New results on (F)BSDE of quadratic growth

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1 Cross hedging, optimal investment, exponential utility

for convex constraints: (N. El Karoui, R. Rouge '00; J. Sekine '02; J. Cvitanic, J. Karatzas '92, Kramkov, Schachermayer '99, Mania, Schweizer '05, Pham '07, Zariphopoulou '01,...)

maximal expected exponential utility from terminal wealth

$$V(x) = \sup_{\pi \in \mathcal{A}} EU(x + X_T^\pi + H) = \sup_{\pi \in \mathcal{A}} E(-\exp(-\alpha(x + \int_0^T \pi_s [dW_s + \theta_s ds] + H)))$$

wealth on $[0, T]$ by investment strategy π :

$$\int_0^T \langle \pi_u, \frac{dS_u}{S_u} \rangle = \int_0^T \pi_u [dW_u + \theta_u du] = X_T^\pi,$$

H liability or derivative, correlated to financial market S

$\pi \in \mathcal{A}$ subject to π taking values in C **closed**

aim: use BSDE to represent optimal strategy π^*

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2 Martingale optimality

Idea: Construct family of processes $Q^{(\pi)}$ such that

(form 1)

$$\begin{array}{rcl} Q_0^{(\pi)} & = & \text{constant}, \\ Q_T^{(\pi)} & = & -\exp(-\alpha(x + X_T^\pi + H)), \\ Q^{(\pi)} & & \text{supermartingale}, \quad \pi \in \mathcal{A}, \\ Q^{(\pi^*)} & & \text{martingale, for (exactly) one} \quad \pi^* \in \mathcal{A}. \end{array}$$

Then

$$\begin{aligned} E(-\exp(-\alpha[x + X_T^\pi + H])) &= E(Q_T^{(\pi)}) \\ &\leq E(Q_0^\pi) \\ &= E(Q_0^{(\pi^*)}) \\ &= E(-\exp(-\alpha[x + X_T^{(\pi^*)} + H])). \end{aligned}$$

Hence π^* optimal strategy.

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3 Solution method based on BSDE

Introduction of BSDE into problem

Find generator f of BSDE

$$Y_t = H - \int_t^T Z_s dW_s - \int_t^T f(s, Z_s) ds, \quad Y_T = H,$$

such that with

$$Q_t^{(\pi)} = -\exp(-\alpha[x + X_t^\pi + Y_t]), \quad t \in [0, T],$$

we have

(form 2)

$$\begin{aligned} Q_0^{(\pi)} &= -\exp(-\alpha(x + Y_0)) = \text{constant}, && \text{(fulfilled)} \\ Q_T^{(\pi)} &= -\exp(-\alpha(x + X_T^\pi + H)) && \text{(fulfilled)} \\ Q^{(\pi)} & \quad \text{supermartingale}, \quad \pi \in \mathcal{A}, \\ Q^{(\pi^*)} & \quad \text{martingale, for (exactly) one} \quad \pi^* \in \mathcal{A}. \end{aligned}$$

This gives solution of valuation problem.

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4 Construction of generator of BSDE

How to determine f:

Suppose f generator of BSDE. Then by Ito's formula

$$\begin{aligned} Q_t^{(\pi)} &= -\exp(-\alpha[x + X_t^\pi + Y_t]) \\ &= Q_0^{(\pi)} + M_t^{(\pi)} + \int_0^t \alpha Q_s^{(\pi)}[-\pi_s\theta_s - f(s, Z_s) + \frac{\alpha}{2}(\pi_s - Z_s)^2]ds, \end{aligned}$$

with a local martingale $M^{(\pi)}$.

