Game theoretic analysis of incomplete markets [1]

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Highlights

• Generalizations of classical BS and CRR formulae with more **rough assumptions** on the underlying assets evolution: interval model.
• **Transaction costs** included.
• **Emergence of risk neutral probabilities in minmax** (robust control) evaluations.
• A natural **unique selection among multiple risk neutral measures** arising in incomplete markets for options specified by **sub-modular functions**.
• Continuous time limit leading to **nonlinear degenerate and/or fractional Black-Scholes** type equations.
• Explicit formulae and new numeric schemes. **Identification of pre-Markov models**.

Geometric risk-neutral laws, I

Def. A probability law $\mu \in \mathcal{P}(E)$ on $E \subset \mathbf{R}^d$ is *risk-neutral* if the origin is its barycenter: $\int_E \xi \mu(d\xi) = 0$. Denote $\mathcal{P}_{rn}(E)$ the set of risk-neutral laws.
More generally: for a compact $E \subset \mathbf{R}^n$ and a continuous $F : E \to \mathbf{R}^d$ let

$$\mathcal{P}(E; F) = \{\mu \in \mathcal{P}(E) : (F, \mu) = \int F(x)\mu(dx) = 0\}.$$

$\mathcal{P}_{rn}(E) = \mathcal{P}(E; Id)$.
Def. E is called *weakly* (resp. *strongly*) *positively complete*, if there exists no $\omega \in \mathbf{R}^d$ such that $(\omega, \xi) > 0$ (resp. $(\omega, \xi) \geq 0$) for all $\xi \in E$.
Geometrically: E does not belong to any open (respectively closed) half-space of $\mathbf{R}^d$.
If $E \subset \mathbf{R}^d$ is a compact convex set, then E is weakly positively complete if and only if it contains the origin.

Geometric risk-neutral laws, II

Proposition

Let $E \subset \mathbf{R}^n$ *be compact and a mapping* $F : E \to \mathbf{R}^d$ *continuous.*
(i) The set $\mathcal{P}(E; F)$ *is not empty if and only if* $F(E)$ *is weakly positively complete in* $\mathbf{R}^d$*.*
(ii) Let E' *be the support of a measure* $\mu \in \mathcal{P}(E; F)$*. If* $F(E')$ *does not coincide with the origin, then it is strongly positively complete in the subspace* $\mathbf{R}^m \subset \mathbf{R}^d$ *generated by* $F(E')$*.*

Proposition

Let $E \subset \mathbf{R}^n$ *be compact, a mapping* $F = (F^1, \cdots, F^d) : E \to \mathbf{R}^d$ *be continuous and* μ *be an extreme point of the set* $\mathcal{P}(E; F)$*. Then* μ *is a linear combination of not more than* $d + 1$ *Dirac measures.*

Geometric risk-neutral laws, III

Proposition

Example. Let $E = \{\xi_1, \cdots, \xi_{d+1}\}$ *be strongly positively complete in* $\mathbf{R}^d$*. Then there exists a unique risk-neutral probability law* $\{p_1, \cdots, p_{d+1}\}$ *on* $\{\xi_1, \cdots, \xi_{d+1}\}$*:* p_i *equals the ratio of the volume of the pyramids* $\Pi[\{0 \cup \{\hat{\xi}_i\}\}]$ *to the whole volume* $\Pi[\xi_1, \cdots, \xi_{d+1}]$*.*

($\{\hat{\xi}_i\}$ denotes the family $\xi_1, \cdots, \xi_{d+1}$ with ξ taken out).

Theorem

Let a compact set $E \subset \mathbf{R}^d$ *be strongly positively complete. Then the extreme points of the set of risk-neutral probabilities on* E *are the Dirac mass at zero (only when* E *contains the origin) and the risk-neutral measures with support on families of size* $m+1$*,* $0 < m \leq d$*, that generate a subspace of dimension* m *and are strongly positively complete in this subspace.*

Underlying Game, I

$$\Pi[\xi_1, \cdots, \xi_{d+1}](f) = \min_{\gamma \in \mathbf{R}^d} \max_i [f(\xi_i) - (\xi_i, \gamma)], \tag{1}$$

and

$$\underline{\Pi}[\xi_1, \cdots, \xi_{d+1}](f) = \max_{\gamma \in \mathbf{R}^d} \min_i [f(\xi_i) + (\xi_i, \gamma)], \tag{2}$$

where $\xi_1, \cdots, \xi_{d+1}$ are $d+1$ vectors in $\mathbf{R}^d$ in general position (origin is in the interior of their convex hull).
A remarkable fact: expressions (1) and (2) are linear in f and the minimizing γ is unique and also depends linearly on f.

