Game theoretic analysis of incomplete markets \(^1\)

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Highlights

- Generalizations of classical BS and CRR formulae with more rough assumptions on the underlying assets evolution: interval model.
- Transaction costs included.
- Emergence of risk neutral probabilities in minmax (robust control) evaluations.
- A natural unique selection among multiple risk neutral measures arising in incomplete markets for options specified by sub-modular functions.
- Continuous time limit leading to nonlinear degenerate and/or fractional Black-Scholes type equations.
- Explicit formulae and new numeric schemes. Identification of pre-Markov models.
Geometric risk-neutral laws, I

Def. A probability law \( \mu \in \mathcal{P}(E) \) on \( E \subset \mathbb{R}^d \) is risk-neutral if the origin is its barycenter: \( \int_E \xi \mu(d\xi) = 0 \). Denote \( \mathcal{P}_{rn}(E) \) the set of risk-neutral laws.

More generally: for a compact \( E \subset \mathbb{R}^n \) and a continuous \( F : E \to \mathbb{R}^d \) let

\[
\mathcal{P}(E; F) = \{ \mu \in \mathcal{P}(E) : (F, \mu) = \int F(x) \mu(dx) = 0 \}.
\]

\( \mathcal{P}_{rn}(E) = \mathcal{P}(E; \text{Id}) \).

Def. \( E \) is called weakly (resp. strongly) positively complete, if there exists no \( \omega \in \mathbb{R}^d \) such that \( (\omega, \xi) > 0 \) (resp. \( (\omega, \xi) \geq 0 \)) for all \( \xi \in E \).

Geometrically: \( E \) does not belong to any open (respectively closed) half-space of \( \mathbb{R}^d \).

If \( E \subset \mathbb{R}^d \) is a compact convex set, then \( E \) is weakly positively complete if and only if it contains the origin.
Proposition

Let $E \subset \mathbb{R}^n$ be compact and a mapping $F : E \rightarrow \mathbb{R}^d$ continuous.

(i) The set $\mathcal{P}(E; F)$ is not empty if and only if $F(E)$ is weakly positively complete in $\mathbb{R}^d$.

(ii) Let $E'$ be the support of a measure $\mu \in \mathcal{P}(E; F)$. If $F(E')$ does not coincide with the origin, then it is strongly positively complete in the subspace $\mathbb{R}^m \subset \mathbb{R}^d$ generated by $F(E')$.

Proposition

Let $E \subset \mathbb{R}^n$ be compact, a mapping $F = (F^1, \cdots, F^d) : E \rightarrow \mathbb{R}^d$ be continuous and $\mu$ be an extreme point of the set $\mathcal{P}(E; F)$. Then $\mu$ is a linear combination of not more than $d + 1$ Dirac measures.
Proposition

Example. Let $E = \{\xi_1, \cdots, \xi_{d+1}\}$ be strongly positively complete in $\mathbb{R}^d$. Then there exists a unique risk-neutral probability law $\{p_1, \cdots, p_{d+1}\}$ on $\{\xi_1, \cdots, \xi_{d+1}\}$: $p_i$ equals the ratio of the volume of the pyramids $\prod[\{0 \cup \{\hat{\xi}_i\}\}]$ to the whole volume $\prod[\xi_1, \cdots, \xi_{d+1}]$.

($\{\hat{\xi}_i\}$ denotes the family $\{\xi_1, \cdots, \xi_{d+1}\}$ with $\xi$ taken out).

Theorem

Let a compact set $E \subset \mathbb{R}^d$ be strongly positively complete. Then the extreme points of the set of risk-neutral probabilities on $E$ are the Dirac mass at zero (only when $E$ contains the origin) and the risk-neutral measures with support on families of size $m + 1$, $0 < m \leq d$, that generate a subspace of dimension $m$ and are strongly positively complete in this subspace.
Underlying Game, I

\[ \Pi[\xi_1, \ldots, \xi_{d+1}](f) = \min_{\gamma \in \mathbb{R}^d} \max_i [f(\xi_i) - (\xi_i, \gamma)], \]

(1)

and

\[ \Pi[\xi_1, \ldots, \xi_{d+1}](f) = \max_{\gamma \in \mathbb{R}^d} \min_i [f(\xi_i) + (\xi_i, \gamma)], \]

(2)

where \( \xi_1, \ldots, \xi_{d+1} \) are \( d + 1 \) vectors in \( \mathbb{R}^d \) in general position (origin is in the interior of their convex hull). A remarkable fact: expressions (1) and (2) are linear in \( f \) and the minimizing \( \gamma \) is unique and also depends linearly on \( f \).
Underlying Game II

