Biased random walks on random paths and critical random trees

THE GEOMETRY OF DISCRETE RANDOM STRUCTURES
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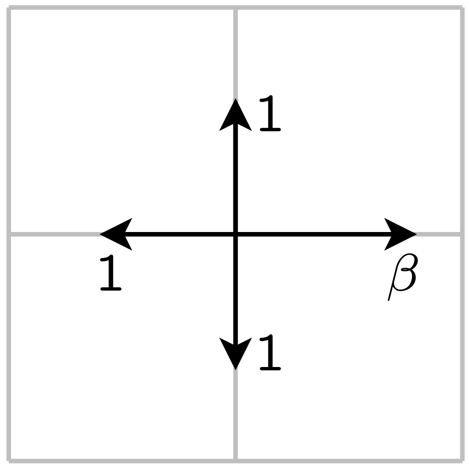
BIASED RANDOM WALKS

Consider the β-biased random walk $(X_n)_{n\geq 0}$ on the integer lattice $\mathbb{Z}^d$ (we will always assume $\beta \geq 1$):

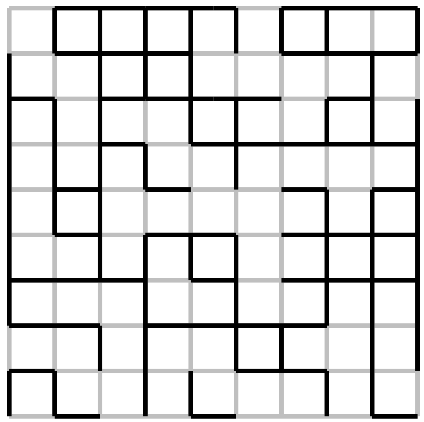


That is, the random walk that is β times more likely to jump in first-coordinate direction than in any other direction. We have

$$X_n = \frac{(\beta-1)n\mathbf{e}_1}{\beta+2d-1} + c_{\beta,d}N(0,I)n^{1/2} + o(n^{1/2}).$$

SUPERCRITICAL PERCOLATION

Bond percolation on integer lattice $\mathbb{Z}^d$ ($d \geq 2$), parameter $p > p_c$: e.g. $p = 0.53$,

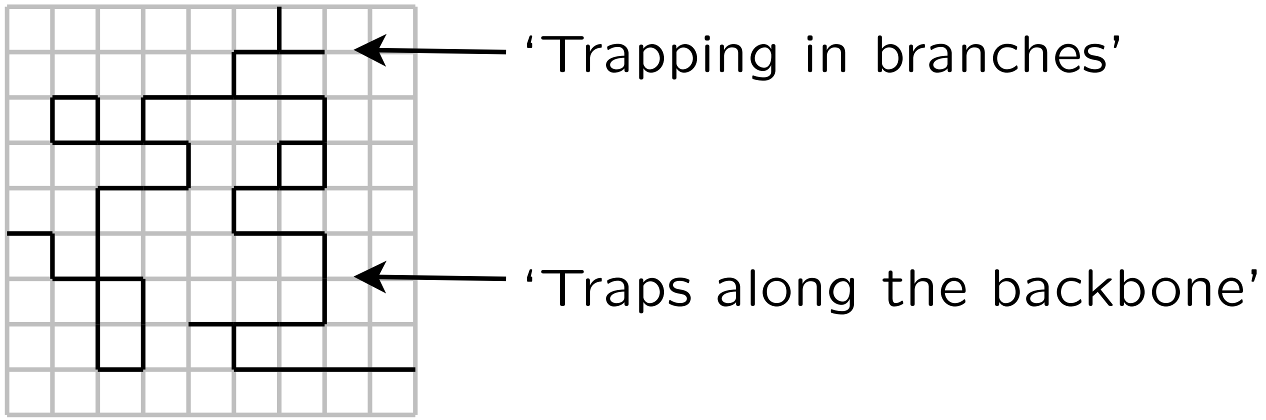


If $\beta = 1$, then the random walk is diffusive for $\mathbf{P}$-a.e. environment [Barlow, Sidoravicius/Sznitman, Biskup/Berger, Mathieu/Piatnitski].