Underlying Game II

Proposition

$\xi_1, \cdots, \xi_{d+1}$ are $d+1$ vectors in $\mathbf{R}^d$ in general position. Then

$$\Pi[\xi_1, \cdots, \xi_{d+1}](f) = \underline{\Pi}[\xi_1, \cdots, \xi_{d+1}](f) = \mathbf{E} f(\xi), \quad (3)$$

where $\mathbf{E}$ is with respect to the unique risk neutral law on $\{\xi_i\}$, and the minimum in (1) is attained on the single γ_0:

$$\gamma_0 = \mathbf{E}[f(\xi) r(\xi)] \quad (4)$$

with explicitly defined vectors $r(\xi)$,

$$|\gamma_0| \leq \|f\| \frac{1}{d} \frac{S(\xi_1, \cdots, \xi_{d+1})}{V(\xi_1, \cdots, \xi_{d+1})}, \quad (5)$$

where $S(\xi_1, \cdots, \xi_{d+1})$ is the surface volume of the pyramid $\Pi[\xi_1, \cdots, \xi_{d+1}]$ and $V(\xi_1, \cdots, \xi_{d+1})$ is its volume.

Underlying Game III

For a compact E and a continuous function f define

$$\Pi[E](f) = \inf_{\gamma \in \mathbf{R}^d} \max_{\xi \in E} [f(\xi) - (\xi, \gamma)] \tag{6}$$

and

$$\underline{\Pi}[E](f) = \sup_{\gamma \in \mathbf{R}^d} \min_{\xi \in E} [f(\xi) - (\xi, \gamma)]. \tag{7}$$

Theorem

Let a compact set $E \subset \mathbf{R}^d$ be strongly positively complete. Then

$$\Pi[E](f) = \max_{\mu} \mathbf{E}_{\mu} f(\xi), \quad \underline{\Pi}[E](f) = \min_{\mu} \mathbf{E}_{\mu} f(\xi) \tag{8}$$

where max *(resp.* min*) is taken over all extreme points μ of risk-neutral laws on E given by Proposition 1,* inf *in* (6) *is attained on some γ (satisfying the estimates above).*

Underlying Game: nonlinear extension

$$\Pi[\xi_1, \cdots, \xi_k](f) = \min_{\gamma \in \mathbf{R}^d} \max_{\xi_1, \cdots, \xi_k} [f(\xi_i, \gamma) - (\xi_i, \gamma)]. \qquad (9)$$

Theorem

Let $\{\xi_1, \cdots, \xi_k\} \subset \mathbf{R}^d$, $k > d$, in general position.
Let the function $f(\xi, \gamma)$ be bounded below and Lipshitz continuous in γ with a Lipschitz constant $\varkappa$, which is small enough.
Then the minimum in (9) is finite, is attained on some γ_0 and

$$\Pi[\xi_1, \cdots, \xi_k](f) = \max_I \mathbf{E}_I f(\xi, \gamma_I), \qquad (10)$$

where max *as above and γ_I is the corresponding (unique) optimal value (solving a fixed point equation).*

Other extensions: infinite-dimensional setting with one-dimensional projections, random geometry.

Mixed strategies with linear constraints, I

Equivalent form of the result above:

$$\Pi[E](f) = \inf_{\gamma \in \mathbf{R}^d} \max_{\mu \in \mathcal{P}(E)} \mathbf{E}_\mu [f(\xi) - (\gamma, \xi)] = \max_{\mu \in \mathcal{P}_{rn}(E)} \mathbf{E}_\mu f(\xi). \tag{11}$$

Let $E \subset \mathbf{R}^d$ be a compact set and $\tilde{\mathcal{P}}(E)$ a closed convex subset of $\mathcal{P}(E)$ (the main example is a set of type $\mathcal{P}(E; F)$). Let

$$\widetilde{\Pi}[E](f) = \inf_{\gamma \in \mathbf{R}^d} \max_{\mu \in \tilde{\mathcal{P}}(E)} \mathbf{E}_\mu [f(\xi) - (\gamma, \xi)]$$

$$= \inf_{\gamma \in \mathbf{R}^d} \max_{\mu \in \tilde{\mathcal{P}}(E)} \left[\int f(\xi)\mu(d\xi) - (\gamma, \int \xi \mu(d\xi)) \right]. \tag{12}$$

Let B denote the linear mapping $\tilde{\mathcal{P}}(E) \to \mathbf{R}^d$ given by

$$B\mu = \mathbf{E}_\mu \xi = \int \xi \mu(d\xi)$$

(barycenter or the center of mass).

Mixed strategies with linear constraints, II

The following main result extends Theorem 2 to the case of mixed strategies with constraints.

