Optimal control of SDEs associated with general Lévy generators

Jiang-Lun Wu

Department of Mathematics, Swansea University, UK

EPSRC Workshop on *Topics on Control*
Institute of Mathematics, Warwick University, UK
30th Nov. – 2nd Dec., 2011

Based on joint works with Jonathan Bennett

[1] Explicit construction of SDEs associated with polar-decomposed Lévy measures and application to stochastic optimization, Frontiers of Mathematics in China **2** (2007), 539–558.

[2] An optimal control problem associated with SDEs driven by Lévy-type processes, Stochastic Analysis and Applications, **26** (2008), 471–494.

[3] Stochastic control of SDEs associated with Lévy generators and application to financial optimization, Frontiers of Mathematics in China **5** (2010), 89–102.

and [4] JLW, Harry Zheng, On an optimal portfolio-comsumption problem associated with Lévy-type generators, in progress.

A fairly large class of Markov processes on $\mathbb{R}^d$ are governed by Lévy generator, either via martingale problem (cf e.g. D W Stroock, “Markov Processes from K. Itô’s Perspectives”, Princeton Univ Press 2003 or V.N. Kolokoltsov, “Markov Processes, Semigroups and Generators”, de Gruyter, 2011) or via Dirichlet form (cf e.g. N Jacob,“Pseudo-Differential Operators and Markov Processes III” Imperial College Press, 2005)

$$
\begin{aligned}
Lf(t,x) &:= \frac{1}{2}a^{i,j}(t,x)\partial_i\partial_j f(t,x) + b^i(t,x)\partial_i f(t,x) \\
&\quad + \int_{\mathbb{R}^d\setminus\{0\}} \{f(t,x+z) - f(t,x) \\
&\qquad - \frac{z1_{\{|z|<1\}}\cdot\nabla f(t,x)}{1+|z|^2}\}\nu(t,x,dz)
\end{aligned}
$$

where $a(t,x) = (a^{i,j}(t,x))_{d\times d}$ is non-negative definite symmetric and $\nu(t,x,dz)$ is a Lévy kernel, i.e.,

$\forall (t,x) \in [0,\infty) \times \mathbb{R}^d$, $\nu(t,x,\cdot)$ is a σ-finite measure on $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ such that

$$\int_{\mathbb{R}^d\setminus\{0\}} \frac{|z|^2}{1+|z|^2}\nu(t,x,dz) < \infty.$$

For such L, in order to get rid of variable dependence on ν, N El Karoui and J P Lepeltier (Z. Wahr. verw. Geb. 39 (1977)) construct a bimeasurable bijection

$$c : [0,\infty) \times \mathbb{R}^d \times U \to \mathbb{R}^d \setminus \{0\}$$

such that

$$\int_U 1_A(c(t,x,y))\lambda(dy) = \int_{\mathbb{R}^d\setminus\{0\}} 1_A(z)\nu(t,x,dz), \quad \forall (t,x)$$

for $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. Where $(U, \mathcal{B}(U))$ is a Lusin space and λ is a σ-finite measure on it. Actually, we can construct c explicitly in case ν has a polar decomposition (with the stable-like case as a concrete example). It is well-known (cf e.g. Theorem I.8.1 in N Ikeda and S Watanabe's book): $\exists$ a Poisson random measure

$$N : \mathcal{B}([0,\infty)) \times \mathcal{B}(U) \times \Omega \to \mathbb{N} \cup \{0\} \cap \{\infty\}$$

on any given probability set-up $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\geq 0})$ with $\mathbf{E}(N(dt, dy, \cdot)) = dt\lambda(dy)$, and

$$\tilde{N}(dt, dy, \omega) := N(dt, dy, \omega) - dt\lambda(dy)$$

being the associated compensating $\{\mathcal{F}_t\}_{t\geq 0}$-martingale measure.

We then can formulate a jump SDE associated with L

$$dS_t = b(t, S_t)dt + \sigma(t, S_t)dW_t + \int_U c(t, S_{t-}, y)\tilde{N}(dt, dy)$$

where $\sigma(t, x)$ is a $d \times m$-matrix such that

$$\sigma(t, x)\sigma^T(t, x) = a(t, x)$$

and $\{W_t\}_{t\in[0,\infty)}$ is an m-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion. We shall consider such equation in the following general formulation

$$\begin{aligned} dS_t &= b(t,S_t)dt + \sigma(t,S_t)dW_t \\ &\qquad + \int_{U\setminus U_0} c_1(t,S_{t-},z)\tilde{N}(dt,dz) \\ &\qquad + \int_{U_0} c_2(t,S_{t-},z)N(dt,dz) \end{aligned}$$

where $U_0 \in \mathcal{B}(U)$ with $\lambda(U_0) < \infty$ is arbitrarily fixed.

