Random Parking and Rubber Elasticity

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Rubber Elasticity

Let $d, n \in \mathbf{N}$ (e.g. $d = n = 3$).

Suppose $D \subset \mathbf{R}^d$ is a bounded domain. D represents a piece of rubber.

Let $\mathcal{L} \subset \mathbb{R}^d$ be a locally finite point process.

$\mathcal{L} \cap D$ the locations of individual "molecules". For $x, y \in \mathcal{L}$ write $x \sim y$ if they are Delaunay neighbours, let $\mathcal{T}$ be the Delaunay triangulation.

Each $u \in C(D, \mathbb{R}^n)$ represents a deformation of the rubber.
$u(x)$ is the location of $x \in D$ under deformation.

We'll define a class of energy functionals $F : C(D, \mathbb{R}^n) \to \mathbb{R}$.

$F(u)$, the energy of deformation u.
Let $u_{\mathcal{L}}$ be affine on each $T \in \mathcal{T}$ with $u_{\mathcal{L}} \equiv u$ on $\mathcal{L}$

The energy functional

$$F_{\mathcal{L}}^{D}(u) = \sum_{x,y\in\mathcal{L},x\sim y,[x,y]\subset D} |y-x|^d f\left(\frac{u(y)-u(x)}{|y-x|}\right)$$

$$+ \sum_{T\in\mathcal{T}(\mathcal{L}),T\subset D} |T| g(\nabla u_{\mathcal{L}}|_T)$$

Assume we are given $f \in C(\mathbb{R}^n, \mathbb{R}^+)$ the bond energy

and $g \in C(M^{n\times d}, \mathbb{R}^+)$ the cell energy (given).

Assume growth bounds on f, g: for some $p > 1$, $C > 0$,

$$C^{-1} \leq \frac{f(z)}{|z|^p} \leq C, \quad |z| \geq 1$$

$$g(\Lambda) \leq C\|\Lambda\|^p, \quad \|\Lambda\| \geq 1.$$

Desirable properties of $\mathcal{L}$, our point process in $\mathbb{R}^d$

$\mathcal{L}$ is *stationary* if $x + \mathcal{L} \stackrel{\mathcal{D}}{=} \mathcal{L}$, for all $x \in \mathbb{R}^d$.

$\mathcal{L}$ is *isotropic* if $R\mathcal{L} \stackrel{\mathcal{D}}{=} \mathcal{L}$ for all $R \in SO_d$.

$\mathcal{L}$ is in *general position* (or just *general*) if no $d+1$ points of $\mathcal{L}$ lie in the same hyperplane, and no $d+2$ points are in the same hypershpere.

Let $\mathcal{A}_{lf}$ be the class of locally finite point configurations in $\mathbb{R}^d$.

$\mathcal{L}$ is *ergodic* if for all $A \subset \mathcal{A}_{lf}$ with $T_x(A) = A$ for all $x \in \mathbb{R}^d$ (where T_x is translation by x) we have $P[\mathcal{L} \in A] \in \{0,1\}$.

For $0 < a < b$ let $\mathcal{A}_{a,b} \subset \mathcal{A}_{lf}$ be the class of ξ such that

$$\begin{aligned} x, y \in \xi \quad &\Longrightarrow \quad |x-y| > a \quad \text{(hard core condition)} \\ x \in \mathbb{R}^d \quad &\Longrightarrow \quad \xi \cap B(x,b) \neq \emptyset \quad \text{(no empty space condition)} \\ B(x,b) \quad &:= \quad \{y : |y-x| \leq b\}. \end{aligned}$$

Let $\mathcal{A}_{a,\infty}$ be those ξ satisfying just the hard core condition.

Gamma-convergence

If F_n and F are real-valued functions on some metric space X.
we say for $x \in X$ that $F_n \xrightarrow{\Gamma} F$ at x if
(a) For all sequences $x_n \to x$ we have $\liminf F_n(x_n) \geq F(x)$, and
(b) $\exists$ sequence $x_n \to x$ with $F_n(x_n) \to F(x)$.

We say $F_n \xrightarrow{\Gamma} F$ if $F_n \xrightarrow{\Gamma} F$ at x for all $x \in X$.

Set $Q_r = [-r/2, r/2]^d$.

