Optimal trade execution under price sensitive risk preferences

Stefan Ankirchner, Thomas Kruse

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Unwinding large positions is part of day-to-day business

... of banks, insurance companies, funds, energy companies, ...

**How to sell / buy?**

- Not too fast
  
  split orders over time to reduce liquidation costs

- Not too slow
  
  reduce market risk of open position

**Stochastic control problem:** What is the optimal trade-off?
Which lottery do you choose?

Lottery 1:

2000

\[ \frac{3}{4} \]

0

\[ \frac{1}{4} \]

Lottery 2:

3000

\[ \frac{1}{4} \]

1000

\[ \frac{1}{4} \]

\[ \frac{1}{4} \]
Skewed versus unskewed proceed distribution
Wanted:

A model such that ...

- allows to introduce skewness in the revenue distribution
- the trading speed is price sensitive
- time consistent strategies
- numerically efficient
- the relative trading rate is independent of the remaining position size
Model: Trading rates determine remaining position

- \( T \) = time horizon
- \( z_t \) = trading rate at \( t \in [0, T] \)
- \( x_t \) = position size at \( t \in [0, T] \)

\[
x_t = x_0 - \int_0^t z_u \, du
\]

Constraint: \( x_T = 0 \)
Non-influenced forward price dynamics

\[ dS_t = \sigma(S_t) dW_t \]

Realized price at \( t \):

\[ \tilde{S}_t = S_t - \eta z_t, \]

where \( \eta > 0 \) is the price impact parameter.

Price impact is LAT:

- linear
- absolute
- temporary
Liquidity costs grow quadratically

Realized proceeds / costs up to $t$:

$$R_t = \int_0^t z_u \tilde{S}_u du = \int_0^t z_u S_u du - \int_0^t \eta z_u^2 du.$$ 

Expected realized proceeds / costs:

$$E[R_t] = x_0 S_0 - E \left[ \int_0^t \eta z_u^2 du \right]$$

book value  liqu. costs

A risk neutral agent closes the position linearly!
Model: Measuring risk

Risk functional:

\[ \int_0^T \lambda(S_t) x_t^2 dt \]

Possible choices for \( \lambda(s) \):

- Long position: \( \lambda(s) = \max[c \ast (\bar{s} - s), 0]^2 \)
- Short position: \( \lambda(s) = \max[c \ast (s - \bar{s}), 0]^2 \)

Interpretation: time average of the value-at-risk squared
Interpret the risk functional as value-at-risk

Risk functional: \[ \int_0^T \lambda(S_t)x_t^2 dt \]

Let \( c = 5\% \) quantile of mark-to-market losses up \( t + \Delta \) of a long position \( x \):

\[ P(x(S_{t+\Delta} - S_0) \leq c) = 5\% . \]

Then

\[ c = x \left( S_t e^{-\sigma \sqrt{\Delta}} a - \sigma^2 / 2\Delta - S_0 \right) , \]

where

- \( a = 95\% \)–quantile of the standard normal distribution
- \( \Delta = \) holding period
Model: gain + value function

**Gain function:** expected liquidity costs + risk

\[
J(t, s, x; (z_r)) = E \left[ \int_t^T \eta z_r^2 + \lambda(S_r)x_r^2 \, dr \right] \bigg|_{S_t = s, x_t = x}
\]

**Value function:**

\[
V(t, s, x) = \inf_{(z_r) \in A_t(x)} J(t, s, x; (z_r)).
\]
\[ V(t, s, x) \text{ solves} \]

\[
- \frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial s^2} - \lambda(s)x^2 - \inf_{z \in \mathbb{R}} (\eta z^2 - \frac{\partial V}{\partial x} z) = 0 \quad (1)
\]

with terminal condition

\[
\lim_{t \uparrow T} V(t, s, x) = \begin{cases} 
\infty & \text{for } x \neq 0 \\
0 & \text{for } x = 0.
\end{cases}
\]

**Variable reduction:** \( V(t, s, x) = l(t, s) \frac{\eta}{T-t} x^2 \). Then

\[
- \frac{\partial l}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 l}{\partial s^2} - \frac{l}{T-t} + \frac{l^2}{T-t} - (T - t) \frac{\lambda(s)}{\eta} = 0, \quad (2)
\]

with \( l(T, s) = 1 \).
Explicit solutions for price-insensitive risk

Definition:

\[ I_c(t) = \begin{cases} \sqrt{\frac{c}{\eta}}(T - t) \coth \left( \sqrt{\frac{c}{\eta}}(T - t) \right) & \text{if } c > 0 \\ 1 & \text{if } c = 0. \end{cases} \]