Proposition

$\xi_1, \cdots, \xi_{d+1}$ are $d + 1$ vectors in $\mathbb{R}^d$ in general position. Then

$$\Pi[\xi_1, \cdots, \xi_{d+1}](f) = \Pi[\xi_1, \cdots, \xi_{d+1}](f) = Ef(\xi), \quad (3)$$

where $E$ is with respect to the unique risk neutral law on $\{\xi_i\}$, and the minimum in (1) is attained on the single $\gamma_0$:

$$\gamma_0 = E[f(\xi)r(\xi)] \quad (4)$$

with explicitly defined vectors $r(\xi)$,

$$|\gamma_0| \leq \|f\| \frac{1}{d} \frac{S(\xi_1, \cdots, \xi_{d+1})}{V(\xi_1, \cdots, \xi_{d+1})}, \quad (5)$$

where $S(\xi_1, \cdots, \xi_{d+1})$ is the surface volume of the pyramid $\Pi[\xi_1, \cdots, \xi_{d+1}]$ and $V(\xi_1, \cdots, \xi_{d+1})$ is its volume.
Underlying Game III

For a compact $E$ and a continuous function $f$ define

$$\Pi[E](f) = \inf_{\gamma \in \mathbb{R}^d} \max_{\xi \in E} [f(\xi) - (\xi, \gamma)]$$

and

$$\Pi[E](f) = \sup_{\gamma \in \mathbb{R}^d} \min_{\xi \in E} [f(\xi) - (\xi, \gamma)].$$

Theorem

Let a compact set $E \subset \mathbb{R}^d$ be strongly positively complete. Then

$$\Pi[E](f) = \max_{\mu} \mathbb{E}_\mu f(\xi), \quad \Pi[E](f) = \min_{\mu} \mathbb{E}_\mu f(\xi)$$

where $\max$ (resp. $\min$) is taken over all extreme points $\mu$ of risk-neutral laws on $E$ given by Proposition 1, $\inf$ in (6) is attained on some $\gamma$ (satisfying the estimates above).
Underlying Game: nonlinear extension

\[ \Pi[\xi_1, \cdots, \xi_k](f) = \min_{\gamma \in \mathbb{R}^d} \max_{\xi_1, \cdots, \xi_k} [f(\xi_i, \gamma) - (\xi_i, \gamma)]. \quad (9) \]

**Theorem**

Let \( \{\xi_1, \cdots, \xi_k\} \subset \mathbb{R}^d \), \( k > d \), in general position.
Let the function \( f(\xi, \gamma) \) be bounded below and Lipshitz continuous in \( \gamma \) with a Lipschitz constant \( \kappa \), which is small enough.

Then the minimum in (9) is finite, is attained on some \( \gamma_0 \) and

\[ \Pi[\xi_1, \cdots, \xi_k](f) = \max_{I} \mathbf{E}_I f(\xi, \gamma_I), \quad (10) \]

where \( \max \) as above and \( \gamma_I \) is the corresponding (unique) optimal value (solving a fixed point equation).

Other extensions: infinite-dimensional setting with one-dimensional projections, random geometry.
Mixed strategies with linear constraints, 1

Equivalent form of the result above:

$$\Pi[E](f) = \inf_{\gamma \in \mathbb{R}^d} \max_{\mu \in \mathcal{P}(E)} E_\mu[f(\xi) - (\gamma, \xi)] = \max_{\mu \in \mathcal{P}_m(E)} E_\mu f(\xi).$$  \hspace{1cm} (11)

Let $E \subset \mathbb{R}^d$ be a compact set and $\tilde{\mathcal{P}}(E)$ a closed convex subset of $\mathcal{P}(E)$ (the main example is a set of type $\mathcal{P}(E; F)$). Let

$$\tilde{\Pi}[E](f) = \inf_{\gamma \in \mathbb{R}^d} \max_{\mu \in \tilde{\mathcal{P}}(E)} E_\mu[f(\xi) - (\gamma, \xi)]$$

$$= \inf_{\gamma \in \mathbb{R}^d} \max_{\mu \in \tilde{\mathcal{P}}(E)} \left[ \int f(\xi)\mu(d\xi) - (\gamma, \int \xi\mu(d\xi)) \right].$$  \hspace{1cm} (12)

Let $B$ denote the linear mapping $\tilde{\mathcal{P}}(E) \rightarrow \mathbb{R}^d$ given by

$$B \mu = E_\mu \xi = \int \xi\mu(d\xi)$$

(barycenter or the center of mass).
Mixed strategies with linear constraints, II

The following main result extends Theorem 2 to the case of mixed strategies with constraints.