Sufficient Maximum Principle

Framstad,Øksendal, Sulem (J Optim Theory Appl 121 (2004))

Øksendal, Sulem ("Applied Stochastic Control of Jump-Diffusions", Springer, 2005); Math Finance 19 (2009); SIAM J Control Optim 2010; Commun Stoch Anal 4 (2010)

Start with a controlled jump Markov process

$$S_t = S_t^{(u)}, \quad t \in [0, T]$$

for any arbitrarily fixed $T \in (0, \infty)$, by the following

$$\begin{aligned} dS_t &= b(t, S_t, u_t)dt + \sigma(t, S_t, u_t)dW_t \\ &\quad + \int_{U \setminus U_0} c_1(t, S_{t-}, u_{t-}, z)\tilde{N}(dt, dz) \\ &\quad + \int_{U_0} c_2(t, S_{t-}, u_{t-}, z)N(dt, dz) \end{aligned} \tag{1}$$

where the control process $u_t = u(t, \omega)$, taking values in a given Borel set $\mathcal{U} \in \mathcal{B}(\mathbb{R}^d)$, is assumed to be $\{\mathcal{F}_t\}$-predictable and cádlág.

The performance criterion is

$$J(u) := \mathbf{E}\left(\int_0^T f(t, S_t, u_t)dt + g(S_T)\right), \quad u \in \mathcal{A}$$

for $\mathcal{A}$ the totality of all admissible controls, and for

$$f : [0, T] \times \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}$$

being continuous, and for $g : \mathbb{R}^d \to \mathbb{R}$ being concave. The objective is to achieve the following

$$J(\hat{u}) = \sup_{u \in \mathcal{A}} J(u)$$

referring $\hat{u}$ being the optimal control of the system. Moreover, if $\hat{S}_t = S_t^{(\hat{u})}$ is the solution to the jump type SDE (1) corresponding to $\hat{u}$, then the pair $(\hat{S}, \hat{u})$ is called *the optimal pair*.

Now the Hamiltonian is defined

$$H : [0, T] \times \mathbb{R}^d \times \mathcal{U} \times \mathbb{R}^d \times \mathbb{R}^{d\otimes m} \times \mathcal{R} \to \mathbb{R}$$

via

$$\begin{aligned} & H(t, r, u, p, q, n^{(1)}, n^{(2)}) \\ = \; & f(t, r, u) + \mu(t, r, u)p + \frac{1}{2}\sigma^T(t, r, u)q \\ & + \int_{U\setminus U_0} n^{(1)}(t, z)c_1(t, r, u, z)\lambda(dz) \\ & + \int_{U_0} [n^{(2)}(t, z)c_2(t, r, u, z) + c_2(t, r, u, z)p]\lambda(dz) \end{aligned}$$

where $\mathcal{R}$ is the collection of all $\mathbb{R}^{d\otimes d}$-valued processes $n : [0, \infty) \times \Omega \to \mathbb{R}^{d\otimes d}$ such that the two integrals in the above formulation converge absolutely.

It is known that the adjoint equation corresponding to an admissible pair (S, u) is the BSDE

$$\begin{aligned} dp(t) &= -\nabla_r H(t, S_t, u_t, p(t), q(t), n^{(1)}(t,\cdot), n^{(2)}(t,\cdot))dt \\ &\quad + q(t)dW_t + \int_{U\setminus U_0} n^{(1)}(t-, z)\tilde{N}(dt, dz) \\ &\quad + \int_{U_0} n^{(2)}(t-, z)N(dt, dz) \end{aligned}$$

with terminal condition

$$p(T) = \triangledown g(S_T)\,.$$

Theorem ([3])

Given an admissible pair $(\hat{S}, \hat{u})$. Suppose $\exists$ an $\{\mathcal{F}_t\}$-adapted solution $(\hat{p}(t), \hat{q}(t), \hat{n}(t,z))$ to the BSDE s.t. for $u \in \mathcal{A}$

$$\mathbf{E}[\int_0^T (\hat{S}_t - S_t^{(u)})^T \{\hat{q}(t)\hat{q}(t)^T + \int_{U_0} [tr(\hat{n}(t,z)\hat{n}(t,z)^T)\lambda(dz)]\} \times (\hat{S}_t - S^{(u)}(t))dt] < \infty\,,$$

$$\mathbf{E}[\int_0^T \hat{p}^T(t)\{\int_{U_0} [tr(c(t, S_{t-}, u_t, z)c^T(t, S_{t-}, u_t, z))\lambda(dz)] + \sigma(t, S_t, u_t)\sigma^T(t, S_t, u_t)\}\hat{p}(t)dt] < \infty,$$