Recall we assume f (bond energy) and g (cell energy) satisfy growth bounds of order $p > 1$. We now state a

Homogenization result (Alicandro, Cicalese and Gloria 2011)

$$F_{\mathcal{L}}^{D}(u) = \sum_{x\sim y, [x,y]\subset D} |y-x|^d f\left(\frac{u(y)-u(x)}{|y-x|}\right) + \sum_{T\in\mathcal{T}(\mathcal{L}):T\subset D} |T| g(\nabla u_{\mathcal{L}}|_T)$$

Suppose $0 < a < b$ and $\mathcal{L}$ is stationary, ergodic, general and a.s. in $\mathcal{A}_{a,b}$. Then as $\varepsilon \downarrow 0$ we have $F_{\varepsilon\mathcal{L}}^{D} \xrightarrow{\Gamma} F_{\rm hom}^{D}$ on $L^p(D, \mathbb{R}^n)$, where

$$F_{\rm hom}^{D}(u) = \begin{cases} \int_D W_{\rm hom}(\nabla u(x))dx, & u \in W^{1,p}(D,\mathbb{R}^n). \\ +\infty & \text{otherwise} \end{cases}$$

$$W_{\rm hom}(\Lambda) = \lim_{r\to\infty} r^{-d} \inf\{F_{\mathcal{L}}^{Q_r}(u) : u(x) = \Lambda \cdot x \text{ on } Q_r \setminus Q_{r-2b}\} \qquad (1)$$

Idea - divide D into cubes of side $\delta = \delta(\epsilon)$ with $\varepsilon \ll \delta \ll 1$. Take $u_\varepsilon = u$ (approx. affine) near boundary of each cube. Let u_ε optimise the energy inside each cube, subject to this constraint. Discuss (1) later.

Existence of a stationary ergodic general $\mathcal{L}$ in $\mathcal{A}_{a,b}$: Random parking.

Let $\rho > 0$. Let $X_1, X_2, \ldots$ be independent uniform random vectors in D.

X_n is *accepted*, unless $\exists m \leq n$ with X_m accepted and $|X_n - X_m| \leq \rho$.

Let $\xi^D = \{\text{accepted } X_i\}$. (random parking process on D). It has the a-hardcore and b-no-empty space properties on D for any $a < \rho < b$.

ξ^{Q_r} has weak limit ξ on $\mathbb{R}^d$ (stationary ergodic, general, in $\mathcal{A}_{a,b}$).

ξ obtained from parking protocol for homogeneous Poisson point process $\{(X_i, T_i)\}$ in $\mathbb{R}^d \times \mathbb{R}^+$, where T_i is arrival time of X_i.

Parking protocol on this Poisson process well-defined by a first passage percolation argument (Penrose 2001).

Subadditivity

Suppose $\mathcal{E}(S, R)$ is a real-valued energy functional defined for all locally finite $S \subset \mathbb{R}^d$, and rectangles R, e.g. p-weighted travelling salesman cost

$$\mathcal{E}_{TSP,p}(S, R) = \min\{\sum_{i=1}^{n} |x_i - x_{i-1}|^p : S \cap R = \{x_1, \ldots, x_n\}, x_0 = x_n\}$$

Known since BHH (1959) that there exists β such that as $r \to \infty$,

$$r^{-d}\mathcal{E}_{TSP,1}(\mathcal{H}, Q_r) \to \beta$$

where $\mathcal{H}$ is a homogeneous PPP on $\mathbb{R}^d$.
We shall describe generic properties of $\mathcal{E}$ guaranteeing such convergence for $\mathcal{E}_{TSP,p}$ and many other examples, e.g. the minimal matching and minimal spanning tree. (cf. Redmond and Yukich (1994), Yukich (1999)).

