

Denote $\Delta := \{(t,s) \in R^2, s \leq t\}$. This equation generates a semi-flow $u : \Delta \times R^d \times \Omega \to R^d$ when the solution exists uniquely. First, we give the definition of the random periodic solution

Definition 3

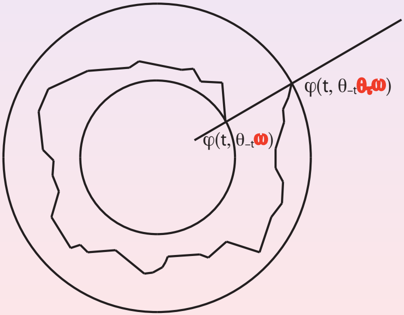
A random periodic solution of period τ of a semi-flow $u : \Delta \times R^d \times \Omega \to R^d$ is a $\mathcal{B}(R) \otimes \mathcal{F}$- measurable map $\varphi : (-\infty, \infty) \times \Omega \to R^d$ such that

$$u(t+\tau, t, \varphi(t,\omega), \omega) = \varphi(t+\tau, \omega) = \varphi(t, \theta_\tau \omega), \tag{7}$$

for any $t \in R$ and $\omega \in \Omega$.

Remark 1

The graph of $\varphi(\cdot,\theta_{-\cdot}\omega)$ *is closed. If the semiflow starts at a point on the graph* $\varphi(\cdot,\theta_{-\cdot}\omega)$*, after period time* τ*, it will come back to the graph* $\varphi(\cdot,\theta_{-\cdot}\theta_\tau\omega)$.



Remark 2

If τ can be any positive number, such φ is actually the stationary solution.
(Take $t = 0$ and define $Y^*(\omega) := \varphi(0,\omega)$ and $\Phi(\tau, x, \omega) = u(\tau, 0, x, \omega)$, we will get

$$\Phi(\tau, Y^*(\omega), \omega) = Y^*(\theta_\tau \omega), \ for\ any\ \tau \geq 0.)$$

Remark 3

There are only a few light touch to random periodicity in literature in very special cases:
Chojnowska-Michalik, Periodic distributions for linear equations with general additive noise, 1990 Hilbert space 1989
Klunger, Periodicity and Sharkovsky's theorem for random dynamical systems, 2001.

Some Assumptions

• A be an $d \times d$ matrix. Suppose that $T_t = e^{-At}$ is a hyperbolic linear flow. So R^d has a direct sum decomposition:

$$R^d = E^s \oplus E^u,$$

where

$$E^s = span\{v : v \text{ is an eigenvector for an eigenvalue } \lambda \text{ with } Re(\lambda) < 0\},$$

$$E^u = span\{v : v \text{ is an eigenvector for an eigenvalue } \lambda \text{ with } Re(\lambda) > 0\}.$$

Denote μ_m the real part of an eigenvalue of A with the largest negative real part, and μ_{m+1} with the smallest positive real part.

• Let $W(t)$, $t \in R$ be an M-dimensional Brownian motion and the filtered Wiener space is $(\Omega, \mathcal{F}, (\mathcal{F}^t)_{t\in R}, P)$. Here $\mathcal{F}_s^t := \sigma(W_u - W_v, s \leq v \leq u \leq t)$ and $\mathcal{F}^t := \vee_{s\leq t}\mathcal{F}_s^t$.

• Suppose $B_0(s)$ is an $d \times M$ matrix and is globally bounded $\sup_{-\infty<s<\infty} \|B_0(s)\| < \infty$.

First note the solution of the initial value problem (6) is given by the following variation of constant formula:

$$\begin{aligned} & u(t,s,x,\omega) \\ = \quad & T_{t-s}x + \int_s^t T_{t-r}F(r,u(r,s,x,\omega))dr + \int_s^t T_{t-r}B_0(r)dW(r). \quad (8) \end{aligned}$$

Coupled forward-backward infinite horizon stochastic integral equation

We define the projections onto each subspace by

$$P^- : R^d \to E^s,\ P^+ : R^d \to E^u.$$

$Y : (-\infty, \infty) \times \Omega \to R^d$ is $\mathcal{B}(R) \otimes \mathcal{F}$-measurable map satisfying:

$$\begin{aligned} & Y(t,\omega) \\ = \;& \int_{-\infty}^{t} T_{t-s}P^- F(s, Y(s,\omega))ds - \int_{t}^{\infty} T_{t-s}P^+ F(s, Y(s,\omega))ds \\ + \;& (\omega)\int_{-\infty}^{t} T_{t-s}P^- B_0(s)\, dW(s) - (\omega)\int_{t}^{\infty} T_{t-s}P^+ B_0(s)\, dW(s) \quad (9) \end{aligned}$$

for all $\omega \in \Omega$, $t \in (-\infty, \infty)$.

