Optimal trade execution under price sensitive risk preferences

Stefan Ankirchner, Thomas Kruse

universitätbonn
Rheinische
Friedrich-Wilhelms-
Universität Bonn

EPSRC Workshop: Topics In Control
December 1, 2011
Warwick

Unwinding large positions is part of day-to-day business

. . . of banks, insurance companies, funds, energy companies, . . .

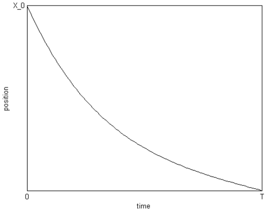
How to sell / buy?

- Not too fast

  split orders over time to reduce liquidation costs

- Not too slow

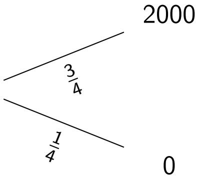
  reduce market risk of open position



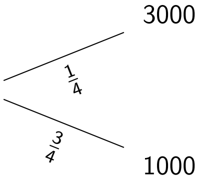
Stochastic control problem: What is the optimal trade-off?

Which lottery do you choose?

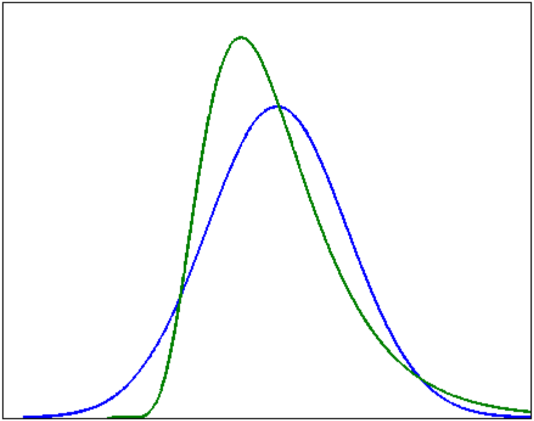
Lottery 1:



Lottery 2:



Skewed versus unskewed proceed distribution



Wanted:

A model such that ...

- allows to introduce skewness in the revenue distribution
- the trading speed is price sensitive
- time consistent strategies
- numerically efficient
- the relative trading rate is independent of the remaining position size

Model: Trading rates determine remaining position

- T = time horizon
- z_t = trading rate at $t \in [0, T]$
- x_t = position size at $t \in [0, T]$

$$x_t = x_0 - \int_0^t z_u du$$

Constraint: $x_T = 0$

Model: Trading rates determine price impact

Non-influenced forward price dynamics

$$dS_t = \sigma(S_t)dW_t$$

Realized price at t:

$$\tilde{S}_t = S_t - \eta z_t,$$

where $\eta > 0$ is the price impact parameter.

Price impact is **LAT**:

- linear
- absolute
- temporary

Liquidity costs grow quadratically

Realized proceeds / costs up to t:

$$R_t = \int_0^t z_u \tilde{S}_u du = \int_0^t z_u S_u du - \int_0^t \eta z_u^2 du.$$

Expected realized proceeds / costs:

$$E[R_t] = \underbrace{x_0 S_0}_{\text{book value}} - \underbrace{E \int_0^t \eta z_u^2 du}_{\text{liqu. costs}}$$

A risk neutral agent closes the position linearly!

Model: Measuring risk

Risk functional:

$$\int_0^T \lambda(S_t) x_t^2 \, dt$$

Possible choices for $\lambda(s)$:

- Long position: $\lambda(s) = \max[c * (\bar{s} - s), 0]^2$
- Short position: $\lambda(s) = \max[c * (s - \bar{s}), 0]^2$

Interpretation: time average of the value-at-risk squared

Interpret the risk functional as value-at-risk

$$\text{Risk functional:} \qquad \int_0^T \lambda(S_t) x_t^2 dt$$

Let $c = 5\%$ quantile of mark-to-market losses up $t + \Delta$ of a **long position** x:

$$P(x(S_{t+\Delta} - S_0) \leq c) = 5\%.$$

Then

$$c = x \left(S_t e^{-\sigma\sqrt{\Delta}\ a - \sigma^2/2\Delta} - S_0 \right),$$

where

- $a = 95\%-$quantile of the standard normal distribution
- $\Delta =$ holding period

Model: gain + value function

Gain function: expected liquidity costs + risk

$$J(t,s,x;(z_r)) = E\left[\int_t^T \eta z_r^2 + \lambda(S_r)x_r^2 dr \middle| S_t = s, x_t = x\right]$$