$Q^{(\pi)}$ satisfies **(form 2)** iff for

$$q(\cdot, \pi, z) = -f(\cdot, z) - \pi\theta + \frac{\alpha}{2}(\pi - z)^2, \quad \pi \in \mathcal{A}, z \in \mathbb{R},$$

we have

(form 3)
$$\begin{aligned} q(\cdot, \pi, z) &\geq 0, \quad \pi \in \mathcal{A} \quad \text{(supermartingale)} \\ q(\cdot, \pi^*, z) &= 0, \quad \text{for (exactly) one } \pi^* \in \mathcal{A} \quad \text{(martingale)}. \end{aligned}$$

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4 Construction of generator of BSDE

Now

$$
\begin{aligned}
q(\cdot,\pi,z) &= -f(\cdot,z) - \pi\theta + \frac{\alpha}{2}(\pi - z)^2 \\
&= -f(\cdot,z) + \frac{\alpha}{2}(\pi - z)^2 - (\pi - z)\cdot\theta + \frac{1}{2\alpha}\theta^2 \quad - z\theta - \frac{1}{2\alpha}\theta^2 \\
&= -f(\cdot,z) + \frac{\alpha}{2}[\pi - (z + \frac{1}{\alpha}\theta)]^2 \quad - z\theta - \frac{1}{2\alpha}\theta^2.
\end{aligned}
$$

Under **non-convex constraint** $p \in C$:

$$
[\pi - (z + \frac{1}{\alpha}\theta)]^2 \geq \text{dist}^2(C, z + \frac{1}{\alpha}\theta).
$$

with **equality** for at least one possible choice of π^* due to **closedness** of C.
Hence **(form 3)** is solved by the choice (predictable selection)

(form 4)

$$
\begin{aligned}
f(\cdot,z) &= \tfrac{\alpha}{2}\text{dist}^2(C, z + \tfrac{1}{\alpha}\theta) - z\cdot\theta - \tfrac{1}{2\alpha}\theta^2 \quad \text{(supermartingale)} \\
\pi^* &: \text{dist}(C, z + \tfrac{1}{\alpha}\theta) = \text{dist}(\pi^*, z + \tfrac{1}{\alpha}\theta) \quad \text{(martingale)}.
\end{aligned}
$$

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5 Summary of results, exponential utility

Solve utility optimization problem

$$\sup_{\pi \in \mathcal{A}} EU(x + X_T^\pi + H)$$

by considering FBSDE

$$\begin{aligned} dX_t^\pi &= \pi_t [dW_t + \theta_t dt], \quad X_0^\pi = x, \\ dY_t &= Z_t dW_t + f(t, Z_t) dt, \quad Y_T = H \end{aligned}$$

with generator as described before; determine π^* by previsible selection; coupling through requirement of martingale optimality

$$\begin{aligned} \sup_{\pi \in \mathcal{A}} EU(x + X_T^\pi + H) &= EU(x + X_T^{\pi^*} + H), \\ U'(x + X_t^{\pi^*} + Y_t) & \qquad \text{martingale}. \end{aligned}$$

for general U: forward part depends on π^*, get fully coupled FBSDE

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6 Cross hedging, optimal investment, utility on $\mathbb{R}$

Lit: Mania, Tevzadze (2003)

$U : \mathbb{R} \to \mathbb{R}$ strictly increasing and concave; maximal expected utility from terminal wealth

$$(1)\ V(x) = \sup_{\pi \in \mathcal{A}} EU(x + X_T^\pi + H)$$

wealth on $[0, T]$ by investment strategy π :

$$\int_0^T \langle \pi_u, \frac{dS_u}{S_u} \rangle = \int_0^T \pi_u [dW_u + \theta_u du] = X_T^\pi,$$

H liability or derivative, correlated to financial market S, W $d-$dimensional Wiener process, W^1 first d_1 components of W

$\pi \in \mathcal{A}$ subject to convex constraint $\pi = (\pi^1, 0)$, π^1 d_1-dimensional, hence incomplete market

aim: use FBSDE system to describe optimal strategy π^*

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7 Verification theorems

Thm 1
Assume U is three times differentiable, U' regular enough. If there exists π^* solving (1), and Y is the predictable process for which $U'(X^{\pi^*}+Y)$ is square integrable martingale, then with $Z = \frac{d}{dt}\langle Y, W\rangle$

$$(\pi^*)^1 = -\theta^1 \frac{U'}{U''}(X^{\pi^*}+Y) - Z^1.$$

Pf:

$$\alpha \;=\; \mathbb{E}(U'(X_T^{\pi^*}+H)|\mathcal{F}_\cdot),\; Y = (U')^{-1}(\alpha) - X^{\pi^*}.$$