Proposition

Suppose that \( \lambda(s) = c \). Then the value function is given by

\[ V(t, x) = \frac{\eta}{T - t} I_c(t)x^2 \]

and the optimal trading speed by

\[ z_t = I_c(t) \frac{x_t}{T - t}. \]

Proof. follows from Kratz, Schöneborn 2009.
Insensitive risk: optimal liquidation paths

Optimal trading speed:

\[ z_t = I_c(t) \frac{x_t}{T - t} \]

linear closure

The factor \( I_c(t) \) inflates linear trading!

**Figure**: Inflator and position paths
Price-sensitive risk: Inflator solves a PDE

\[- \frac{\partial I}{\partial t} - \frac{1}{2} \sigma^2 \frac{\partial^2 I}{\partial s^2} - \frac{I}{T - t} + \frac{l^2}{T - t} - (T - t) \frac{\lambda(s)}{\eta} = 0, \quad (3)\]

**Theorem**

There exists a unique viscosity solution $I$ of (3) on $[0, T) \times (0, \infty)$ such that

- $I \geq 1$
- $I$ has polynomial growth in $s$
- boundary conditions

\[
\lim_{t \to T \atop s \to s_0} I(t, s) = 1 \quad \text{for all } s_0 \in (0, \infty),
\]

\[
\lim_{t \to t_0 \atop s \searrow 0} I(t, s) = I_{\lambda(0)}(t_0) \quad \text{for all } t_0 \in [0, T).
\]

Moreover, $I$ is continuous.
The value function is a quadratic form

**Theorem**

The value function is a quadratic form in $x$:

$$V(t, s, x) = l(t, s) \frac{\eta}{T - t} x^2.$$  

The optimal trading speed is given by

$$z(t, s, x) = l(t, s) \frac{x}{T - t}.$$  

Associated position trajectory

$$x_t = x_0 \exp \left( - \int_0^t l(u, S_u) \frac{T}{T - u} du \right).$$
Long position: Inflator increases as prices fall

- long position $x_0 > 0$
- risk weight $\lambda(s) = \max[c \ast (\bar{s} - s), 0]^2$

Parameters: $S_0 = 50$, $\bar{s} = 50$, $c = 0.01$
Optimal trading speed

\[ z(t, s, x) = l(t, s) \left( x \frac{x}{T - t} \right). \]

inflator linear closure
Trading speed depends on price evolvement

Figure: Price dependence of liquidation paths for $c = 0.03$
Skewness in proceeds / costs

Figure: Histograms of realized proceeds
Can we solve the discrete problem explicitly?

Discrete value function:

\[ V_N^n(s, x) := \inf_{(z_k) \in A_k(x)} E \left[ \sum_{k=n}^{N-1} \eta^N z_k^2 + \lambda^N(S_k^N)x_k^2 \bigg| S_N^n = s, x_n = x \right] . \]

Proposition

*The value function is a quadratic form*

\[ V_N^n(s, x) = a_N^n(s)x^2, \]

where \( a_N^n \) is defined via the function recursion

\[
\begin{align*}
a_{N-1}^N(s) &= \eta^N + \lambda^N(s), \\
a_N^n(s) &= \frac{\eta^N E[a_{n+1}(S_{n+1}^N)|S_n = s]}{\eta^N + E[a_{n+1}(S_{n+1}^N)|S_n = s]} + \lambda^N(s).
\end{align*}
\]
The discrete value fct converges

**Theorem**

We have $V^N \rightarrow V$ pointwise in $[0, T) \times (0, \infty) \times \mathbb{R}$ as $N \rightarrow \infty$. 

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Optimal trade execution


Conclusion

- We present a liquidation model with a price sensitive risk functional
- A device that allows to introduce skewness in the revenue / cost distribution
- Trading speed increases if prices move into an unfavorable direction
- Inflator is characterized in terms of a PDE
- A flexible and numerically efficient way to derive time consistent liquidation paths
Thank you!