Differentiable approximation of Lévy and fractional processes

Haidar Al-Talibi, Astrid Hilbert, Vassili Kolokoltsov

Institutionen för datavetenskap, fysik och matematik
Linnéuniversitetet

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Previous Results and Physical Background

S. Albeverio, A. H., V- Kolokoltsov

Systems of stochastic Newton / Hamilton equations in Euclidean space given by:

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\begin{aligned}
dx(t) &= v(t)dt, \qquad x(0) = x_0, \\
dv(t) &= (-\beta v(t)dt) + K(x(t))dt + dw_t, \qquad v(0) = v_0
\end{aligned}
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where w is standard Brownian motion, K allows for a strong solution, $t \geq 0$, and $\beta \in \boldsymbol{R}$.

Qualitative problems, asymptotic behaviour for small times and parameters.

The solution $(x(t), v(t))$ is a degenerate diffusion on the cotangent bundle with possibly hypoelliptic generator.

Generalizations: Geodesic flow and driving Lévy processes

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The OU-process

For the physical Ornstein Uhlenbeck theory of motion, given by a second order SDE on $\boldsymbol{R}^d$, the solution of the corresponding system on the cotangent bundle ($\boldsymbol{R}^{2d}$) is given by:

$$v_t = e^{-\beta t} v_0 + \int_0^t e^{-\beta(t-u)} \, dB_u,$$

which is called Ornstein-Uhlenbeck velocity process, and

$$x_t = x_0 + \int_0^t e^{-\beta s} v_0 ds + \int_0^t \int_0^s e^{-\beta s} e^{\beta u} \, dB_u ds, \tag{1}$$

which is called Ornstein-Uhlenbeck position process. The initial values are given by $(x_0, v_0) = (x(0), v(0))$ and $t \geq 0$.

In Nelson's notation the noise B is Gaussian with variance $2\beta^2 D$ with $2\beta^2 D = 2\frac{\beta k T}{m}$ and physical constants k, T, m in order to match Smolouchwsky's constants.

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The Smoluchowski-Kramers Limit

Let (x, v) be the solution of the system

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where w is standard Brownian motion in $\boldsymbol{R}^\ell$.

Theorem Let (x, v) satisfy the equation above and assume that b is a function in $\boldsymbol{R}^\ell$ satisfying a global Lipschitz condition. Moreover assume that w is standard BM and y solves the equation

$$dy(t) = b(y(t), t)dt + dw(t) \qquad y(0) = x_0 \ .$$

Then for all x_0 with probability one

$$\lim_{\beta \to \infty} x(t) = y(t),$$

uniformly for t in compact subintervals of $[0, \infty)$.

Remark: Ramona Westermann: smoother noise $\frac{1}{\delta}\int_0^t \xi_{\frac{s}{\delta}} ds$, ξ Gaussian
"Application to manifolds" D. Elworthy Abel Symposium 2005

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Modified OU-process

Here we introduce a modified Ornstein-Uhlenbeck position process driven by βX_t, where $\{X_t\}_{t\geq 0}$ is an α-stable Lévy process, $0 < \alpha < 2$ and $\beta > 0$ is a scaling parameter as above

$$x_t = x_0 + \int_0^t e^{-\beta s} v_0 ds + \int_0^t \int_0^s e^{-\beta(s-u)} \beta b(x_s) du ds + \int_0^t \int_0^s e^{-\beta(s-u)} \beta dX_u ds. \quad (2)$$

For arbitrary Lévy processes Y the characteristic function is of the form $\phi_{Y_t}(u) = e^{t\eta(u)}$ for each $u \in \boldsymbol{R},\ t \geq 0$, η being the Lévy-symbol of $Y(1)$.

e.g. Applebaum, Samorodnitsky and Taqqu, Sato

We concentrate on α-stable Lévy processes with Lévy-symbol:

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Time change

Proposition Assume that Y is an α-stable Lévy process, $0 < \alpha < 2$, and g is a continuous function on the interval $[s, t] \subset T \subsetneq \boldsymbol{R}$.

Let η be the Lévy symbol of Y_1 and
ξ be the Lévy symbol of $\psi(t) = \int_s^t g(r)\, dY_r$.