If $\beta > 1$, then the walk is directionally transient for $\mathbf{P}$-a.e. environment. Moreover, there exists a $\beta_c \in (1, \infty)$ such that:
if $\beta < \beta_c$, then the biased random walk has positive speed,
if $\beta > \beta_c$, then the biased random walk has zero speed,
[Berger/Gantert/Peres, Sznitman, Fribergh/Hammond].

CRITICAL PERCOLATION

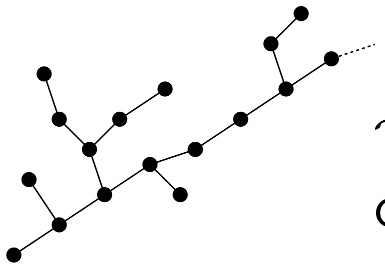
Close to $p = p_c$, physicists [Barma/Dhar] have identified two types of trapping:



Motivated by this, we consider biased random walk on:

- a critical Galton-Watson tree conditioned to survive,
- the range of a random walk.

TRAPPING IN CRITICAL BRANCHES



$\mathcal{T}^*$ - family tree of critical Galton-Watson process conditioned to survive. Offspring distribution in domain of attraction of α-stable law, $\alpha \in (1,2]$. Bias away from root.

[cf. C.] The unbiased walk ($\beta = 1$), when rescaled as

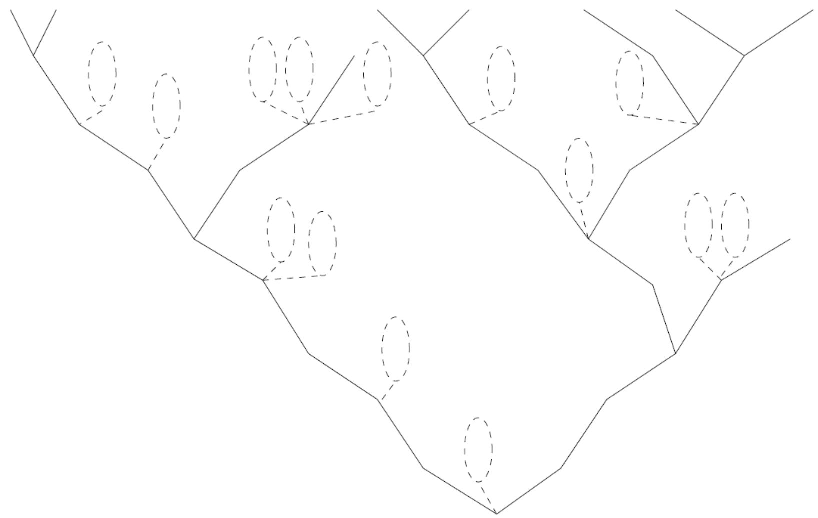
$$\left(\alpha_n^{-1} X_{\lfloor tn\alpha_n \rfloor}\right)_{t\geq 0},$$

converges in distribution to a non-trivial diffusion, where

$$\alpha_n \sim n^{(\alpha-1)/\alpha}\ell(n).$$

How does the biased random walk behave?
(Joint with Alexander Fribergh and Takashi Kumagai.)

SUPERCRITICAL ARGUMENT [BEN AROUS/FRIBERGH/GANTERT/HAMMOND]



Decompose into backbone (solid lines) and traps (dashed lines).

Linear progress on backbone $\Rightarrow$ regeneration arguments.

Expected time to leave trap with base at level i:

$$1 + \beta^{-i} \sum_{x \sim y \in T} c(x,y) \approx \beta^{h(T)},$$

where $c(x,y) := \beta^{\min\{\text{gen}(x),\text{gen}(y)\}}$.