Theorem

The set $\tilde{\mathcal{P}}(E) \cap \mathcal{P}_{rn}(E)$ is empty if and only if the set $B(\tilde{\mathcal{P}}(E))$ is not weakly positively complete, in which case $\tilde{\Pi}[E](f) = -\infty$. Otherwise

$$\tilde{\Pi}[E](f) = \inf_{\gamma \in \mathbf{R}^d} \max_{\mu \in \tilde{\mathcal{P}}(E)} \mathbf{E}_\mu [f(\xi) - (\gamma, \xi)]$$

$$= \max_{\mu \in \tilde{\mathcal{P}}(E) \cap \mathcal{P}_{rn}(E)} \mathbf{E}_\mu f(\xi).$$

Interval model for a market

Market with several securities in discrete time $k = 1, 2, ...$:
The risk-free bonds (bank account), priced B_k,
and J common stocks, $J = 1, 2...$, priced S_k^i, $i \in \{1, 2, ..., J\}$.
$B_{k+1} = \rho B_k$, $\rho \geq 1$ is a constant interest rate,
$S_{k+1}^i = \xi_{k+1}^i S_k^i$, where $\xi_k^i, i \in \{1, 2, ..., J\}$, are unknown sequences taking values in some fixed intervals
$M_i = [d_i, u_i] \subset \mathbf{R}$ (*interval model*).
This model generalizes the colored version of the classical CRR model in a natural way.
In the latter a sequence ξ_k^i is confined to take values only among two boundary points d_i, u_i, and it is supposed to be random with some given distribution.

Rainbow (or colored) European Call Options

A premium function f of J variables specifies the type of an option.

Standard examples ($S^1, S^2, ..., S^J$ represent the expiration values of the underlying assets, and $K, K_1, ..., K_J$ represent the strike prices):

Option delivering the best of J risky assets and cash

$$f(S^1, S^2, ..., S^J) = \max(S^1, S^2, ..., S^J, K), \tag{13}$$

Calls on the maximum of J risky assets

$$f(S^1, S^2, ..., S^J) = \max(0, \max(S^1, S^2, ..., S^J) - K), \tag{14}$$

Multiple-strike options

$$f(S^1, S^2, ..., S^J) = \max(0, S^1 - K_1, S^2 - K_2, ..., S^J - K_J), \tag{15}$$

Portfolio options

$$f(S^1, S^2, ..., S^J) = \max(0, n_1 S^1 + n_2 S^2 + ... + n_J S^J - K), \tag{16}$$

Spread options: $f(S^1, S^2) = \max(0, (S^2 - S^1) - K)$.

Investor's (seller of an option) control: one step

X_k the capital of the investor at the time $k = 1, 2, \dots$. At each time $k-1$ the investor determines his portfolio by choosing the numbers γ_k^i of common stocks of each kind to be held so that the structure of the capital is represented by the formula

$$X_{k-1} = \sum_{i=1}^{J} \gamma_k^i S_{k-1}^i + (X_{k-1} - \sum_{i=1}^{J} \gamma_k^i S_{k-1}^i),$$

where the expression in bracket corresponds to the part of his capital laid on the bank account. The control parameters γ_k^i can take all real values, i.e. short selling and borrowing are allowed. The value ξ_k becomes known in the moment k and thus the capital at the moment k becomes

$$X_k = \sum_{i=1}^{J} \gamma_k^i \xi_k^i S_{k-1}^i + \rho(X_{k-1} - \sum_{i=1}^{J} \gamma_k^i S_{k-1}^i).$$

Investor's control: n step game

If n is the *maturity date*, this procedures repeats n times starting from some initial capital $X = X_0$ (selling price of an option) and at the end the investor is obliged to pay the premium f to the buyer.
Thus the (final) income of the investor equals

$$G(X_n, S_n^1, S_n^2, ..., S_n^J) = X_n - f(S_n^1, S_n^2, ..., S_n^J). \qquad (17)$$

The evolution of the capital can thus be described by the dynamic n-step game of the investor (strategies are sequences of real vectors $(\gamma_1, ..., \gamma_n)$ (with $\gamma_j = (\gamma_j^1, ..., \gamma_j^J)$)) with the Nature (characterized by unknown parameters ξ_k^i).
A position of the game at any time k is characterized by $J + 1$ non-negative numbers $X_k, S_k^1, ..., S_k^J$ with the final income specified by the function

$$G(X, S^1, ..., S^J) = X - f(S^1, ..., S^J) \qquad (18)$$

Robust control (guaranteed payoffs, worst case scenario)