Theorem (cont'd)

and $\forall t \in [0, T]$

$$H(t, \hat{S}_t, \hat{u}_t, \hat{p}(t), \hat{q}(t), \hat{n}(t, .)) = \sup_{u \in \mathcal{A}} H(t, \hat{S}_t, u_t, \hat{p}(t), \hat{q}(t), \hat{n}(t, .)). \quad (2)$$

If $\hat{H}(r) := max_{u \in \mathcal{A}} H(t, r, u, \hat{p}(t), \hat{q}(t), \hat{n}(t, \cdot))$ exists and is a concave function of r, then $(\hat{S}, \hat{u})$ is an optimal pair.

Remark For (2), it suffices that the function

$$(r, u) \to H(t, r, u, \hat{p}(t), \hat{q}(t), \hat{n}(t, \cdot))$$

is concave, $\forall t \in [0, T]$.

Optimal control problem

Benth, Karlsen, Reikvam (Finance Stoch 5 (2001); Stochastics Stochastics Rep 74(2002))

Ishikawa (Appl Math Optim 50 (2004))

Jakobsen, Karlsen (JDE 212 (2005); NoDEA 13 (2006))

Start with a Lévy type process

$$
\begin{aligned}
Z_t \;=\; & \mu t + \int_0^t \theta(s) dW_s + \int_0^t \int_{U \setminus U_0} c_1(z) \tilde{N}(ds, dz) \\
& + \int_0^t \int_{U_0} c_2(z) N(ds, dz)
\end{aligned}
$$

where μ is a constant, $\theta : [0, T] \to \mathbb{R}$ and $c_1, c_2 : U \to \mathbb{R}$ are measurable. Here assume that

$$
\int_{U_0} (e^{c_2(z)} - 1) \lambda(dz) < \infty.
$$

We are concerned with the following 1-dimensional linear SDE

$$
\begin{aligned}
dS_t &= b(t)S_t dt + \frac{1}{2}\sigma(t)^2 S_t dt + \sigma(t) S_t dW_t \\
&+ S_t \int_U (e^{c_1(z)} - 1 - c_1(z) 1_{\{U \setminus U_0\}}(z)) \lambda(dz) dt \\
&+ S_{t-} \int_U (e^{c_1(z)} - 1) \tilde{N}(dt, dz).
\end{aligned}
$$

Based on the driving processes Z_t and S_t, we construct two processes X_t and Y_t with $X_0 = x$, $Y_0 = y$, via

$$
\begin{aligned}
X_t = x - G_t + \int_0^t \sigma(s)\pi_s X_s dW_s + L_t \\
+ \int_0^t (r + ([b(s) + \frac{1}{2}\sigma(s)^2 + \int_{U\setminus U_0} (e^{c_1(z)} \\
-1 - c_1(z))\lambda(dz)] - r)\pi_s) X_s ds \\
+ \int_0^t \pi_{s-} X_{s-} \int_{U\setminus U_0} (e^{c_1(z)} - 1)\tilde{N}(ds, dz) \\
+ \int_0^t \pi_{s-} X_{s-} \int_{U_0} (e^{c_2(z)} - 1) N(ds, dz)
\end{aligned}
$$

and

$$
Y_t = ye^{-\beta t} + \beta \int_0^t e^{-\beta(t-s)} dG_s
$$

respectively, where

$$G_t := \int_0^t g_s ds$$

with $(g_t)_{t\geq 0}$ being a nondecreasing $\{\mathcal{F}_t\}$-adapted cádlág process of finite variation such that $0 \leq \sup_{t\geq 0} g_t < \infty$, L_t is a nondecreasing, nonnegative, and $\{\mathcal{F}_t\}$-adapted cádlág process, and $\pi_t \in [0, 1]$ is $\{\mathcal{F}_t\}$-adapted cádlág. The triple (G_t, L_t, π_t) is referred as the parameter process.

Remark The background for X_t being the self-financing investment policy according to the portfolio π_t:

$$\frac{dX_t}{X_{t-}} = (1 - \pi_t)\frac{dB_t}{B_t} + \pi_t \frac{dS_t}{S_{t-}}$$

with B_t standing for the riskless bond $dB_t = rB_t dt$.