Theorem

The set \( \tilde{P}(E) \cap P_{rn}(E) \) is empty if and only if the set \( B(\tilde{P}(E)) \) is not weakly positively complete, in which case \( \tilde{\Pi}[E](f) = -\infty \). Otherwise

\[
\tilde{\Pi}[E](f) = \inf_{\gamma \in \mathbb{R}^d} \max_{\mu \in \tilde{P}(E)} E[ f(\xi) - (\gamma, \xi) ]
\]

\[
= \max_{\mu \in \tilde{P}(E) \cap P_{rn}(E)} E[ f(\xi) ].
\]
Market with several securities in discrete time $k = 1, 2, ...$:
The risk-free bonds (bank account), priced $B_k$, 
and $J$ common stocks, $J = 1, 2..., $ priced $S^i_k$, $i \in \{1, 2, ..., J\}$.
$B_{k+1} = \rho B_k$, $\rho \geq 1$ is a constant interest rate,
$S^i_{k+1} = \xi^i_{k+1} S^i_k$, where $\xi^i_k$, $i \in \{1, 2, ..., J\}$, are unknown sequences taking values in some fixed intervals
$M_i = [d_i, u_i] \subset \mathbb{R}$ (interval model).
This model generalizes the colored version of the classical CRR model in a natural way.
In the latter a sequence $\xi^i_k$ is confined to take values only among two boundary points $d_i, u_i$, and it is supposed to be random with some given distribution.
Rainbow (or colored) European Call Options

A premium function $f$ of $J$ variables specifies the type of an option.

Standard examples ($S^1, S^2, \ldots, S^J$ represent the expiration values of the underlying assets, and $K, K_1, \ldots, K_J$ represent the strike prices):

- Option delivering the best of $J$ risky assets and cash
  $$f(S^1, S^2, \ldots, S^J) = \max(S^1, S^2, \ldots, S^J, K),$$  
  (13)

- Calls on the maximum of $J$ risky assets
  $$f(S^1, S^2, \ldots, S^J) = \max(0, \max(S^1, S^2, \ldots, S^J) - K),$$  
  (14)

- Multiple-strike options
  $$f(S^1, S^2, \ldots, S^J) = \max(0, S^1 - K_1, S^2 - K_2, \ldots, S^J - K_J),$$  
  (15)

- Portfolio options
  $$f(S^1, S^2, \ldots, S^J) = \max(0, n_1 S^1 + n_2 S^2 + \ldots + n_J S^J - K),$$  
  (16)

- Spread options: $f(S^1, S^2) = \max(0, (S^2 - S^1) - K)$.  

Investor’s (seller of an option) control: one step

Let $X_k$ be the capital of the investor at the time $k = 1, 2, \ldots$. At each time $k - 1$ the investor determines his portfolio by choosing the numbers $\gamma_k^i$ of common stocks of each kind to be held so that the structure of the capital is represented by the formula

$$X_{k-1} = \sum_{i=1}^{J} \gamma_k^i S_{k-1}^i + (X_{k-1} - \sum_{i=1}^{J} \gamma_k^i S_{k-1}^i),$$

where the expression in bracket corresponds to the part of his capital laid on the bank account. The control parameters $\gamma_k^i$ can take all real values, i.e. short selling and borrowing are allowed. The value $\xi_k$ becomes known in the moment $k$ and thus the capital at the moment $k$ becomes

$$X_k = \sum_{i=1}^{J} \gamma_k^i \xi_k^i S_{k-1}^i + \rho(X_{k-1} - \sum_{i=1}^{J} \gamma_k^i S_{k-1}^i).$$
Investor’s control: n step game

If $n$ is the *maturity date*, this procedures repeats $n$ times starting from some initial capital $X = X_0$ (selling price of an option) and at the end the investor is obliged to pay the premium $f$ to the buyer. Thus the (final) income of the investor equals

$$G(X_n, S_n^1, S_n^2, ..., S_n^J) = X_n - f(S_n^1, S_n^2, ..., S_n^J).$$

The evolution of the capital can thus be described by the dynamic $n$-step game of the investor (strategies are sequences of real vectors $(\gamma_1, ..., \gamma_n)$ (with $\gamma_j = (\gamma_j^1, ..., \gamma_j^J)$)) with the Nature (characterized by unknown parameters $\xi_k^i$).