Minmax payoff (guaranteed income) with the final income G in a one step game with the initial conditions $X, S^1, ..., S^J$ is given by the *Bellman operator*

$$\mathbf{B} G(X, S^1, ..., S^J)$$

$$= \max_\gamma \min_\xi G(\rho X + \sum_{i=1}^J \gamma^i \xi^i S^i - \rho \sum_{i=1}^J \gamma^i S^i, \xi^1 S^1, ..., \xi^J S^J),$$

and the guaranteed income in the n step game with the initial conditions $X_0, S_0^1, ..., S_0^J$ is

$$\mathbf{B}^n G(X_0, S_0^1, ..., S_0^J).$$

Reduced Bellman operator

Clearly for G of form $G(X, S^1, \cdots, S^J) = X - f(S^1, \cdots, S^J)$,

$$\mathbf{B} G(X, S^1, ..., S^J)$$

$$= X - \frac{1}{\rho} \min_{\gamma} \max_{\xi} [f(\xi^1 S^1, \xi^2 S^2, \cdots, \xi^J S^J) - \sum_{j=1}^{J} \gamma^j S^j (\xi^j - \rho)],$$

and hence

$$\mathbf{B}^n G(X, S^1, \cdots, S^J) = X - (\mathcal{B}^n f)(S^1, \cdots, S^J),$$

where the *reduced Bellman operator* is defined as:

$$(\mathcal{B} f)(z) = \frac{1}{\rho} \min_{\gamma} \max_{\{\xi^j \in [d_j, u_j]\}} [f(\xi \circ z) - (\gamma, \xi \circ z - \rho z)]. \quad (19)$$

Here $(\xi \circ z)^i = \xi^i z^i$ - Hadamard product.

Hedging

Main definition. A strategy $\gamma_1^i, ..., \gamma_n^i, i = 1, ..., J$, of the investor is called a *hedge*, if for any sequence $(\xi_1, ..., \xi_n)$ (with $\xi_j = (\xi_j^1, ..., \xi_j^J)$) the investor is able to meet his obligations, i.e.

$$G(X_n, S_n^1, ..., S_n^J) \geq 0.$$

The minimal value of the capital X_0 for which the hedge exists is called the *hedging price* H of an option.

Theorem (*Game theory for option pricing.*)

The minimal value of X_0 for which the income of the investor is not negative (and which by definition is the hedge price H) is given by

$$H^n = (\mathcal{B}^n f)(S_0^1, ..., S_0^J). \tag{20}$$

Risk-neutral evaluation for options: setting

A linear change of variables yields

$$(\mathcal{B}f)(z^1, ..., z^J) = \frac{1}{\rho} \min_{\gamma} \max_{\{\eta \in [z^i(d_i - \rho), z^i(u_i - \rho)]\}} [f(\eta + \rho z) - (\gamma, \eta)]. \tag{21}$$

Assuming f is convex, we are in the setting above with

$$\Pi = \Pi_{z,\rho} = \times_{i=1}^{J} [z^i(d_i - \rho), z^i(u_i - \rho)],$$

with vertices

$$\eta_I = \xi_I \circ z - \rho z, \quad \xi_I = \{d_i|_{i \in I}, u_j|_{j \notin I}\},$$

parametrized by all subsets (including the empty one) $I \subset \{1, \dots, J\}$.
Above theory reduces our dynamic game to a controlled Markov jump problem:

Risk-neutral evaluation for options: result

Theorem

suppose the vertices ξ_I are in general position: for any J subsets $I_1, \cdots, I_J$, the vectors $\{\xi_{I_k} - \rho\mathbf{1}\}_{k=1}^J$ are independent in $\mathbf{R}^J$. Then

$$(\mathcal{B}f)(z) = \max_{\{\Omega\}} \mathbf{E}_\Omega f(\xi \circ z), \quad z = (z^1, \cdots, z^J), \tag{22}$$

where $\{\Omega\}$ is the collection of subsets $\Omega = \xi_{I_1}, \cdots, \xi_{I_{J+1}}$ of the set of vertices of Π, of size $J+1$, such that their convex hull contains $\rho\mathbf{1}$ as an interior point, and where $\mathbf{E}_\Omega$ denotes the expectation with respect to the unique probability law $\{p_I\}$, $\xi_I \in \Omega$, on the set of vertices of Π, which is supported on Ω and is risk neutral with respect to $\rho\mathbf{1}$, that is

$$\sum_{I \subset \{1,\ldots,J\}} p_I \xi_I = \rho\mathbf{1}. \tag{23}$$

Sub-modular payoffs

A function $f : \mathbf{R}_+^2 \to \mathbf{R}_+$ is called *sub-modular*, if the inequality

$$f(x_1, y_2) + f(x_2, y_1) \geq f(x_1, y_1) + f(x_2, y_2)$$

holds whenever $x_1 \leq x_2$ and $y_1 \leq y_2$. A function $f : \mathbf{R}_+^d \to \mathbf{R}_+$ is called *sub-modular* if

$$f(x \bigvee y) + f(x \bigwedge y) \leq f(x) + f(y),$$

where $\bigvee$ (respectively $\bigwedge$) denotes the Pareto (coordinate-wise) maximum (respectively minimum).