Equivalence of the random periodic solution and solution of infinite horizon stochastic integral equations

Theorem 1

The coupled forward-backward infinite horizon stochastic integral equation (13) has one solution if and only if

$$u(t+\tau, t, Y(t,\omega), \omega) = Y(t+\tau, \omega) = Y(t, \theta_\tau \omega) \text{ for any } t \in R \quad a.s.$$

Ref: Feng, Zhao and Zhou, JDE 2011.
Alternative analytic method studying periodic solutions from Poincaré's geometric method.

Existence Theorem

Theorem 2

Assume above conditions on A *and* B_0. *Let* $F : (-\infty, \infty) \times R^d \to R^d$ *be a continuous map, globally bounded and the Jacobian* $\nabla F(t, \cdot)$ *be globally bounded, and* F *and* B_0 *also satisfy Condition (P) and there exists a constant* $L_1 > 0$ *such that* $||B_0(s_1) - B_0(s_2)|| \leq L_1|s_1 - s_2|^{\frac{1}{2}}$. *Then there exists at least one* $\mathcal{B}(R) \otimes \mathcal{F}$*-measurable map* $Y : (-\infty, +\infty) \times \Omega \to R^d$ *satisfying the integral equation (13) and* $Y(t + \tau, \omega) = Y(t, \theta_\tau \omega)$ *for any* $t \in R$ *and* $\omega \in \Omega$.

Consider SPDE of parabolic type on a bounded domain D on R^d with a smooth boundary:

$$\begin{aligned} du(t,x) &= \mathcal{L}u(t,x)\,dt + F(t,u(t,x))\,dt + \sum_{k=1}^{\infty} \sigma_k(t)\phi_k(x)dW^k(t), \quad t \geq s, \\ u(s) &= \psi \in L^2(D), \\ u(t)|_{\partial D} &= 0. \end{aligned} \tag{10}$$

Here $\mathcal{L}$ is the second order differential operator on D,

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i} + c(x).$$

Assume

Condition (L). The coefficients a_{ij}, c are smooth functions on $\overline{D}$, $a_{ij} = a_{ji}$, and there exists a constant $\gamma > 0$ such that $\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \geq \gamma|\xi|^2$ for any $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in R^d$.

Here $\phi_k,\ k \geq 1$ is a complete orthonormal system of eigenfunction of $\mathcal{L}$ with corresponding eigenvalues $\mu_k,\ k \geq 1$ and from the uniformly elliptic condition, we have

$$\|\nabla\phi_k\|_{L^2(D)} \leq C\sqrt{|\mu_k|};$$

W^k are mutually independent one-dimensional standard Brownian motions and

$$\sum_{k=1}^{\infty} \sigma_k^2(t) < \infty. \tag{11}$$

Assume F and σ_k satisfy:

Condition (P) *There exists a constant* $\tau > 0$ *such that for any* $t \in R$, $u \in R^d$

$$F(t,u) = F(t+\tau,u),\ \sigma_k(t) = \sigma_k(t+\tau).$$

Denote $T_t = e^{\mathcal{L}t}$ which is a hyperbolic linear flow induced by $\mathcal{L}$ and $L^2(D)$ has a direct sum decomposition:

$$L^2(D) = E^s \oplus E^u,$$

where

$$E^s = span\{v : v \text{ is a generalized eigenvector for an eigenvalue } \lambda \text{ with } \lambda < 0\},$$

$$E^u = span\{v : v \text{ is a generalized eigenvector for an eigenvalue } \lambda \text{ with } \lambda > 0\}.$$

We also define the projections onto each subspace by

$$P^+ : L^2(D) \to E^u,\ P^- : L^2(D) \to E^s.$$

The solution of the initial value problem (6) is given by the following variation of constant formula:

$$
\begin{aligned}
& u(t,s,\psi(x),\omega) \\
&= \int_D K(t-s,x,y)\psi(y)dy + \int_s^t \int_D K(t-r,x,y)F(s,u(r,s,\psi(y),\omega))dydr \\
&+ \int_s^t \int_D K(t-r,x,y)\sum_{k=1}^{\infty}\sigma_k(r)\phi_k(y)dydW^k(r), \qquad (12)
\end{aligned}
$$

where $K(t,x,y)$ is the heat kernel of the operator $\mathcal{L}$. Note

$$K(t,x,y) = \sum_{i=1}^{\infty} e^{\mu_i t}\phi_i(x)\phi_i(y).$$

Note

$$T_t\psi(x) = \int_D K(t,x,y)\psi(y)dy.$$

Consider the coupled infinite horizon stochastic integral equation

$$\begin{aligned} & Y(t,\omega)(x) \\ = & \int_{-\infty}^{t} T_{t-s}P^{-}F(s,Y(s,\omega))(x)ds - \int_{t}^{\infty} T_{t-s}P^{+}F(s,Y(s,\omega))(x)ds \\ & + \int_{-\infty}^{t} T_{t-s}P^{-}(\sum_{k=1}^{\infty}\sigma_k(s)\phi_k)(x)\,dW^k(s) \\ & - \int_{t}^{\infty} T_{t-s}P^{+}(\sum_{k=1}^{\infty}\sigma_k(s)\phi_k)(x)\,dW^k(s) \end{aligned} \tag{13}$$