A position of the game at any time $k$ is characterized by $J + 1$ non-negative numbers $X_k, S_k^1, ..., S_k^J$ with the final income specified by the function

$$G(X, S^1, ..., S^J) = X - f(S^1, ..., S^J)$$
Robust control (guaranteed payoffs, worst case scenario)

Minmax payoff (guaranteed income) with the final income $G$ in a one step game with the initial conditions $X, S^1, ..., S^J$ is given by the Bellman operator

$$B^G(X, S^1, ..., S^J)$$

$$= \max_{\gamma} \min_{\xi} G(\rho X + \sum_{i=1}^{J} \gamma^i \xi^i S^i - \rho \sum_{i=1}^{J} \gamma^i S^i, \xi^1 S^1, ..., \xi^J S^J),$$

and the guaranteed income in the $n$ step game with the initial conditions $X_0, S^1_0, ..., S^J_0$ is

$$B^n G(X_0, S^1_0, ..., S^J_0).$$
Reduced Bellman operator

Clearly for $G$ of form $G(X, S^1, \cdots, S^J) = X - f(S^1, \cdots, S^J)$,

$$B G(X, S^1, \ldots, S^J)$$

$$= X - \frac{1}{\rho} \min \max_{\xi} [f(\xi^1 S^1, \xi^2 S^2, \ldots, \xi^J S^J) - \sum_{j=1}^J \gamma^j S^j(\xi^j - \rho)],$$

and hence

$$B^n G(X, S^1, \cdots, S^J) = X - (B^n f)(S^1, \cdots, S^J),$$

where the reduced Bellman operator is defined as:

$$(B f)(z) = \frac{1}{\rho} \min_{\gamma} \max_{\{\xi^i \in [d_i, u_i]\}} [f(\xi \circ z) - (\gamma, \xi \circ z - \rho z)]. \quad (19)$$

Here $$(\xi \circ z)^i = \xi^i z^i$$ - Hadamard product.
Hedging

Main definition. A strategy \( \gamma_1^i, ..., \gamma_n^i, i = 1, ..., J \), of the investor is called a hedge, if for any sequence \((\xi_1, ..., \xi_n)\) (with \(\xi_j = (\xi_1^j, ..., \xi_J^j)\)) the investor is able to meet his obligations, i.e.

\[
G(X_n, S_1^n, ..., S_J^n) \geq 0.
\]

The minimal value of the capital \(X_0\) for which the hedge exists is called the hedging price \(H\) of an option.

**Theorem** (Game theory for option pricing.)

The minimal value of \(X_0\) for which the income of the investor is not negative (and which by definition is the hedge price \(H\)) is given by

\[
H^n = (B^n f)(S_0^1, ..., S_0^J).
\]  \(\text{(20)}\)
Risk-neutral evaluation for options: setting

A linear change of variables yields

\[(Bf)(z^1, \ldots, z^J) = \frac{1}{\rho} \min_{\rho} \max_{\gamma} \{ \eta \in [z^i(d_i - \rho), z^i(u_i - \rho)] \} \left[ f(\eta + \rho z) - (\gamma, \eta) \right]. \tag{21} \]

Assuming \( f \) is convex, we are in the setting above with

\[ \Pi = \Pi_{z, \rho} = \times_{i=1}^J [z^i(d_i - \rho), z^i(u_i - \rho)], \]

with vertices

\[ \eta_I = \xi_I \oplus z - \rho z, \quad \xi_I = \{ d_i | i \in I, u_j | j \notin I \}, \]

parametrized by all subsets (including the empty one) \( I \subset \{1, \ldots, J\} \).

Above theory reduces our dynamic game to a controlled Markov jump problem:
Risk-neutral evaluation for options: result

**Theorem**

suppose the vertices $\xi_i$ are in general position: for any $J$ subsets $I_1, \cdots, I_J$, the vectors $\{\xi_{I_k} - \rho 1\}_{k=1}^J$ are independent in $\mathbb{R}^J$. Then

$$(Bf)(z) = \max_{\{\Omega\}} \mathbb{E}_{\Omega} f(\xi \circ z), \quad z = (z^1, \cdots, z^J),$$  \hspace{1cm} (22)$$

where $\{\Omega\}$ is the collection of subsets $\Omega = \xi_{I_1}, \cdots, \xi_{I_{J+1}}$ of the set of vertices of $\Pi$, of size $J + 1$, such that their convex hull contains $\rho 1$ as an interior point, and where $\mathbb{E}_{\Omega}$ denotes the expectation with respect to the unique probability law $\{p_I\}$, $\xi_i \in \Omega$, on the set of vertices of $\Pi$, which is supported on $\Omega$ and is risk neutral with respect to $\rho 1$, that is

$$\sum_{I \subset \{1, \ldots, J\}} p_I \xi_I = \rho 1.$$  \hspace{1cm} (23)$$
Sub-modular payoffs

A function \( f : \mathbb{R}^2_+ \rightarrow \mathbb{R}_+ \) is called *sub-modular*, if the inequality

\[
 f(x_1, y_2) + f(x_2, y_1) \geq f(x_1, y_1) + f(x_2, y_2)
\]

holds whenever \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \). A function \( f : \mathbb{R}^d_+ \rightarrow \mathbb{R}_+ \) is called *sub-modular* if

\[
 f(x \vee y) + f(x \wedge y) \leq f(x) + f(y),
\]

where \( \vee \) (respectively \( \wedge \)) denotes the Pareto (coordinate-wise) maximum (respectively minimum).