Remark

If f is twice continuously differentiable, then it is sub-modular if and only if $\frac{\partial^2 f}{\partial z_i \partial z_j} \leq 0$ for all $i \neq j$.

As one easily sees, the payoffs of the first three examples of rainbow options, given at the beginning, are sub-modular.

Example J=2 (two colors)

The polyhedron Π is then a rectangle. From sub-modularity of f it follows that the maximum is always achieved either on

$$\Omega_d = \{(d_1, d_2), (d_1, u_2), (u_1, d_2)\},$$

or on

$$\Omega_u = \{(d_1, u_2), (u_1, d_2), (u_1, u_2)\}.$$

and $\mathcal{B}f$ reduces either to $\mathbf{E}_{\Omega_u}$ or to $\mathbf{E}_{\Omega_d}$ depending on a certain 'correlation coefficient' of possible jumps.

Example J=2 (two colors) continued

Theorem

Let $J = 2$, f be convex sub-modular, and denote

$$\kappa = \frac{(u_1 u_2 - d_1 d_2) - \rho(u_1 - d_1 + u_2 - d_2)}{(u_1 - d_1)(u_2 - d_2)}. \tag{24}$$

If $\kappa \geq 0$, then $(\mathcal{B}f)(z_1, z_2)$ equals

$$\frac{\rho - d_1}{u_1 - d_1} f(u_1 z_1, d_2 z_2) + \frac{\rho - d_2}{u_2 - d_2} f(d_1 z_1, u_2 z_2) + \kappa f(d_1 z_1, d_2 z_2),$$

If $\kappa \leq 0$, the $(\mathcal{B}f)(z_1, z_2)$ equals

$$\frac{u_1 - \rho}{u_1 - d_1} f(d_1 z_1, u_2 z_2) + \frac{u_2 - \rho}{u_2 - d_2} f(u_1 z_1, d_2 z_2) + |\kappa| f(u_1 z_1, u_2 z_2),$$

Example J=2 (two colors) completed

By linearity, the powers of $\mathcal{B}$ can be found. Say, if $\kappa = 0$,

$$C_h = \rho^{-n} \sum_{k=0}^{n} C_n^k$$

$$\left(\frac{\rho - d_1}{u_1 - d_1}\right)^k \left(\frac{\rho - d_2}{u_2 - d_2}\right)^{n-k} f(u_1^k d_1^{n-k} S_0^1, d_2^k u_2^{n-k} S_0^2).$$

(two-dimensional version of CRR formula).
Important: risk neutral selector.

$J > 2$ colors: reduction to a linear Bellman

Notation: for a set $I \subset \{1, 2, ..., J\}$, $f_I(z)$ (resp. $\tilde{f}_I(z)$) is $f(\xi^1 z_1, \cdots, \xi^J z_J)$ with $\xi^i = d_i$ for $i \in I$ and $\xi_i = u_i$ for $i \notin I$ (resp. $\xi^i = u_i$ for $i \in I$ and $\xi_i = d_i$ for $i \notin I$).

Theorem

Let f be convex and sub-modular. If $\sum_{i=1}^J \frac{\rho - d_i}{u_i - d_i} < 1$ or $\sum_{i=1}^J \frac{u_i - \rho}{u_i - d_i} < 1$, then respectively

$$(\mathcal{B}f)(z) = \frac{1}{\rho}\left[\tilde{f}_\emptyset(z) + \sum_{j=1}^J \frac{\rho - d_j}{u_j - d_j}(\tilde{f}_j(z) - \tilde{f}_\emptyset)\right], \qquad (25)$$

$$(\mathcal{B}f)(z) = \frac{1}{\rho}\left[f_\emptyset(z) + \sum_{j=1}^J \frac{u_j - \rho}{u_j - d_j}(f_j(z) - f_\emptyset)\right]. \qquad (26)$$

Again $\mathcal{B}$ is linear implying a *multi-color extension of CRR formula*.