Value function:

$$V(t,s,x) = \inf_{(z_r)\in\mathcal{A}_t(x)} J(t,s,x;(z_r)).$$

HJB Equation

$V(t,s,x)$ **solves**

$$-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial s^2} - \lambda(s)x^2 - \inf_{z\in\mathbb{R}}(\eta z^2 - \frac{\partial V}{\partial x} z) = 0 \qquad (1)$$

with terminal condition

$$\lim_{t\nearrow T} V(t,s,x) = \begin{cases} \infty \text{ for } x \neq 0 \\ 0 \text{ for } x = 0. \end{cases}$$

Variable reduction: $V(t,s,x) = I(t,s)\frac{\eta}{T-t}x^2$. Then

$$-\frac{\partial I}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 I}{\partial s^2} - \frac{I}{T-t} + \frac{I^2}{T-t} - (T-t)\frac{\lambda(s)}{\eta} = 0, \qquad (2)$$

with $I(T,s) = 1$.

Explicit solutions for price-insensitive risk

Definition:

$$l_c(t) = \begin{cases} \sqrt{\frac{c}{\eta}}(T-t)\coth\left(\sqrt{\frac{c}{\eta}}(T-t)\right) & \text{if } c > 0 \\ 1 & \text{if } c = 0. \end{cases}$$

Proposition

Suppose that $\lambda(s) = c$. Then the value function is given by

$$V(t,x) = \frac{\eta}{T-t} l_c(t) x^2$$

and the optimal trading speed by

$$z_t = l_c(t)\frac{x_t}{T-t}.$$

Proof. follows from Kratz, Schöneborn 2009.

Insensitive risk: optimal liquidation paths

Optimal trading speed:

$$z_t = I_c(t) \underbrace{\frac{x_t}{T-t}}_{\text{linear closure}}$$

The factor $I_c(t)$ inflates linear trading!

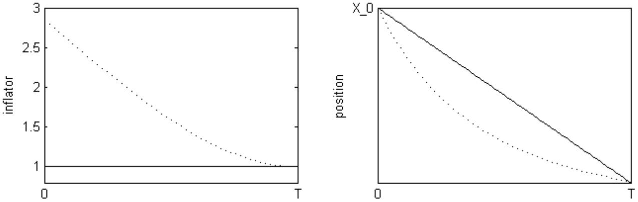


Figure: Inflator and position paths

Price-sensitive risk: Inflator solves a PDE

$$-\frac{\partial I}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 I}{\partial s^2} - \frac{I}{T-t} + \frac{I^2}{T-t} - (T-t)\frac{\lambda(s)}{\eta} = 0, \qquad (3)$$

Theorem

There exists a unique viscosity solution I of (3) on $[0,T) \times (0,\infty)$ such that

- $I \geq 1$
- *I has polynomial growth in s*
- *boundary conditions*

$$\lim_{\substack{t \nearrow T \\ s \to s_0}} I(t,s) = 1 \qquad \text{for all } s_0 \in (0,\infty),$$
$$\lim_{\substack{t \to t_0 \\ s \searrow 0}} I(t,s) = I_{\lambda(0)}(t_0) \qquad \text{for all } t_0 \in [0,T).$$

Moreover, I is continuous.

The value function is a quadratic form

Theorem

The value function is a quadratic form in x:

$$V(t,s,x) = I(t,s)\frac{\eta}{T-t}x^2.$$

The optimal trading speed is given by

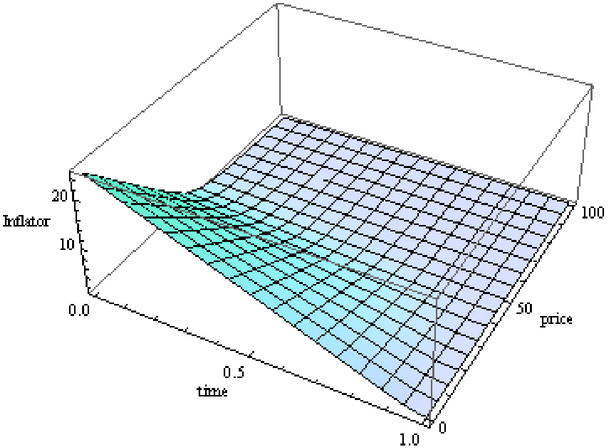
$$z(t,s,x) = I(t,s)\frac{x}{T-t}.$$

Associated position trajectory

$$x_t = x_0 \exp\left(-\int_0^t \frac{I(u,S_u)}{T-u}du\right).$$

Long position: Inflator increases as prices fall

- long position $x_0 > 0$
- risk weight $\lambda(s) = \max[c * (\bar{s} - s), 0]^2$