Use Itô's formula and martingale property. Find

$$Y = H - \int_\cdot^T Z_s dW_s - \int_\cdot^T f(s, X_s^{\pi^*}, Y_s, Z_s)ds,$$

with

$$f(s, X_s^{\pi^*}, Y_s, Z_s) = -\frac{1}{2}\frac{U^{(3)}}{U''}(X^{\pi^*}+Y)|\pi_s^* + Z_s|^2 - \pi_s^*\theta_s.$$

Use variational maximum principle to derive formula for π^*.

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7 Verification theorems

From preceding theorem derive the FBSDE system

Thm 2
Assumptions of Thm 1; then optimal wealth process X^{π^*} given as component X of solution (X,Y,Z) of fully coupled FBSDE system

$$\begin{aligned} X &= x - \int_0^{\cdot} (\theta_s^1 \frac{U'}{U''}(X_s+Y_s) + Z_s^1) dW_s^1 - \int_0^{\cdot} (\theta_s^1 \frac{U'}{U''}(X_s+Y_s) + Z_s^1)\theta_s^1 ds, \\ Y &= H - \int_{\cdot}^T Z_s dW_s \\ & - \int_{\cdot}^T [|\theta_s^1|^2((-\frac{1}{2}\frac{U^{(3)}U'^2}{(U'')^3} + \frac{U'}{U''})(X_s+Y_s) + Z_s^1 \cdot \theta_s^1) \\ & \qquad -\frac{1}{2}|Z_s^2|^2 \frac{U^{(3)}}{U''}(X_s+Y_s)] ds. \quad (2) \end{aligned}$$

Pf:
Use expression for f and formula for π^* from Thm 1.

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8 Representation of optimal strategy

Invert conclusion of Thm 2 to give representation of optimal strategy

Thm 3
Let (X, Y, Z) be solution of (2), $U(X_T + H)$ integrable, $U'(X_T + H)$ square integrable. Then

$$(\pi^*)^1 = -\theta^1 \frac{U'}{U''}(X+Y) + Z^1$$

is optimal solution of (1).

Pf:
By concavity for any admissible π

$$U(X^\pi + Y) - U(X+Y) \leq U'(X+Y)(X^\pi - X).$$

Now prove that

$$U'(X+Y)(X^\pi - X) = U'(X^{\pi^*} + Y)(X^\pi - X^{\pi^*}) \quad \text{is a martingale!}$$

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9 Utility function on $\mathbb{R}_+$

Replace $U'(X^{\pi^*} + Y)$ with $X^{\pi^*} U'(X^{\pi^*}) \exp(\tilde{Y})$. Then (X, Y, Z) satisfies (3) if and only if $(X, \tilde{Y}, \tilde{Z})$ satisfies (4) ($\tilde{Z} = \frac{d}{dt}\langle W, \tilde{Y}\rangle$):

Thm 4
Let $(X, \tilde{Y}, \tilde{Z})$ be solution of the fully coupled FBSDE

$$\begin{aligned} X &= x - \int_0^\cdot (\frac{U'}{U''}(X_s)(\theta_s^1 + \tilde{Z}_s^1) dW_s^1 - \int_0^\cdot (\frac{U'}{U''}(X_s)(\theta_s^1 + \tilde{Z}_s^1)\theta_s^1 ds, \\ \tilde{Y} &= \ln(\frac{U'(X_T + H)}{U'(X_T)}) - \int_\cdot^T \tilde{Z}_s dW_s \\ & \quad - \int_\cdot^T [|\tilde{Z}_s^1 + \theta_s^1|^2((1 - \frac{1}{2}\frac{U^{(3)}U'}{(U'')^2})(X_s) - \frac{1}{2}|\tilde{Z}_s|^2] ds. \quad (4) \end{aligned}$$

such that $U(X_T^{\pi^*} + H)$ is integrable and $X^{\pi^*} U'(X^{\pi^*}) \exp(\tilde{Y})$ is a true martingale. Then

$$(\pi^*)^1 = -\frac{U'}{U''}(X)(Z^1 + \theta^1)$$

solves (1).