Example J=3 (three colors), I

When conditions of the above theorem do not hold the reduced Bellman operator does not turn to a linear form, even though essential simplifications still have place for submodular payoffs. Introduce the following coefficients:

$$\alpha_I = 1 - \sum_{j \in I} \frac{u_j - r}{u_j - d_j}, \text{where} \quad I \subset \{1, 2, ..., J\}.$$

In particular, in case $J = 3$

$$\begin{aligned} \alpha_{12} &= \left(1 - \frac{u_1 - r}{u_1 - d_1} - \frac{u_2 - r}{u_2 - d_2}\right) \\ \alpha_{13} &= \left(1 - \frac{u_1 - r}{u_1 - d_1} - \frac{u_3 - r}{u_3 - d_3}\right) \\ \alpha_{23} &= \left(1 - \frac{u_2 - r}{u_2 - d_2} - \frac{u_3 - r}{u_3 - d_3}\right). \end{aligned} \tag{27}$$

Example J=3 (three colors), II

Theorem

Conditions of Theorem 8 do not hold.
If $\alpha_{12} \geq 0$, $\alpha_{13} \geq 0$ *and* $\alpha_{23} \geq 0$, *then*

$$(\mathcal{B}f)(\mathbf{z}) = \frac{1}{r} \max(I, II, II),$$

$$I = -\alpha_{123} f_{\{1,2\}}(\mathbf{z}) + \alpha_{13} f_{\{2\}}(\mathbf{z}) + \alpha_{23} f_{\{1\}}(\mathbf{z}) + \frac{u_3 - r}{u_3 - d_3} f_{\{3\}}(\mathbf{z}),$$

$$II = -\alpha_{123} f_{\{1,3\}}(\mathbf{z}) + \alpha_{12} f_{\{3\}}(\mathbf{z}) + \alpha_{23} f_{\{1\}}(\mathbf{z}) + \frac{u_2 - r}{u_2 - d_2} f_{\{2\}}(\mathbf{z}),$$

$$III = -\alpha_{123} f_{\{2,3\}}(\mathbf{z}) + \alpha_{12} f_{\{3\}}(\mathbf{z}) + \alpha_{13} f_{\{2\}}(\mathbf{z}) + \frac{u_1 - r}{u_1 - d_1} f_{\{1\}}(\mathbf{z}).$$

For the cases (i) $\alpha_{ij} \leq 0$, $\alpha_{jk} \geq 0$, $\alpha_{ik} \geq 0$, *and (ii)* $\alpha_{ij} \geq 0$, $\alpha_{jk} \leq 0$, $\alpha_{ik} \leq 0$, *where* $\{i, j, k\}$ *is an arbitrary permutation of the set* $\{1, 2, 3\}$, *similar explicit formulae are available.*

Transaction costs

Extended state space (at time $m-1$):

$$X_{m-1}, S^j_{m-1}, v_{m-1} = \gamma^j_{m-1}, \quad j = 1, \cdots, J.$$

New state at time m becomes

$$X_m, \quad S^j_m = \xi^j_m S^j_{m-1}, \quad v_m = \gamma^j_m, \quad j = 1, \cdots, J,$$

$$X_m = \sum_{j=1}^{J} \gamma^j_m \xi^j_m S^j_{m-1} + \rho(X_{m-1} - \sum_{j=1}^{J} \gamma^j_m S^j_{m-1}) - g(\gamma_m - v_{m-1}, S_{m-1}). \tag{28}$$

New reduced Bellman operator:

$$(\mathcal{B}f)(z, v) = \min_{\gamma} \max_{\xi} [f(\xi \circ z, \gamma) - (\gamma, \xi \circ z - \rho z) + g(\gamma - v, z)]. \tag{29}$$

Other extensions

American and real options,
Path dependent payoffs,
Time dependent data
Nonlinear jump pattern, where the reduced Bellman operator becomes

$$(\mathcal{B}f)(z) = \min_{\gamma} \max_{i \in \{1,\cdots,k\}} [f(g_i(z)) - (\gamma, g_i(z) - \rho z)], \quad z = (z^1, ..., z^J), \tag{30}$$

or equivalently

$$(\mathcal{B}f)(z) = \min_{\gamma} \max_{\eta_i \in \{g_i(z)\}, i=1,\cdots,k} [f(\eta_i + \rho z) - (\gamma, \eta_i)]. \tag{31}$$

Upper and Lower values; intrinsic risk I

The upper value (or the upper expectation) $\overline{\mathbf{E}}f$ of a random variable f is defined as the minimal capital of the investor such that he/she has a strategy that guarantees that at the final moment of time, his capital is enough to buy f, i.e.

$$\overline{\mathbf{E}}f = \inf\{\alpha : \exists\gamma : \forall\xi,\ X^{\alpha}_{\gamma}(\xi) - f(\xi) \geq 0\}.$$

Dually, the lower value (or the lower expectation) $\underline{\mathbf{E}}f$ of a random variable f is defined as the maximum capital of the investor such that he/she has a strategy that guarantees that at the final moment of time his capital is enough to sell f, i.e.

$$\underline{\mathbf{E}}f = \sup\{\alpha : \exists\gamma : \forall\xi,\ X^{\alpha}_{\gamma}(\xi) + f(\xi) \geq 0\}.$$

One says that the prices are consistent if $\overline{\mathbf{E}}f \geq \underline{\mathbf{E}}f$. If these prices coincide, we are in a kind of abstract analog of a complete market. In the general case, upper and lower prices are also referred to as a seller and buyer prices respectively.