By Itô formula, the generator A to (X_t, Y_t) is

$$
\begin{aligned}
Av(x,y) &= -\alpha v - \beta y v_y + \sigma(t)\pi x v_{xx} \\
&\quad + \{(r + \pi([b(t) + \frac{1}{2}\sigma(t)^2 \\
&\quad + \int_{U \setminus U_0} (e^{c_1(z)} - 1 - c_1(z))\lambda(dz)] - r))x v_x \\
&\quad + \int_{U \setminus U_0} (v(x + \pi x(e^{c_1(z)} - 1), y) \\
&\qquad - v(x,y) - \pi x v_x (e^{c_1(z)} - 1))\lambda(dz) \\
&\quad + \int_{U_0} (v(x + \pi x(e^{c_2(z)} - 1), y) - v(x,y))\lambda(dz)\} \\
&\quad + u(g) - g(v_x - \beta v_y)
\end{aligned}
$$

for any $v \in C^{2,2}(\mathbb{R} \times \mathbb{R})$ and for $\pi \in [0,1], g \in [0, M_1]$.

Now we define the value function

$$v(x, y) := \sup_{(\pi.,g.,L.)\in\mathcal{A}} \mathbf{E}^{(X^{(\pi.,g.,L.)},Y^{(\pi.,g.,L.)})}[\int_0^\infty e^{-\alpha s}u(g_s)ds]$$

where the supremum is taken over all admissible controls and u is a utility function, i.e., u is strictly increasing, differential, and concave on $[0, \infty)$ such that

$$u(0) = u'(\infty) = 0, \quad u(\infty) = u'(0) = \infty.$$

We also denote that

$$\begin{aligned} k(\gamma,\rho) \quad := \quad & \max_{\pi}\Big\{\gamma(r+\pi([b(t)+\frac{1}{2}\sigma(t)^2 \\ & +\int_{U\setminus U_0}(e^{c_1(z)}-1-c_1(z))\lambda(dz)]-r)) \\ & +\sigma(t)\pi\rho+\int_{U\setminus U_0}[(1+\pi(e^{c_1(z)}-1))^{\gamma} \\ & -1-\gamma\pi(e^{c_1(z)}-1)) \\ & +\int_{U_0}(1+\pi(e^{c_2(z)}-1))^{\gamma}-1]\lambda(dz)\Big\}. \end{aligned}$$

Theorem ([2])

[i](Existence) v is well-defined, i.e., there exists an optimal control $(\pi^*, g^*, L^*) \in \mathcal{A}$ such that

$$v(x,y) = \mathbf{E}^{(X^{(\pi^*,g^*,L^*)}, Y^{(\pi^*,g^*,L^*)})}[\int_0^\infty e^{-\alpha s} u(g_s^*) ds].$$

Furthermore, v is a constrained viscosity solution to the following Hamilton-Jacobi-Bellman integro-variational inequality

$$\max \left\{ v_x 1_{\{x \le 0\}}, \sup_{(\pi,g) \in \mathcal{A}} \{Av\}, (\beta v_y - v_x) 1_{\{x \ge 0\}} \right\} = 0$$

in $D_\beta := \{(x,y) : y > 0, y + \beta x > 0\}$, and

$$v = 0 \quad \text{outside of} \quad D_\beta.$$

Theorem (cont'd)

[ii] (Uniqueness) For $\gamma > 0$ and each $\rho \geq 0$ choose $\alpha > 0$ s.t. $k(\gamma, \rho) < \alpha$. Then the HJB integro-variational inequality admits at most one constrained viscosity solution.

Theorem (JLW and H. Zheng [4])

Assume the following dynamic programming principle hold for the value function v: $\forall t \geq 0$ and for any stopping time τ

$$
\begin{aligned}
v(x,y) \quad = \quad & sup_{(\pi,g,L)\in\mathcal{A}}\mathbf{E}\left[\int_0^{t\wedge\tau} e^{-\alpha s}u(g_s^*)ds\right. \\
& \left. + e^{-\alpha(t\wedge\tau)}v(X_{t\wedge\tau}^{(\pi,g,L)}, Y_{t\wedge\tau}^{(\pi,g,L)})\right].
\end{aligned}
$$

Then, v is the unique, constrained (subject to a gradient constraint) viscosity solution of the following integro-differential HJB equation

$$
\max\left\{\beta v_y - v_x, \sup_{(\pi,g)\in\mathcal{A}}\{Av\}\right\} = 0
$$

in $D := \{(x,y) : x > 0, y > 0\}$.

Further discussion on the properties of v is under way.