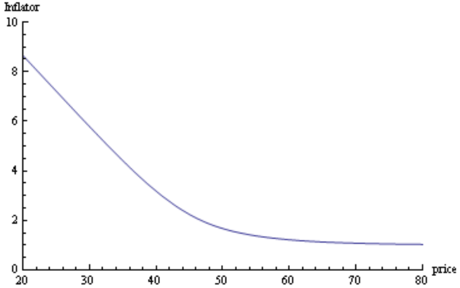


Parameters: $S_0 = 50$, $\bar{s} = 50$, $c = 0.01$

Optimal trading speed

Optimal trading speed

$$z(t,s,x) \quad = \quad \underbrace{I(t,s)}_{\text{inflator}} \quad \underbrace{\frac{x}{T-t}}_{\text{linear closure}}.$$



Trading speed depends on price evolvement

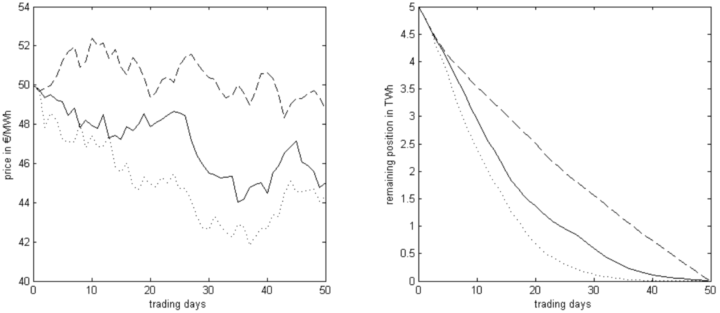


Figure: Price dependence of liquidation paths for $c = 0.03$

Skewness in proceeds / costs

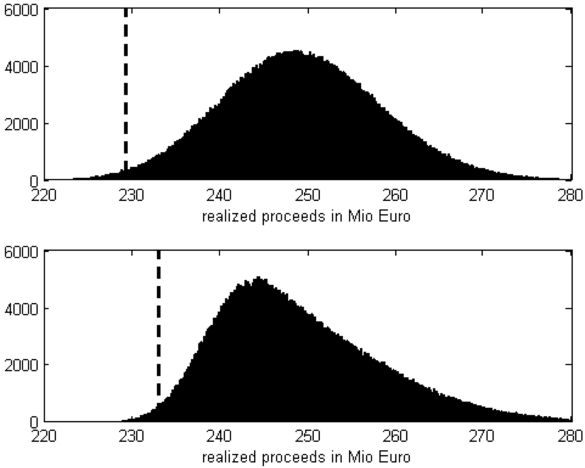


Figure: Histograms of realized proceeds

Can we solve the discrete problem explicitly?

Discrete value function:

$$V_n^N(s,x) := \inf_{(z_k)\in\mathcal{A}_k(x)} E\left[\sum_{k=n}^{N-1} \eta^N z_k^2 + \lambda^N(S_k^N)x_k^2 \middle| S_n^N = s, x_n = x\right].$$

Proposition

The value function is a **quadratic form**

$$V_n^N(s,x) = a_n^N(s)x^2,$$

where a_n^N is defined via the function recursion

$$a_{N-1}^N(s) = \eta^N + \lambda^N(s), \quad a_n^N(s) = \frac{\eta^N E[a_{n+1}^N(S_{n+1}^N)|S_n = s]}{\eta^N + E[a_{n+1}^N(S_{n+1}^N)|S_n = s]} + \lambda^N(s).$$

The discrete value fct converges

Theorem

We have $V^N \to V$ *pointwise in* $[0, T) \times (0, \infty) \times \mathbb{R}$ *as* $N \to \infty$.

Literature

- Ankirchner, Kruse. *Optimal trade execution under price-sensitive risk preferences*. 2011.
- Ankirchner, Kruse. *Price-sensitive liquidation in continuous-time*. 2011.
- Kratz, Schöneborn. *Optimal liquidation in dark pools.* SSRN 2010.

Conclusion

- We present a liquidation model with a price sensitive risk functional
- A device that allows to introduce skewness in the revenue / cost distribution
- Trading speed increases if prices move into an unfavorable direction
- Inflator is characterized in terms of a PDE
- A flexible and numerically efficient way to derive time consistent liquidation paths

Thank you!