Upper and Lower values; intrinsic risk II

Our setting:

$$(\mathcal{B}_{low} f)(z) = \max_{\gamma} \min_{\{\xi^j \in \{d_j, u_j\}\}} [f(\xi \circ z) - (\gamma, \xi \circ z - \rho z)], \quad (32)$$

$$(\mathcal{B}_{low} f)(z) = \min_{\{\Omega\}} \mathbf{E}_{\Omega} f(\xi \circ z), \quad z = (z^1, \cdots, z^J). \quad (33)$$

The difference between lower and upper prices can be considered as a measure of intrinsic risk of an incomplete market.
Cash-back methodology for dealing with intrinsic risk.
Link with coherent measure of risk.

Identification of pre-Markov chains, I

Example: multi-nomial model of stock prices: in each period the price is multiplied by one of n given positive numbers $a_1 < \cdots < a_n$.
Risk-neutrality for a probability law $\{p_1, \cdots, p_n\}$ on these multipliers: $\sum_{i=1}^n p_i a_i = \rho$.
Suppose the prices of certain contingent claims specified by payoffs f from a family F are given yielding

$$\sum_{i=1}^{n} p_i f(a_i) = \omega(f), \quad f \in F.$$

If the family F is rich enough, one can expect to be able to identify a unique eligible risk-neutral probability law, so that max in the r.h.s. of (**??**) disappears.

Identification of pre-Markov chains, II

Assume $n-2$ premia of European calls (with different strike prices) are given. Choose $a_2, \cdots, a_{n-1}$ to coincide with strike prices of these call options. Then

$$\begin{cases} p_1 + \cdots + p_n = 1, \\ a_1 p_1 + \cdots + a_n p_n = \rho \\ (a_3 - a_2) p_3 + (a_4 - a_2) p_4 + \cdots (a_n - a_2) p_n = \omega_3 \\ \quad \ldots \\ (a_{n-1} - a_{n-2}) p_{n-1} + (a_n - a_{n-2}) p_n = \omega_{n-1} \\ (a_n - a_{n-1}) p_n = \omega_n \end{cases} \tag{34}$$

with certain ω_j.
The determinant of this system is $\prod_{k=2}^{n} (a_k - a_{k-1})$. The system is of triangular type, and thus explicitly solvable.

Identification of pre-Markov chains, III

To simplify further: assume equal spacing: $a_k - a_{k-1} = \Delta$ for all $k = 2, \cdots, n$ and $\Delta > 0$. Then system (34) reduces to the system of type

$$\begin{cases} x_1 + \cdots + x_n = b_1, \\ x_2 + 2x_3 + \cdots + (n-1)x_n = b_2 \\ x_3 + 2x_4 + \cdots + (n-2)x_n = b_3 \\ \quad \ldots \\ x_{n-1} + 2x_n = b_{n-1} \\ x_n = b_n \end{cases} \tag{35}$$

(where $x_k = \Delta p_k, b_1 = \Delta, b_2 = \rho - 1,\ b_j = \omega_j$ for $j > 2$).

Identification of pre-Markov chains, IV

Explicit solution

$$\begin{cases} x_n = b_n, \\ x_{n-1} = b_{n-1} - 2b_n \\ x_k = b_k - 2b_{k+1} + b_{k+2}, \quad k = 2, \cdots, n-2, \\ x_1 = b_1 - b_2 + b_3. \end{cases} \tag{36}$$

Similarly with colored options or interest rate models.

Continuous time limit

$$g_i(z) = z + \tau^{\alpha} \phi_i(z), \quad i = 1, \cdots, k, \tag{37}$$

with some functions ϕ_i and a constant $\alpha \in [1/2, 1]$.
Introducing

$$p_i^I(z) = \lim_{\tau \to 0} p_i^I(z, \tau)$$

yields

$$rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}) + \frac{1}{2} \max_I \sum_{i \in I} p_i^I(z) \left(\frac{\partial^2 f}{\partial z^2} \phi_i(z), \phi_i(z) \right) \tag{38}$$

in case $\alpha = 1/2$, and the trivial first order equation

$$rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}) \tag{39}$$

in case $\alpha > 1/2$.