$h(T)$ has exponential tails $\Rightarrow$ time in traps has polynomial $\Rightarrow$ polynomial rate of escape. [cf. Zindy]

SUPERCRITICAL RESULT [BEN AROUS/FRIBERGH/GANTERT/HAMMOND]

If offspring distribution Z satisfies:

$$\mathbf{E}Z > 1,\ \mathbf{E}Z^2 < \infty,\ \mathbf{P}(Z = 0) > 0,$$

and the drift satisfies:

$$\beta > \beta_c := \frac{1}{f'(q)},$$

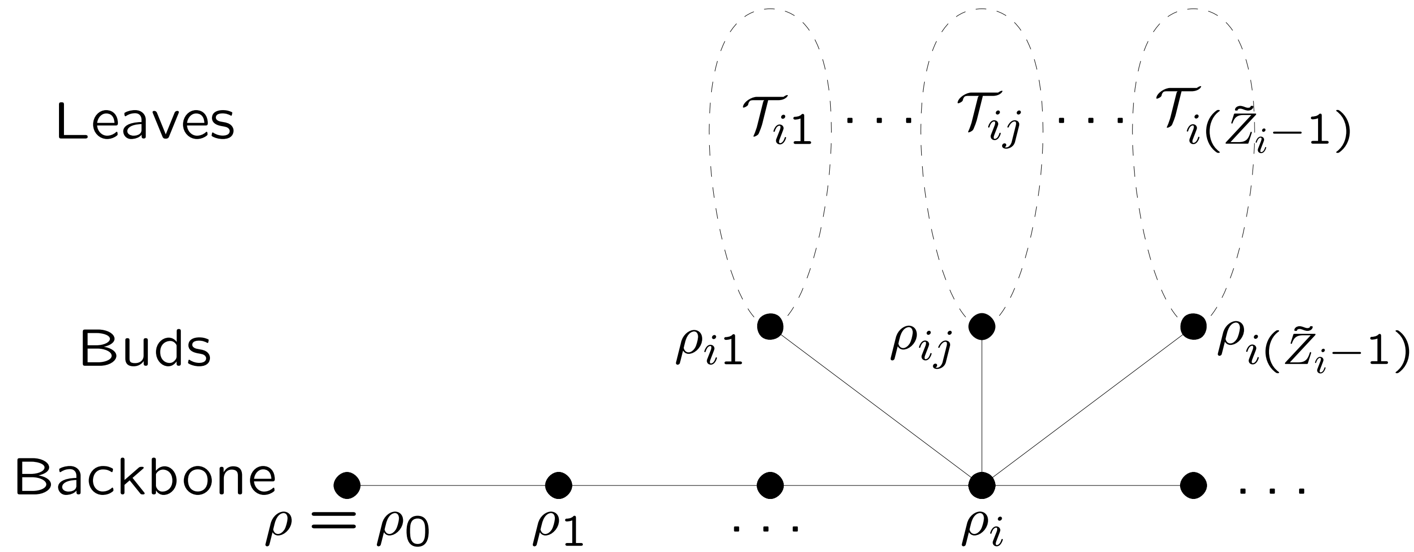
where q is the extinction probability and $f(x) := \mathbf{E}x^Z$, then

$$\frac{\log |X_n|}{\log n} \to \gamma := \frac{\log \beta_c}{\log \beta}, \quad \mathbb{P}_\rho\text{-a.s.}$$

Finer subsequential distributional limits established. Also distributional limits in randomly biased case [Ben Arous/Hammond, Hammond].

CRITICAL STRUCTURE

Let $\mathcal{T}^*$ be a critical Galton-Watson tree (i.e. $\mathbf{E}Z = 1$), with offspring distribution in domain of attraction of α-stable law, $\alpha \in (1,2]$, conditioned to survive:



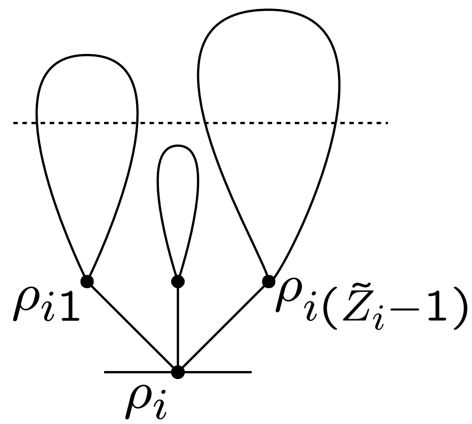
$$\mathbf{P}\left(\tilde{Z} = k\right) = k\mathbf{P}(Z = k), \quad k \geq 1.$$

$\mathcal{T}_{ij}$ are unconditioned Galton-Watson trees, offspring dist. Z.

