Control of complex systems

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- "to develop mathematical foundations and applications for the control theory of complex systems"
- I seek two 2-year postdocs:
  - one on the mathematical foundations of control of probability distributions for spatially extended stochastic systems,
  - the other on computer tests and demonstrations of control procedures.

Control of probability distributions v trajectories

- e.g. climate v weather, “democratic” socio-economics v Orwell’s big brother, management v control
- To control trajectories of a chaotic dynamical system requires control with higher gain than the Lyapunov exponent, but to modify the probability distribution is more feasible (cf. Ruelle’s formula for derivative of SRB measure wrt time-dependent control)
- Start with open loop control (feedback loops raise questions about how to model the effects of observations, which in principle should collapse the probability distribution), and stochastic dynamics, in particular probabilistic cellular automata (PCA)
- I need to learn more about what is already done in stochastic control theory (I did read Mark Davis’ book): what I say may be already known or wrong or useless!

Sensitivity of stationary probability

- For a Markov chain with transition operator P depending smoothly on parameters λ, suppose for $\lambda=0$ there is a non-degenerate stationary probability $\pi_0=\pi_0 P_0$, i.e. $(I-P_0)_Z$ is invertible, where Z=zero-charge measures, then for all small λ there is a non-degenerate stationary probability π_λ and $\pi' = \pi P'(I-P)^{-1}$.
- If π_0 attracts exponentially then so does π_λ for small enough λ.

Response to time-dependent control

- Suppose time-dependent transition operator P_t, near an exponentially mixing one, then there is a unique time-dependent probability π_t and

  $\pi'_t = \pi_{t-1}P'_{t-1} + \pi_{t-2}P'_{t-2}P_{t-1} + \pi_{t-3}P'_{t-3}P_{t-2}P_{t-1} + \ldots$
- To make sense of all the above for PCA, need a suitable metric on space of probabilities on a large product space.

How to measure distance between multivariate probability distributions

- S countable set
- For s in S, (X_s,d_s) Polish (complete separable metric) space of diameter $\leq \Omega$
- $X = \Pi X_s$ with product topology
- Ⓟ = Borel probabilities on X; want a metric on Ⓟ
- All standard metrics are useless when |S| is large, e.g. "Total variation convergence essentially never occurs for particle systems" (Liggett, 1985). Same for Jeffreys-Jensen-Shannon, Hellinger, Fisher information, projective, transportation (Vasserstein, Kantorovich, Rubinstein) metrics.

Dobrushin metric

- BC = bounded continuous functions $f:X \rightarrow R$
- $\Delta_s(f) = \sup\, (f(x)-f(y))/d_s(x_s,y_s)$
  over x,y in X with $x_r = y_r$ for all $r \neq s$, $x_s \neq y_s$.
- $|f| = \Sigma\, \Delta_s(f)$, *Dobrushin semi-norm*
- F = {f in BC: $|f| < \infty$}, *Dobrushin's functions*
- Z = Borel zero-charge measures μ on X, i.e. $\mu(X)=0$
- $|\mu| = \sup\, \mu(f)/|f|$ over non-constant f in F
- $(Z,|.|)$ is a Banach space
- For ρ,σ in Ⓟ:
  $D(\rho,\sigma) = |\rho-\sigma|$, *Dobrushin metric,*
  makes Ⓟ a complete metric space (of diameter = $\sup\, \mathrm{diam}_s(X_s)$)
- Not purely information theoretic; reflects metrics on the X_s.

Applications to PCA

- Probability p_s^x on X_s for new state x_s' of site s in S given current state x in X
- Transition probability $p^x = \Pi\, p_s^x$
- Transition operator P on f in BC:

  $(Pf)(x) = p^x(f)$
- Induces P on ρ in Ⓟ by $(\rho P)(f) = \rho(Pf)$
- Want to bound $|P|$ on Z

Dobrushin's dependency matrix

- For ρ,σ probabilities on X_r, let
  $D_r(\rho,\sigma) = \sup (\rho(g)-\sigma(g))/|g|$
  over non-constant Lipschitz functions $g: X_r \to R$, $|g|$ = best Lipschitz constant
- For r,s in S, let $K_{rs} = \sup D_r(p_r^x,p_r^y)/d_s(x_s,y_s)$
  over x,y in X with $x_q = y_q$ for all $q \neq s$, $x_s \neq y_s$.
- Then $|P| \leq |K|_\infty$.
- In particular, $|K|_\infty < 1$ implies P has a unique stationary probability π and it attracts exponentially
- e.g. Stavskaya for $\lambda>½$, NEC voter for λ in (⅓,⅔)
- Same if $|K^t|_\infty \leq Cr^t$ for some $r<1$, $D(\sigma P^t,\pi) \leq Cr^t D(\sigma,\pi)$

More on the exponentially mixing regime

- Exponentially attracting stationary probability is stable to perturbation:

  $D(\sigma P^t,\pi) \leq C(r+C|P-P_0|)^t D(\sigma,\pi)$
- Can use Dobrushin metric to define C^1 dependence of P on parameters λ and deduce C^1 dependence of π on λ with $\pi' = \pi P' (I-P)^{-1}$.
- And time-dependent response formula converges
- Question: range of control?

Beyond the exponentially mixing regime

- For S infinite can get non-unique stationary probability, or non-mixing ones, even if finite truncations would not.
- e.g. Stavskaya with λ small, NEC voter with λ small or near 1. [demonstrate]
- Questions of control to influence selection, range of control... [demonstrate effect of boundary control on NEC voter]