Then we have

$$\xi(u) = \int_s^t \eta(ug(r))\, dr \, .$$

cf. Lukacs

For $g(\ell) = e^{\beta(\ell - t)}, \ell \geq 0$ and the α-stable process X as above the symbol of $Z_t = \int_s^t e^{\beta(r-t)}\, dX_r$ is:

$$\xi(u) = \int_s^t e^{\alpha\beta(r-t)}\, dr \cdot \eta(u) = \frac{1}{\alpha\beta}\left(1 - e^{-\alpha\beta t}\right)\eta(u)$$

with η as above, and $0 \leq s \leq t$.

Time Change - α-stable case

Recall: For the α-stable process X, $0 < \alpha < 2$, the symbol of $Z_t = \int_s^t e^{\beta(r-t)}\, dX_r$ is:

$$\frac{1}{\alpha\beta}\left(1 - e^{-\alpha\beta t}\right)\eta(u)$$

with η, η_1 as above respectively, and $0 \leq s \leq t$.

We are thus lead to introduce the time change:

$$\tau^{-1}(t) = \frac{1}{\alpha\beta}\left(1 - e^{-\alpha\beta t}\right)$$

which is actually deterministic.

This means that X_t and $Z_{\tau^{-1}(t)}$ have the same distribution.

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Approximation Theorem

Theorem 1 Let x be the position process (2) and assume that b is a function in $\boldsymbol{R}^\ell$ satisfying a global Lipschitz condition. Moreover assume that X an α-stable and y solves the equation

$$dy(t) = b(y(t), t)dt + dX(t) \qquad y(0) = x_0 \ .$$

Then for all x_0 with probability one

$$\lim_{\beta \to \infty} x(t) = y(t),$$

uniformly for t in compact subintervals of $[0, \infty)$.

For $b = 0$ the increment of the OU position process is

$$x_{t_2} - x_{t_1} = \int_{t_1}^{t_2} e^{-\beta s} v_0 ds + \int_{t_1}^{t_2} \int_0^s e^{-\beta s} e^{\beta u} \beta \, dX_u ds. \tag{3}$$

For the part of the double integral which reveals the limiting increment we use partial integration to have

$$e^{-\beta s} \beta \int_{t_1}^{t_2} \int_{t_1}^{s} e^{\beta u} \, dX_u ds = -e^{-\beta t_2} \int_{t_1}^{t_2} e^{\beta u} \, dX_u + (X_{t_2} - X_{t_1}) \tag{4}$$

By introducing a time change, on the right hand side of (4) we obtain

$$-e^{-\beta t_2} \int_{t_1}^{t_2} e^{-\beta(t_2-u)} \, dX_u = Z_{\frac{1}{\alpha\beta}(1-e^{-\alpha\beta\Delta t})} = \frac{1}{\sqrt[\alpha]{\beta}} Z_{\frac{1}{\alpha}(1-e^{-\alpha\beta\Delta t})}$$

The time changed process Z is an α-stable Lévy process. If $\beta \to 1$ then $e^{-\alpha\beta\Delta t}$ tends to zero and $Z_{\frac{1}{\alpha}(1-e^{-\alpha\beta\Delta t})}$ converges to $Z_{\frac{1}{\alpha}}$.

The product $\frac{1}{\sqrt[\alpha]{\beta}} Z_{\frac{1}{\alpha}(1-e^{-\beta\alpha\Delta t})}$ tends to zero almost surely for large β.
The result also holds for $b(y(t), t) \neq 0$ e.g. by applying the technique of Nelson to the nonlinear term.

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Approximation Theorem - fractional BM

Recalling the scaled, modified Ornstein Uhlenbeck process:

$$x_t = x_0 + \int_0^t e^{-\beta s} v_0 ds + \int_0^t \int_0^s e^{-\beta(s-u)} \beta b(x_s) du ds + \int_0^t \int_0^s e^{-\beta s} e^{\beta u} \beta dB_u^H ds. \quad (5)$$

where $\{B_t^H\}_{t \geq 0}$ is Fractional Brownian motion with index H, $0 < H < 1$, $\beta > 0$, and b is a function in $\boldsymbol{R}^\ell$ satisfying a linear growth condition.
Existence of a pathwise unique solution via a Girsanov theorem with the Ornstein Uhlenbeck process as reference plus Yamada Watanabe theorem.

Theorem 2 Let the position process x and b be as above. Moreover assume that $\{B_t^H\}_{t \geq 0}$ is Fractional Brownian motion and y solves the equation

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Do a change of measure – only the nonlinear part of the drift -
Ornstein Uhlenbeck part remains.

Existence of solutions Nualart and Ouknine resp. Rascanu.

See also: Boufoussi and C.A.Tudor.

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