Properties of $\mathcal{E}_{TSP,p}(\cdot)$ (and other choices of $\mathcal{E}(\cdot)$ with 'order' p)

- Translation invariant: $\mathcal{E}(x+S, x+R) = \mathcal{E}(S,R)$, all $x \in \mathbb{R}^d$, all S, R.
- Almost subadditive: $\mathcal{E}(S \cup T, R) \leq \mathcal{E}(S,R) + \mathcal{E}(T,R) + C(\mathrm{diam} R)^p$.
- Smooth: $|\mathcal{E}(T,R) - \mathcal{E}(S,R)| \leq C(\mathrm{diam} R)^p (\mathrm{card}((S \triangle T) \cap R))^{1-p/d}$.
- There is an approximate energy functional $\tilde{\mathcal{E}}(S,R)$ defined for all rectangles $R \subset \mathbb{R}^d$ with $\tilde{\mathcal{E}}$ translation invariant, and
- Superadditive: if $R_0 = \cup_{i=1}^n R_i$ (rectangles) then

$$\tilde{\mathcal{E}}(S,R) \geq \sum_{i=1}^m \tilde{\mathcal{E}}(S,R_i)$$

- Close to $\mathcal{E}$ of order p: $r^{-p}|\tilde{\mathcal{E}}(S,Q_r) - \mathcal{E}(S,Q_r)| = o(\mathrm{card}(S \cap Q_r))$.

For $\mathcal{E}_{TSP,p}$ take $\tilde{\mathcal{E}}(S,R)$ to be the TSP cost with 'free travel' outside R.

General LLNs for $\mathcal{E}$

Suppose $p \geq 1$ and $\mathcal{E}$ is TI, almost subadditive, and smooth of order p. Suppose there exists $\tilde{\mathcal{E}}(S, R)$ which is TI, superadditive and close to $\mathcal{E}$ of order p. Then (Redmond/Yukich) there is a constant such that

$$r^{-d}\mathcal{E}(\mathcal{H}, Q_r) \to \gamma \quad a.s.$$

Moreover (Gloria-P. 2012), if $\mathcal{L}$ is stationary and ergodic with $\mathcal{L} \in \mathcal{A}_{a,\infty}$ for some $a > 0$, then there exists $\gamma \in \mathbb{R}$ such that

$$r^{-d}\mathcal{E}(\mathcal{L}, Q_r) \to \gamma \quad a.s.$$

Proof via multiparameter subadditive ergodic theorem (Akcoglu/Krengel).

Example: take $\mathcal{E}(S, R) = \inf\{F_S^R(u) : u(x) = \Lambda x \text{ near } \partial R\}$.

Back to random parking

Results described so far show that if ξ is random parking on $\mathbb{R}^d$ then for $\mathcal{E}$ satisfying hypotheses of subadditivity etc., we have for some γ that

$$r^{-d}\mathcal{E}(\xi, Q_r) \to \gamma$$

e.g. with $\mathcal{E}$ the p-weighted TSP on $\xi \cap Q_R$
or with $\mathcal{E}$ the minmum of $F_\xi^{Q_r}(u)$ given Λ-boundary conditions.

Would like to replace ξ by ξ^{Q_r} in the above results, since (i) any simulation studies would be on a finite region (ii) no physical reason for process generating ξ on $D = Q_r$ to depend on input from outside D
Gloria-P. (2012): can indeed replace ξ by ξ^{Q_r} in the above.

Also: can extend the earlier homogenization result:

$$F^D_{\mathcal{L}}(u) = \sum_{x\sim y,[x,y]\subset D} |y-x|^d f\left(\frac{u(y)-u(x)}{|y-x|}\right) + \sum_{T\in\mathcal{T}(\mathcal{L}):T\subset D} |T| g(\nabla u_{\mathcal{L}}|_T)$$

Suppose ξ^D_ρ is the random parking process in D with hard-core parameter ρ. Then for $\rho > 0$, as $\varepsilon \downarrow 0$ we have $F^D_{\xi^D_{\varepsilon\rho}} \xrightarrow{\Gamma} F^D_{\text{hom}}$ on $L^p(D,\mathbb{R}^n)$, where

$$F^D_{\text{hom}}(u) = \begin{cases} \int_D W_{\text{hom}}(\nabla u(x))dx, & u \in W^{1,p}(D,\mathbb{R}^n). \\ +\infty & \text{otherwise} \end{cases}$$

$$W_{\text{hom}}(\Lambda) = \lim_{r\to\infty} r^{-d} \inf\{F^{Q_r}_{\xi^{Q_r}}(u) : u(x) = \Lambda\cdot x \text{ on } Q_r \setminus Q_{r-2b}\}. \quad (2)$$

Proof relies heavily on exponential stabilization of random parking (Schreiber, P. and Yukich 2007)