BIG TRAP LOCATIONS

Define a level n critical height $h_n := n(\log n)^{-1}$, and set

$$N_n(i) := \#\left\{1 \leq j \leq \tilde{Z}_i - 1 : \; h(\mathcal{T}_{ij}) \geq h_n\right\}.$$



Then, since

$$\mathbf{P}\left(N_n(i) = 0\right) \sim 1 - \frac{\alpha}{(\alpha-1)h_n},$$

the number of backbone vertices from which big traps emanate up to level n is distributed as Binomial$(n, c_\alpha n^{-1} \log n)$. In particular, it grows like $c_\alpha \log n$.

APPROXIMATION BY I.I.D. SUM

Define a hitting time Δ_n by setting

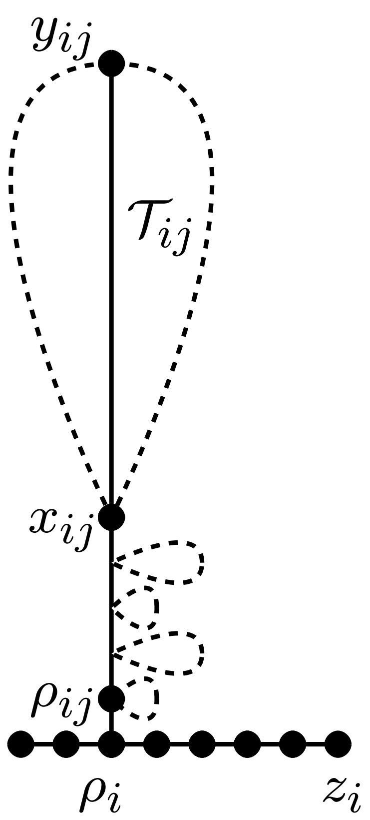
$$\Delta_n := \inf\{m \geq 0 : X_m = \rho_n\}.$$

Let t_i be total time spent by random walk in traps emanating from backbone vertex ρ_i. Can show time spent in small traps (height less than h_n) is negligible. Therefore

$$\Delta_n \approx \sum_{i=0}^{n-1} t_i \mathbf{1}_{\{N_n(i) \geq 1\}}.$$

Since jump process on backbone does not backtrack more than $(\log n)^2$, and big traps are separated by at least n^{ε}, summands are asymptotically independent.

BIG TRAPS VISITED



Let

$$B_i := \left\{ j = 1, \ldots, \tilde{Z}_i - 1 : \, h(\mathcal{T}_{ij}) \geq h_n \right\}$$

and

$$V_i := \left\{ j \in B_i : \, \tau_{x_{ij}} < \tau_{z_i} \right\},$$

where x_{ij} is ‘trap entrance’ and $z_i := \rho_{i+1+h_n^\delta}$, then

$$P_{\rho_i}^{\mathcal{T}^*}(V_i = A) = \frac{1}{1 + \#B_i} \binom{\#B_i}{\#A}^{-1}.$$

TIME SPENT IN BIG TRAP CLUSTER

By first conditioning on B_i and then V_i, it is possible to check that

$$\mathbb{P}_\rho\left(\max_{j\in V_i} h(\mathcal{T}_{ij}) \geq x\right) = q_x^{\alpha-1}L(q_x) \sim \frac{1}{(\alpha-1)x}.$$

NB. This is different to

$$\mathbb{P}_\rho\left(\max_{j=1,\dots,\tilde{Z}_i-1} h(\mathcal{T}_{ij}) \geq x\right) \sim \alpha q_x^{\alpha-1}L(q_x) \sim \frac{\alpha}{(\alpha-1)x}.$$

It follows that

$$\mathbb{P}_\rho\left(t_i \geq x\right) \approx \mathbb{P}_\rho\left(\beta^{\max_{j\in V_i} h(\mathcal{T}_{ij})} \geq x\right) \approx \frac{\log\beta}{(\alpha-1)\log x}.$$