**Remark**

*If \( f \) is twice continuously differentiable, then it is sub-modular if and only if \( \frac{\partial^2 f}{\partial z_i \partial z_j} \leq 0 \) for all \( i \neq j \).*

As one easily sees, the payoffs of the first three examples of rainbow options, given at the beginning, are sub-modular.
Example J$=2$ (two colors)

The polyhedron $\Pi$ is then a rectangle. From sub-modularity of $f$ it follows that the maximum is always achieved either on

$$\Omega_d = \{(d_1, d_2), (d_1, u_2), (u_1, d_2)\},$$

or on

$$\Omega_u = \{(d_1, u_2), (u_1, d_2), (u_1, u_2)\}.$$ 

and $Bf$ reduces either to $E_{\Omega_u}$ or to $E_{\Omega_d}$ depending on a certain 'correlation coefficient' of possible jumps.
Example J=2 (two colors) continued

Theorem
Let $J = 2$, $f$ be convex sub-modular, and denote

$$
\kappa = \frac{(u_1 u_2 - d_1 d_2) - \rho(u_1 - d_1 + u_2 - d_2)}{(u_1 - d_1)(u_2 - d_2)}.
$$

If $\kappa \geq 0$, then $(Bf)(z_1, z_2)$ equals

$$
\frac{\rho - d_1}{u_1 - d_1} f(u_1 z_1, d_2 z_2) + \frac{\rho - d_2}{u_2 - d_2} f(d_1 z_1, u_2 z_2) + \kappa f(d_1 z_1, d_2 z_2),
$$

If $\kappa \leq 0$, the $(Bf)(z_1, z_2)$ equals

$$
\frac{u_1 - \rho}{u_1 - d_1} f(d_1 z_1, u_2 z_2) + \frac{u_2 - \rho}{u_2 - d_2} f(u_1 z_1, d_2 z_2) + |\kappa| f(u_1 z_1, u_2 z_2),
$$
Example J=2 (two colors) completed

By linearity, the powers of $B$ can be found. Say, if $\kappa = 0$,

$$C_h = \rho^{-n} \sum_{k=0}^{n} C_n$$

$$\left( \frac{\rho - d_1}{u_1 - d_1} \right)^k \left( \frac{\rho - d_2}{u_2 - d_2} \right)^{n-k} f(u_1^k d_1^{n-k} S_0^1, d_2^k u_2^{n-k} S_0^2).$$

(two-dimensional version of CRR formula).

Important: risk neutral selector.
$J > 2$ colors: reduction to a linear Bellman

Notation: for a set $I \subset \{1, 2, \ldots, J\}$, $f_i(z)$ (resp. $\tilde{f}_i(z)$) is $f(\xi^1 z_1, \cdots, \xi^J z_J)$ with $\xi^i = d_i$ for $i \in I$ and $\xi_i = u_i$ for $i \notin I$ (resp. $\xi^i = u_i$ for $i \in I$ and $\xi_i = d_i$ for $i \notin I$).

Theorem

Let $f$ be convex and sub-modular. If $\sum_{i=1}^J \frac{\rho - d_i}{u_i - d_i} < 1$ or $\sum_{i=1}^J \frac{u_i - \rho}{u_i - d_i} < 1$, then respectively

$$ (Bf)(z) = \frac{1}{\rho} \left[ \tilde{f}_\emptyset(z) + \sum_{j=1}^J \frac{\rho - d_j}{u_j - d_j} (\tilde{f}_j(z) - \tilde{f}_\emptyset) \right], \quad (25) $$

$$ (Bf)(z) = \frac{1}{\rho} \left[ f_\emptyset(z) + \sum_{j=1}^J \frac{u_j - \rho}{u_j - d_j} (f_j(z) - f_\emptyset) \right]. \quad (26) $$

Again $B$ is linear implying a multi-color extension of CRR formula.
Example J=3 (three colors), 1

When conditions of the above theorem do not hold the reduced Bellman operator does not turn to a linear form, even though essential simplifications still have place for submodular payoffs. Introduce the following coefficients:

$$\alpha_I = 1 - \sum_{j \in I} \frac{u_j - r}{u_j - d_j}, \text{ where } I \subset \{1, 2, \ldots, J\}.$$ 