Continuous time limit: $J = 2$

$$u_i = 1 + \sigma_i \sqrt{\tau}, \quad d_i = 1 - \sigma_i \sqrt{\tau}, \quad i = 1, 2. \tag{40}$$

Hence

$$\frac{u_i - \rho}{u_i - d_i} = \frac{1}{2} - \frac{r}{2\sigma_i}\sqrt{\tau}, \quad i = 1, 2,$$

$$\kappa = -\frac{1}{2} r \sqrt{\tau} (\frac{1}{\sigma_1} + \frac{1}{\sigma_2}).$$

The upper price equation

$$rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}) + \frac{1}{2}\left[\sigma_1^2 z_1^2 \frac{\partial^2 f}{\partial z_1^2} - 2\sigma_1 \sigma_2 z_1 z_2 \frac{\partial^2 f}{\partial z_1 \partial z_2} + \sigma_2^2 z_2^2 \frac{\partial^2 f}{\partial z_2^2}\right]. \tag{41}$$

The lower price equation

$$rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}) + \frac{1}{2}\left[\sigma_1^2 z_1^2 \frac{\partial^2 f}{\partial z_1^2} + 2\sigma_1 \sigma_2 z_1 z_2 \frac{\partial^2 f}{\partial z_1 \partial z_2} + \sigma_2^2 z_2^2 \frac{\partial^2 f}{\partial z_2^2}\right]. \tag{42}$$

Fractional dynamics, I

Example: $J = 2$, sub-modular payoffs.

$$X_{n+1}^{\tau}(z) = X_n^{\tau}(z) + \sqrt{\tau}\phi(X_n^{\tau}(z)), \quad X_0^{\tau}(z) = z,$$

where $\phi(z)$ is one of three points $(z^1 d_1, z^2 u_2), (z^1 u_1, z^2 d_2), (z^1 u_1, z^2 u_2)$ that are chosen with the corresponding risk neutral probabilities. As was shown above, this Markov chain converges, as $\tau \to 0$ and $n = [t/\tau]$ (where $[s]$ denotes the integer part of a real number s), to the diffusion process X_t solving the Black-Scholes type (degenerate) equation (41), i.e. a sub-Markov process with the generator $Lf(x)$ being

$$-rf + r(z, \frac{\partial f}{\partial z}) + \frac{1}{2}\left[\sigma_1^2 z_1^2 \frac{\partial^2 f}{\partial z_1^2} - 2\sigma_1\sigma_2 z_1 z_2 \frac{\partial^2 f}{\partial z_1 \partial z_2} + \sigma_2^2 z_2^2 \frac{\partial^2 f}{\partial z_2^2}\right]. \tag{43}$$

Fractional dynamics, II

Assume now that the times between jumps $T_1, T_2, \cdots$ are i.i.d.:

$$\mathbf{P}(T_i \geq t) \sim \frac{1}{\beta t^\beta}$$

with $\beta \in (0,1)$. It is well known that such T_i belong to the domain of attraction of the β-stable law:

$$\Theta_t^\tau = \tau^{1/\beta}(T_1 + \cdots + T_{[t/\tau]})$$

converge, as $\tau \to 0$, to a β-stable Lévy motion Θ_t, which is a Lévy process on $\mathbf{R}_+$ with the fractional derivative of order β as the generator:

$$Af(t) = -\frac{d^\beta}{d(-t)^\beta} f(t) = -\frac{1}{\Gamma(-\beta)} \int_0^\infty (f(t+r) - f(t)) \frac{dr}{r^{1+\beta}}.$$

Fractional dynamics, III

We are now interested in the process

$$Y_t^\tau(z) = X_{N_t^\tau}^\tau(z),$$

where

$$N_t^\tau = \max\{u : \Theta_u^\tau \leq t\}.$$

The limiting process

$$N_t = \max\{u : \Theta_u \leq t\}$$

is therefore the inverse (or hitting time) process of the β-stable Lévy motion Θ_t.

Fractional dynamics, IV

Theorem

The process Y_t^τ *converges to* $Y_t = X_{N_t}$, *whose averages* $f(T-t,x) = \mathbf{E} f(Y_{T-t}(x))$ *have explicit representation*

$$f(T-t,x) = \int_0^\infty \int_0^\infty \int_0^\infty G_u^-(z_1,z_2;w_1,w_2) Q(T-t,u)\, du dw_1 dw_2,$$

where G^-, *the transition probabilities of* X_t, $Q(t,u)$ *denotes the probability density of the process* N_t.
Moreover, for $f \in C^2_\infty(\mathbf{R}^d)$, $f(t,x)$ *satisfy the (generalized) fractional evolution equation (of Black-Scholes type)*

$$\frac{d^\beta}{dt^\beta} f(t,x) = Lf(t,x) + \frac{t^{-\beta}}{\Gamma(1-\beta)} f(t,x).$$

General case leads to fractional extension of nonlinear Black-Scholes type equation (not worked out rigorously yet).

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