SUMS OF SLOWLY-VARYING VARIABLES

Let $(X_i)_{i=1}^{\infty}$ be independent random variables, with distributional tail $\bar{F}(u) = 1 - F(u) = \mathbf{P}(X_i > u)$ satisfying: $\bar{F}(0) = 1$, $\bar{F}(u) > 0$ for all $u > 0$,

$$\lim_{u \to \infty} \frac{\bar{F}(uv)}{\bar{F}(u)} = 1,$$

for any $v > 0$, and $\bar{F}(u) \to 0$ as $u \to \infty$. If $L(x) := 1/\bar{F}(x)$, then

$$\left(\frac{1}{n} L \left(\sum_{i=1}^{nt} X_i \right) \right)_{t \geq 0} \to (m(t))_{t \geq 0} \,,$$

where $(m(t))_{t \geq 0}$ is an extremal process. In particular, m can be defined as the maximum process of the Poisson point process with intensity measure $x^{-2} dx dt$. See [Darling, Kasahara].

EXTREMAL CONVERGENCE [C./FRIBERGH/KUMAGAI]

If $\beta > 1$, then $(\Delta_n)_{n\geq 0}$ satisfies

$$\left(\frac{(\alpha-1)\log_+\Delta_{nt}}{n\log\beta}\right)_{t\geq 0} \to (m(t))_{t\geq 0}.$$

Moreover, if $(\pi(X_n))_{n\geq 0}$ is the projection of the biased random walk onto the backbone, then

$$\left(\frac{\pi(X_{\lfloor e^{nt}\rfloor})\log\beta}{(\alpha-1)n}\right)_{t\geq 0} \to \left(m^{-1}(t)\right)_{t\geq 0}.$$

Arguments also demonstrate localisation and extremal aging.

RELATED ONE-DIMENSIONAL TRAP MODEL

Let $\tau = (\tau_x)_{x\in\mathbb{Z}}$ be a family of independent and identically distributed strictly positive (and finite) random variables whose distribution has a slowly-varying tail $\bar{F}(u) = \mathbf{P}(\tau_0 > u)$.

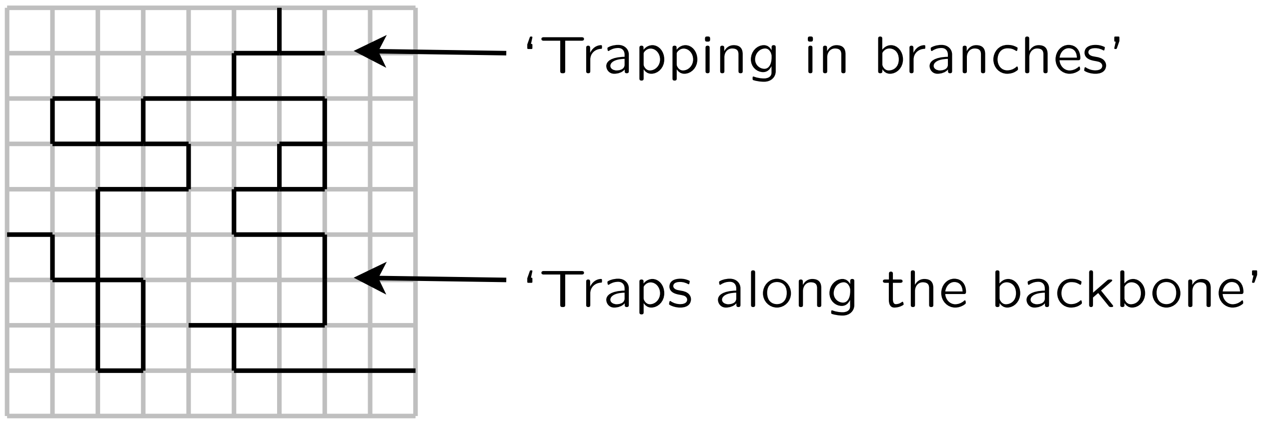
Conditional on τ, let X be a Markov chain on $\mathbb{Z}$ with jump rates