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10 The incomplete case

$U(x) = \frac{1}{p} x^p$ for $x > 0, p < 1$; $(X^{\pi^*}, \tilde{Y}, \tilde{Z})$ solution of (4)

then we saw that $G := X^{\pi^*} U'(X^{\pi^*}) \exp(\tilde{Y})$ is a martingale; X^{π^*} supermartingale, hence $U(X^{\pi^*})$ supermartingale; aim: use these two facts to construct solution by **iteration**

Notation: $\tilde{Z}_t dW_t := \tilde{Z}_t^1 dW_t^1$, $dN_t := \tilde{Z}_t^2 dW_t^2$; wlog: $\theta = 0$ (otherwise measure change, drift θ)

Initialization: Set $\tilde{Z}^0 := 0$ and

$$X^1 := x\mathbb{E}\Big(\frac{1}{1-p}\tilde{Z}^0 * W\Big) = x, \quad G_T^1 := \frac{X_T^1}{(X_T^1 + H)^{1-p}} \leq C|X_T^1|^p = Cx^p.$$

Obtain $(Z^1, N^1) \in \mathcal{H}^2$ via the following consequence of martingale representation

$$G_t^1 = \mathbb{E}[G_T^1|\mathcal{F}_t] = G_T^1 - \int_t^T G_s^1 \left(\frac{1}{1-p}\tilde{Z}_s^1 dW_s + dN_s^1\right)$$

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10 The incomplete case

n+1-*st* iteration

Z^n and N^n already obtained, define

$$X^{n+1} := x\mathbb{E}\big(\frac{1}{1-p}\tilde{Z}^n * W\big), \quad G_T^{n+1} := \frac{X_T^{n+1}}{(X_T^{n+1}+H)^{1-p}} \leq C|X_T^{n+1}|^p$$

with $\mathbb{E}G_T^{n+1} \leq Cx^p < \infty$.

Obtain (Z^{n+1}, N^{n+1}) via the following consequence of martingale representation

$$G_t^{n+1} = \mathbb{E}[G_T^{n+1}|\mathcal{F}_t] = G_T^{n+1} - \int_t^T G_s^{n+1}\left(\frac{1}{1-p}\tilde{Z}_s^{n+1}dW_s + dN_s^{n+1}\right)$$

- we obtain sequence $\big(\int \tilde{Z}_s^n dW_s, N^n\big)_{n\in\mathbb{N}}$
- convergence (possibly along subsequence)?

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10 The incomplete case

Thm 5
For $n \in \mathbb{N}$ we have

$$\mathbb{E}\left[\frac{1}{2}\int_0^T |\frac{1}{1-p}\tilde{Z}_s^n|^2 ds + \langle N^n\rangle_T\right] \leq \frac{1-p^n}{1-p}\big(2\log C + \log x^p\big)$$
$$+\frac{p^{n-1}}{2}\mathbb{E}\left[\frac{1}{2}\int_0^T |\frac{1}{1-p}\tilde{Z}_s^1|^2 ds + \langle N^1\rangle_T\right]$$
$$< \infty.$$

$\Longrightarrow \big(\int \tilde{Z}_s^n dW_s, N^n\big)_{n\in\mathbb{N}}$ bounded in $\mathcal{H}^2$

$\Longrightarrow$ Delbaen & Schachermayer (A compactness principle for bounded sequences of martingales with applications, 1996)

$\exists(\tilde{Z}, N)$ such that $\tilde{Z}^n \to \tilde{Z}$ and $N^n \to N$

Finally, obtain FBSDE solution $(X, \tilde{Y}, \tilde{Z}, N)$ by using $(\tilde{Z}, N)$ for solving for X and $\tilde{Y}$

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