Denote by Y_1 the sum of the last two terms.

Main tools solving infinite horizon stochastic integral equations

Denote

$$\begin{aligned} & C^0_\tau((-\infty,+\infty),L^2(\Omega\times D)) \\ := \quad & \{f\in C^0((-\infty,+\infty),L^2(\Omega\times D)) : f(t+\tau,\omega)=f(t,\theta_\tau\omega)\} \end{aligned}$$

with the norm $\|f\|^2 = \sup_{t\in(-\infty,+\infty)} \int_D E|f(t,\omega,x)|^2 dx < \infty$. First note $Y_1 \in C^0_\tau((-\infty,+\infty),L^2(\Omega\times D))$. So we only need to solve the equation

$$\begin{aligned} Z(t,\omega) \quad = \quad & \int_{-\infty}^{t} T_{t-s}P^- F(s,Z(s,\omega)+Y_1(s,\omega))ds \\ & -\int_t^{\infty} T_{t-s}P^+ F(s,Z(s,\omega)+Y_1(s,\omega))ds. \qquad (14) \end{aligned}$$

in $C^0_\tau((-\infty,+\infty),L^2(\Omega\times D))$.

Set for $z \in C^0_\tau((-\infty, +\infty), L^2(\Omega \times D))$,

$$\begin{aligned} \mathcal{M}(z)(t,\omega) &= \int_{-\infty}^{t} T_{t-s}P^{-}F(s, z(s,\omega) + Y_1(s,\omega))ds \\ &- \int_{t}^{\infty} T_{t-s}P^{+}F(s, z(s,\omega) + Y_1(s,\omega))ds, \end{aligned} \quad (15)$$

And we need to find a fixed point.

Generalized Schauder's fixed point theorem

Theorem 3

Let H be a Banach space, S be a convex subset of H. Assume a map $T : H \to H$ is continuous and $T(S) \subset S$ is relatively compact in H. Then T has a fixed point in H.

Relative compactness of Wiener-Sobolev space

Theorem 4

Let D be a bounded domain in R^d. Consider a sequence $(v_n)_{n\in N}$ of $C^0([0,T],L^2(\Omega\times D))$. Suppose that:

(1) $\sup_{n\in N}\sup_{t\in[0,T]} E\|v_n(t,\cdot)\|^2_{H^1(D)} < \infty$.

(2) $\sup_{n\in N}\sup_{t\in[0,T]} \int_D \|v_n(t,x,\cdot)\|^2_{1,2}dx < \infty$.

(3) There exists a constant $C>0$ such that for any $t_1,t_2\in[0,T]$

$$\sup_n \int_D E|v_n(t_1,x)-v_n(t_2,x)|^2dx < C|t_1-t_2|.$$

(4) (4i) There exists a constant C such that for any $0 < \alpha < \beta < T$, and $h \in R$ with $|h| < \min(\alpha, T - \beta)$ and any $t_1, t_2 \in [0, T]$,

$$\sup_n \int_D \int_\alpha^\beta E|\mathcal{D}_{\theta+h} v_n(t_1, x) - \mathcal{D}_\theta v_n(t_2, x)|^2 d\theta dx < C(|h| + |t_1 - t_2|).$$

(4ii) For any $\epsilon > 0$, there exist $0 < \alpha < \beta < T$ such that

$$\sup_n \sup_{t \in [0,T]} \int_D \int_{[0,T] \setminus (\alpha, \beta)} E|\mathcal{D}_\theta v_n(t, x)|^2 d\theta dx < \epsilon.$$

Then $\{v_n, n \in N\}$ is relatively compact in $C^0([0, T], L^2(\Omega \times D))$.

- $L^2(\Omega)$: Da Prato, Malliavin and Nualart CRAS 1992, Peszant BPASM, 1993, $L^2(\Omega)$
- $L^2([0, T], L^2(\Omega \times D))$: Bally and Saussereau JFA 2004
- $C^0([0, T], L^2(\Omega \times D))$: Feng and Zhao JFA (2012)

Thank you for your attention!