The case of polar-decomposed Lévy measures

Recall the Lévy generator

$$
\begin{aligned}
Lf(t,x) &:= \frac{1}{2}a^{i,j}(t,x)\partial_i\partial_j f(t,x) + b^i(t,x)\partial_i f(t,x) \\
&\quad + \int_{\mathbb{R}^d\setminus\{0\}} \{f(t,x+z) - f(t,x) \\
&\qquad - \frac{z1_{\{|z|<1\}}\cdot\nabla f(t,x)}{1+|z|^2}\}\nu(t,x,dz)
\end{aligned}
$$

and the associated SDE

$$
dS_t = b(t,S_t)dt + \sigma(t,S_t)dW_t + \int_U c(t,S_{t-},y)\tilde{N}(dt,dy)
$$

Here we consider a special case: ν admits a polar-decomposition.

$$(U, \mathcal{B}(U), \lambda) = (\mathrm{S}^{d-1} \times (0, \infty), \lambda)$$

where λ is σ-finite. Now let

m: a finite Borel measure on S^{d-1}

$z : \mathbb{R}^d \times \mathrm{S}^{d-1} \times (0, \infty) \to \mathbb{R}^d \setminus \{0\}$ bimeasurable bijection

$g : \mathbb{R}^d \times \mathrm{S}^{d-1} \times \mathcal{B}((0, \infty)) \to (0, \infty)$ is a positive kernel

Our ν is then taken the form

$$\nu(x, dz) = \int_{\mathrm{S}^{d-1}} \int_0^\infty 1_{dz}(z(x, \theta, r)g(x, \theta, dr)$$

M Tsuchiya, Stoch Stoch Reports 38 (1992)
V. Kolkoltsov, Proc London Maths Soc 80 (2000)
V. Kolkoltsov, *Nonlinear Markov Processes and Kinetic Equations.* (CUP, 2010)

Example (Bass, PTRF (1988); Kolkoltsov) Take

$$z(x,\theta,r) = r\theta \quad \text{and} \quad g(x,\theta,dr) = \frac{dr}{r^{1+\alpha(x)}}$$

then

$$\nu(x,dz) = \frac{dr}{r^{1+\alpha(x)}} m(d\theta)$$

Theorem ([1])

(i) For $d \geq 2$, i.e., for the case that the given σ-finite measure space

$$U = (S^{d-1} \times (0, \infty))$$

the coefficient of the jump term in the SDE associated to $\nu(x, dz)$ is given by $c(t, x, (r, \theta)) = r\theta$;
(ii) For the case when $d = 1$, namely, for the case that the given σ-finite measure space

$$(U, \mathcal{B}(U), \lambda) = ((0, \infty), \mathcal{B}((0, \infty)), \lambda)$$

the coefficient of the jump term in the SDE associated to $\nu(x, dz)$ defined by

$$\nu(x, dz) = \frac{dr}{r^{1+\alpha(x)}} \quad \alpha(x) \in (0, 2), \quad x \in \mathbb{R}$$

is given by $|c(t, x, (r, \theta))| = r$.

As an application, we consider a consumption-portfolio optimzation problem. The wealth process is modelled via

$$\begin{aligned} dS(t) &= \{\rho_t S(t) + (b(t) - \rho_t)u(t) - w(t)\}dt \\ &\quad + \sigma(t)w(t)dW(t) \\ &\quad + w(t-)\int_{0<|r|<1} r\theta \tilde{N}(dt, drd\theta) \\ &\quad + w(t-)\int_{|r|\geq 1} r\theta N(dt, drd\theta). \end{aligned}$$

Our objective is to solve the following consumption-portfolio optimization problem:

$$\sup_{(w,u)\in\mathcal{A}} \mathbf{E}[\int_0^T \exp(-\int_0^t \delta(s)ds)[\frac{w(t)^\gamma}{\gamma}]dt] \tag{3}$$

subject to the terminal wealth constraint

$$S(T) \geq 0 \quad a.s.$$

where $\mathcal{A}$ is the set of predictable consumption-portfolio pairs (w, u) with the control u being tame and the consumption w being nonnegative, such that the above SDE has a strong solution over $[0, T]$.

Theorem ([1])

An optimal control (u^*, w^*) is given by

$$u^*(t,x) = exp\left\{\int_0^t \frac{\delta(s)}{\gamma - 1} ds\right\} f(t)^{\frac{1}{\gamma-1}} x$$

and

$$w^*(t,x) = \hat{\pi} x$$

with $f(t)$ and $\hat{\pi}$ being explicitly constructed.

Thank You!