$$c(x,y) := \begin{cases} \left(\frac{\beta}{\beta+1}\right)\tau_x^{-1}, & \text{if } y = x+1, \\ \left(\frac{1}{\beta+1}\right)\tau_x^{-1}, & \text{if } y = x-1, \end{cases}$$

and $c(x,y) = 0$ otherwise, then

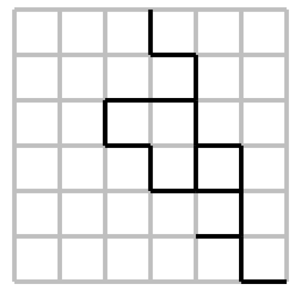
$$\left(\frac{1}{n}X_{\bar{F}^{-1}(1/nt)}\right)_{t\geq 0} \to \left(m^{-1}(t)\right)_{t\geq 0}.$$

RECALL CRITICAL TRAPPING MECHANISMS



RANGE OF A RANDOM WALK

Let $S = (S_n)_{n\in\mathbb{Z}}$ be the two-sided simple random walk on $\mathbb{Z}^d$ starting from 0, built on an underlying probability space with probability measure $\mathbf{P}$. Define the range of the random walk S to be the graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ with vertex set



$$V(\mathcal{G}) := \{S_n : n \in \mathbb{Z}\},$$

and edge set

$$E(\mathcal{G}) := \Big\{\{S_n, S_{n+1}\} : n \in \mathbb{Z}\Big\}.$$

For $\mathbf{P}$-a.e. random walk path, the graph $\mathcal{G}$ is infinite, connected and clearly has bounded degree.

UNBIASED RANDOM WALK ON $\mathcal{G}$ [C.]

Let $d \geq 5$. For $\mathbf{P}$-a.e. realisation of $\mathcal{G}$, the law of

$$\left(n^{-1/2}\mathrm{sgn}(X_{\lfloor tn \rfloor})(d_{\mathcal{G}}(0, X_{\lfloor tn \rfloor})\right)_{t \geq 0},$$

under $\mathbf{P}_0^{\mathcal{G}}$, converges as $n \to \infty$ to the law of $(B_{t\kappa_1(d)})_{t \geq 0}$. Furthermore, the law of

$$\left(n^{-1/4}X_{\lfloor tn \rfloor}\right)_{t \geq 0},$$

under $\mathbb{P}$, converges as $n \to \infty$ to the law of $(W^{(d)}_{B_{t\kappa_2(d)}})_{t \geq 0}$.

NB. Result does not hold in $d = 3, 4$ [C., Shiraishi].

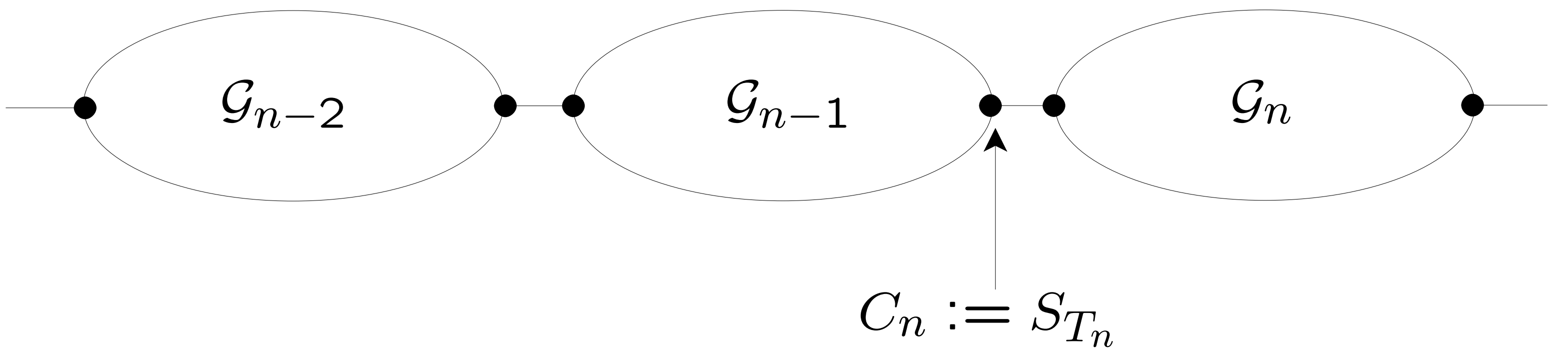
GEOMETRY OF TWO-SIDED RANGE

For $d \geq 5$, $\mathbf{P}$-a.s., the two-sided process S admits an infinite set of cut-times