In particular, in case $J = 3$

$$\begin{align*}
\alpha_{12} &= \left(1 - \frac{u_1-r}{u_1-d_1} - \frac{u_2-r}{u_2-d_2}\right) \\
\alpha_{13} &= \left(1 - \frac{u_1-r}{u_1-d_1} - \frac{u_3-r}{u_3-d_3}\right) \\
\alpha_{23} &= \left(1 - \frac{u_2-r}{u_2-d_2} - \frac{u_3-r}{u_3-d_3}\right).
\end{align*} \quad (27)$$
Example J=3 (three colors), II

Theorem

Conditions of Theorem 8 do not hold.
If $\alpha_{12} \geq 0$, $\alpha_{13} \geq 0$ and $\alpha_{23} \geq 0$, then

$$(Bf)(z) = \frac{1}{r} \max(I, II, III),$$

$I = -\alpha_{123} f_{\{1,2\}}(z) + \alpha_{13} f_{\{2\}}(z) + \alpha_{23} f_{\{1\}}(z) + \frac{u_3 - r}{u_3 - d_3} f_{\{3\}}(z),$

$II = -\alpha_{123} f_{\{1,3\}}(z) + \alpha_{12} f_{\{3\}}(z) + \alpha_{23} f_{\{1\}}(z) + \frac{u_2 - r}{u_2 - d_2} f_{\{2\}}(z),$

$III = -\alpha_{123} f_{\{2,3\}}(z) + \alpha_{12} f_{\{3\}}(z) + \alpha_{13} f_{\{2\}}(z) + \frac{u_1 - r}{u_1 - d_1} f_{\{1\}}(z).$

For the cases (i) $\alpha_{ij} \leq 0$, $\alpha_{jk} \geq 0$, $\alpha_{ik} \geq 0$, and (ii) $\alpha_{ij} \geq 0$, $\alpha_{jk} \leq 0$, $\alpha_{ik} \leq 0$, where $\{i,j,k\}$ is an arbitrary permutation of the set $\{1,2,3\}$, similar explicit formulae are available.
Transaction costs

Extended state space (at time $m - 1$):

$$X_{m-1}, S^j_{m-1}, v_{m-1} = \gamma^j_{m-1}, \quad j = 1, \ldots, J.$$ 

New state at time $m$ becomes

$$X_m, \quad S^j_m = \xi^j_m S^j_{m-1}, \quad v_m = \gamma^j_m, \quad j = 1, \ldots, J,$$

$$X_m = \sum_{j=1}^J \gamma^j_m \xi^j_m S^j_{m-1} + \rho(X_{m-1} - \sum_{j=1}^J \gamma^j_m S^j_{m-1}) - g(\gamma_m - v_{m-1}, S_{m-1}). \tag{28}$$

New reduced Bellman operator:

$$(Bf)(z, v) = \min_{\gamma} \max_{\xi} [f(\xi \circ z, \gamma) - (\gamma, \xi \circ z - \rho z) + g(\gamma - v, z)]. \tag{29}$$
Other extensions

American and real options,
Path dependent payoffs,
Time dependent data
Nonlinear jump pattern, where the reduced Bellman operator becomes

\[(Bf)(z) = \min_{\gamma} \max_{i \in \{1, \ldots, k\}} \left[ f\left(g_i(z)\right) - (\gamma, g_i(z) - \rho z) \right], \quad z = (z^1, \ldots, z^J), \tag{30} \]

or equivalently

\[(Bf)(z) = \min_{\gamma} \max_{\eta_i \in \{g_i(z)\}, i = 1, \ldots, k} \left[ f(\eta_i + \rho z) - (\gamma, \eta_i) \right]. \tag{31} \]
The upper value (or the upper expectation) $\overline{Ef}$ of a random variable $f$ is defined as the minimal capital of the investor such that he/she has a strategy that guarantees that at the final moment of time, his capital is enough to buy $f$, i.e.

$$\overline{Ef} = \inf\{\alpha : \exists \gamma : \forall \xi, X^\alpha_\gamma(\xi) - f(\xi) \geq 0\}.$$

Dually, the lower value (or the lower expectation) $\underline{Ef}$ of a random variable $f$ is defined as the maximum capital of the investor such that he/she has a strategy that guarantees that at the final moment of time his capital is enough to sell $f$, i.e.

$$\underline{Ef} = \sup\{\alpha : \exists \gamma : \forall \xi, X^\alpha_\gamma(\xi) + f(\xi) \geq 0\}.$$

One says that the prices are consistent if $\overline{Ef} \geq \underline{Ef}$. If these prices coincide, we are in a kind of abstract analog of a complete market. In the general case, upper and lower prices are also referred to as a seller and buyer prices respectively.
Upper and Lower values; intrinsic risk II

Our setting:

\[
(B_{low} f)(z) = \max_{\gamma} \min_{\{\xi^j \in \{d_j, u_j\}\}} \left[ f(\xi \circ z) - (\gamma, \xi \circ z - \rho z) \right], \quad (32)
\]

\[
(B_{low} f)(z) = \min_{\{\Omega\}} \mathbb{E}_{\Omega} f(\xi \circ z), \quad z = (z^1, \ldots, z^J). \quad (33)
\]

The difference between lower and upper prices can be considered as a measure of intrinsic risk of an incomplete market.

Cash-back methodology for dealing with intrinsic risk.
Link with coherent measure of risk.
Identification of pre-Markov chains, I

Example: multi-nominal model of stock prices: in each period the price is multiplied by one of \( n \) given positive numbers \( a_1 < \cdots < a_n \).

Risk-neutrality for a probability law \( \{p_1, \cdots, p_n\} \) on these multipliers: \( \sum_{i=1}^{n} p_i a_i = \rho \).

Suppose the prices of certain contingent claims specified by payoffs \( f \) from a family \( F \) are given yielding

\[
\sum_{i=1}^{n} p_i f(a_i) = \omega(f), \quad f \in F.
\]

If the family \( F \) is rich enough, one can expect to be able to identify a unique eligible risk-neutral probability law, so that \( \max \) in the r.h.s. of (??) disappears.
Assume $n - 2$ premia of European calls (with different strike prices) are given. Choose $a_2, \cdots, a_{n-1}$ to coincide with strike prices of these call options. Then

$$
\begin{align*}
& p_1 + \cdots + p_n = 1, \\
& a_1 p_1 + \cdots + a_n p_n = \rho \\
& (a_3 - a_2)p_3 + (a_4 - a_2)p_4 + \cdots (a_n - a_2)p_n = \omega_3 \\
& \quad \cdots \\
& (a_{n-1} - a_{n-2})p_{n-1} + (a_n - a_{n-2})p_n = \omega_{n-1} \\
& (a_n - a_{n-1})p_n = \omega_n
\end{align*}
$$

(34)

with certain $\omega_j$.

The determinant of this system is $\prod_{k=2}^n (a_k - a_{k-1})$. The system is of triangular type, and thus explicitly solvable.
Identification of pre-Markov chains, III

To simplify further: assume equal spacing: \( a_k - a_{k-1} = \Delta \) for all \( k = 2, \ldots, n \) and \( \Delta > 0 \). Then system (34) reduces to the system of type

\[
\begin{align*}
&x_1 + \cdots + x_n = b_1, \\
&x_2 + 2x_3 + \cdots + (n-1)x_n = b_2 \\
&x_3 + 2x_4 + \cdots + (n-2)x_n = b_3 \\
&\quad \vdots \\
&x_{n-1} + 2x_n = b_{n-1} \\
&x_n = b_n
\end{align*}
\]

(35)

(\text{where } x_k = \Delta p_k, b_1 = \Delta, b_2 = \rho - 1, b_j = \omega_j \text{ for } j > 2).
Explicit solution

\[
\begin{align*}
    x_n &= b_n, \\
    x_{n-1} &= b_{n-1} - 2b_n \\
    x_k &= b_k - 2b_{k+1} + b_{k+2}, \quad k = 2, \ldots, n-2, \\
    x_1 &= b_1 - b_2 + b_3.
\end{align*}
\] (36)

Similarly with colored options or interest rate models.
Continuous time limit

\[ g_i(z) = z + \tau^\alpha \phi_i(z), \quad i = 1, \cdots, k, \quad (37) \]

with some functions \( \phi_i \) and a constant \( \alpha \in [1/2, 1] \).

Introducing

\[ p_i^l(z) = \lim_{\tau \to 0} p_i^l(z, \tau) \]

yields

\[ rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}) + \frac{1}{2} \max_i \sum_{i \in I} p_i^l(z) \left( \frac{\partial^2 f}{\partial z^2} \phi_i(z), \phi_i(z) \right) \]

\[ (38) \]

in case \( \alpha = 1/2 \), and the trivial first order equation

\[ rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}) \]

\[ (39) \]

in case \( \alpha > 1/2 \).
Continuous time limit: \( J = 2 \)

\[
    u_i = 1 + \sigma_i \sqrt{\tau}, \quad d_i = 1 - \sigma_i \sqrt{\tau}, \quad i = 1, 2. \tag{40}
\]