$$\mathcal{T} := \{n : S_{(-\infty,n]} \cap S_{[n+1,\infty)} = \emptyset\},$$

which will be denoted $\ldots T_{-2} < T_{-1} < T_0 \leq 0 < T_1 < T_2 < \ldots$.

Under $\hat{\mathbf{P}} := \mathbf{P}(\cdot|0 \in \mathcal{T})$, $\mathcal{G}$ is made up of a string of stationary ergodic finite graphs.



[Bolthausen/Sznitman/Zeitouni] applied this kind of decomposition of the path of S to deduce the diffusivity of a random walk in a particular high-dimensional random environment.

BIASED RANDOM WALK ON $\mathcal{G}$

Given $\beta \geq 1$, assign to each edge $e = \{e_+, e_-\} \in E(\mathcal{G})$ a conductance

$$\mu_e := \beta^{\max\{e_-^{(1)}, e_+^{(1)}\}},$$

where $e_\pm^{(1)}$ is the first coordinate of $e_\pm$.

For jump chain, we have

$$\omega_n^\pm := \mathbf{P}_0^{\mathcal{G}}(J_{m+1} = n \pm 1 | J_m = n) = \frac{1}{\mu(\{C_n\}) R_{\mathcal{G}}(C_n, C_{n\pm 1})}.$$

From this, one can check that

$$(\omega_n^-, \omega_n^0, \omega_n^+)_{n \in \mathbb{Z}}$$

is a stationary, ergodic sequence.

POTENTIAL OF RWRE

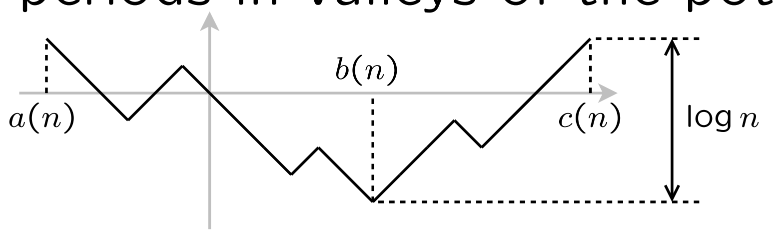
A key quantity in understanding the behaviour of a RWRE is the potential function:

$$R_n := \sum_{i=1}^{n} \log \rho_i,$$

where $\rho_i := \omega_i^- / \omega_i^+$. This satisfies

$$R_n = \log R_{\mathcal{G}}(C_n, C_{n+1}) - \log R_{\mathcal{G}}(C_0, C_1) \sim -C_n^{(1)} \log \beta \sim -S_{n\tau}^{(1)} \log \beta,$$

which has Brownian scaling $\Rightarrow$ Sinai's regime, in which the walker is trapped for long periods in valleys of the potential [Sinai].



LOCALIZATION RESULT [C.]

Fix a bias parameter $\beta > 1$ and $d \geq 5$. If $X = (X_n)_{n\geq 0}$ is the biased random walk on the range $\mathcal{G}$ of the two-sided simple random walk S in $\mathbb{Z}^d$, then there exists an S-measurable random variable L_n taking values in $\mathbb{R}^d$ such that

$$\mathbb{P}\left(\left|\frac{X_n}{\log n} - L_n\right| > \varepsilon\right) \to 0,$$

for any $\varepsilon > 0$. Moreover,

$$L_n \to L_\beta := \frac{L}{\log \beta},$$

in distribution under $\mathbf{P}$, where L is a random variable taking values in $\mathbb{R}^d$ whose distribution can be characterized explicitly.