Hence

\[
    \frac{u_i - \rho}{u_i - d_i} = \frac{1}{2} - \frac{r}{2\sigma_i} \sqrt{\tau}, \quad i = 1, 2,
\]

\[
    \kappa = -\frac{1}{2} r \sqrt{\tau} \left( \frac{1}{\sigma_1} + \frac{1}{\sigma_2} \right).
\]

The upper price equation

\[
    rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}) + \frac{1}{2} \left[ \sigma_1^2 z_1^2 \frac{\partial^2 f}{\partial z_1^2} - 2\sigma_1 \sigma_2 z_1 z_2 \frac{\partial^2 f}{\partial z_1 \partial z_2} + \sigma_2^2 z_2^2 \frac{\partial^2 f}{\partial z_2^2} \right]. \tag{41}
\]

The lower price equation

\[
    rf = \frac{\partial f}{\partial t} + r(z, \frac{\partial f}{\partial z}) + \frac{1}{2} \left[ \sigma_1^2 z_1^2 \frac{\partial^2 f}{\partial z_1^2} + 2\sigma_1 \sigma_2 z_1 z_2 \frac{\partial^2 f}{\partial z_1 \partial z_2} + \sigma_2^2 z_2^2 \frac{\partial^2 f}{\partial z_2^2} \right]. \tag{42}
\]
Fractional dynamics, I

Example: $J = 2$, sub-modular payoffs.

\[ X_{n+1}^\tau(z) = X_n^\tau(z) + \sqrt{\tau} \phi(X_n^\tau(z)), \quad X_0^\tau(z) = z, \]

where $\phi(z)$ is one of three points $(z^1 d_1, z^2 u_2), (z^1 u_1, z^2 d_2), (z^1 u_1, z^2 u_2)$ that are chosen with the corresponding risk neutral probabilities. As was shown above, this Markov chain converges, as $\tau \to 0$ and $n = \lceil t/\tau \rceil$ (where $[s]$ denotes the integer part of a real number $s$), to the diffusion process $X_t$ solving the Black-Scholes type (degenerate) equation (41), i.e. a sub-Markov process with the generator $Lf(x)$ being

\[
-rf + r(z, \frac{\partial f}{\partial z}) + \frac{1}{2} \left[ \sigma^2 z^2 \frac{\partial^2 f}{\partial z^2} - 2\sigma z z_1 \frac{\partial^2 f}{\partial z_1 \partial z_2} + \sigma^2 z^2 \frac{\partial^2 f}{\partial z_2^2} \right].
\]
Assume now that the times between jumps $T_1, T_2, \cdots$ are i.i.d.:

$$P(T_i \geq t) \sim \frac{1}{\beta t^\beta}$$

with $\beta \in (0, 1)$. It is well known that such $T_i$ belong to the domain of attraction of the $\beta$-stable law:

$$\Theta^\tau_t = \tau^{1/\beta}(T_1 + \cdots + T_{[t/\tau]})$$

converge, as $\tau \to 0$, to a $\beta$-stable Lévy motion $\Theta_t$, which is a Lévy process on $\mathbb{R}_+$ with the fractional derivative of order $\beta$ as the generator:

$$Af(t) = -\frac{d^\beta}{d(-t)^\beta}f(t) = -\frac{1}{\Gamma(-\beta)} \int_0^\infty (f(t + r) - f(t)) \frac{dr}{r^{1+\beta}}.$$
Fractional dynamics, III

We are now interested in the process

\[ Y_{\tau t}(z) = X_{N_{\tau t}}(z), \]

where

\[ N_{\tau t} = \max\{u : \Theta_{\tau u} \leq t\}. \]

The limiting process

\[ N_t = \max\{u : \Theta_u \leq t\} \]

is therefore the inverse (or hitting time) process of the \( \beta \)-stable Lévy motion \( \Theta_t \).
Fractional dynamics, IV

Theorem

The process $Y_t^T$ converges to $Y_t = X_{N_t}$, whose averages $f(T - t, x) = \mathbb{E} f(Y_{T-t}(x))$ have explicit representation

$$f(T - t, x) = \int_0^\infty \int_0^\infty \int_0^\infty G_{u^-}(z_1, z_2; w_1, w_2) Q(T - t, u) \, du \, dw_1 \, dw_2,$$

where $G^-$, the transition probabilities of $X_t$, $Q(t, u)$ denotes the probability density of the process $N_t$.

Moreover, for $f \in C^2_\infty(\mathbb{R}^d)$, $f(t, x)$ satisfy the (generalized) fractional evolution equation (of Black-Scholes type)

$$\frac{d^\beta}{dt^\beta} f(t, x) = Lf(t, x) + \frac{t^{-\beta}}{\Gamma(1 - \beta)} f(t, x).$$

General case leads to fractional extension of nonlinear Black-Scholes type equation (not worked out rigorously yet).