Passive realizations of stationary stochastic processes

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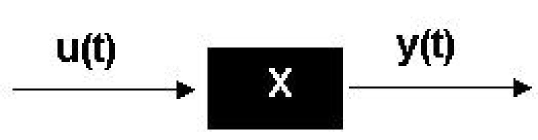
December 2, 2011

Linear Time Invariant Dynamical Systems

The object of our research is Linear Time Invariant Dynamical System with discrete time. Such a system can be schematically represented as a ”black box” X.



Linear Time Invariant Dynamical Systems

The evolution of such Linear System with discrete time and Hilbert spaces of input and output data U, Y, respectively, and state space X can be described by equations

$$\begin{cases} x(t+1) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases}$$

where $x(t) \in X,\ u(t) \in U,\ y(t) \in Y$, and

$$A \in \mathbb{B}(X), \qquad B \in \mathbb{B}(U, X), \qquad C \in \mathbb{B}(X, Y), \qquad D \in \mathbb{B}(U, Y).$$

Linear Time Invariant Dynamical Systems

Let

$$X_\Sigma^c = \bigvee_{k\geq 0} A^k BU, \qquad X_\Sigma^o = \bigvee_{k\geq 0} (A^*)^k C^* Y.$$

System Σ is said to be

controllable	if	$X = X_\Sigma^c$;
observable	if	$X = X_\Sigma^o$;
simple	if	$X = X_\Sigma^c \vee X_\Sigma^o$.

System Σ is *minimal* if and only if it is controllable and observable, i.e.

$$X = X_{\Sigma}^{c} = X_{\Sigma}^{o}.$$

Linear Time Invariant Dynamical Systems

$\mathbb{B}(U, Y)$-valued function θ_Σ defined by formulae

$$\theta_\Sigma(z) = D + zC(I - zA)^{-1}B, \qquad z \in \Lambda_A,$$

where Λ_A is the set of $z \in \mathbb{C}$ for which bounded inverse $(I - zA)^{-1}$ existsis said to be the *transfer function* of the system Σ.

Systems $\Sigma_i = (A_i, B_i, C_i, D_i; X_i, U, Y)$, $i = 1, 2$, are called unitary similar if there exists unitary operator $R \in \mathbb{B}(X_1, X_2)$ such that

$$A_2 = RA_1R^{-1}, \qquad B_2 = RB_1, \qquad C_2 = C_1R^{-1}, \qquad D_2 = D_1.$$

If the main operator A of the system Σ has the property

$$\lim_{n\to\infty} A^n = 0, \qquad \lim_{n\to\infty} (A^*)^n = 0,$$

then system Σ is said to be bi-stable.

If in this case A is contractive operator then it can be written

$$A \in C_{00}$$

Passive Linear Systems

We devote the most attention to **passive** linear dynamical systems. System $\Sigma = (A, B, C, D; X, U, Y)$ is said to be passive system if for any initial condition and for any input data $\{u(t)\} \subset U$ the following inequality is valid

$$\|x(t+1)\|^2 - \|x(t)\|^2 \leq \left(\Phi \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \right)_{U \oplus Y} \qquad (1)$$

where Φ is a "power" operator such that $\Phi = \Phi^*$.

Inequality

$$\|x(t+1)\|^2 - \|x(t)\|^2 \leq \left(\Phi \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \right)_{U \oplus Y}$$

is said to be **condition of passivity** of system Σ and it has the next physical meaning:
– $\| \, . \, \|^2$ is interpreted as an energy;
– left part of inequality is interpreted as an energy variation of internal states at the moment t;
– right part - as an external energy of the system Σ at the moment t.

Condition of passivity

$$\|x(t+1)\|^2 - \|x(t)\|^2 \leq \left(\Phi \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \right)_{U \oplus Y}$$

means that Σ has no internal energy sources, i.e. system does not produce additional energy but can absorb energy.

If in this condition we have equality sign and the same situation is in corresponding inequality for adjoint system $\Sigma^* = (A^*, C^*, B^*, D^*; X, Y, U)$ then system Σ is **conservative**. Such a system saves energy.

Passive Scattering Systems

Passive linear system $\Sigma = (A, B, C, D; X, U, Y)$ is said to be **passive scattering system** if the corresponding "power" operator Φ is such that

$$\Phi = \begin{bmatrix} I_U & 0 \\ 0 & -I_Y \end{bmatrix}.$$

For system of this type condition of passivity takes the following form

$$\|x(t+1)\|^2 - \|x(t)\|^2 \leq \|u(t)\|^2 - \|y(t)\|^2. \qquad (2)$$

Input and output data of passive scattering systems can be interpreted as incoming and outgoing waves, respectively.

Passive Scattering Systems

Condition (2) means that the operator

$$M_\Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \Big(\in \mathbb{B}(X \oplus U, X \oplus Y) \Big)$$

is a contraction.

System Σ is said to be a **conservative scattering system** if opeator M_Σ is unitary, i.e.

$$M_\Sigma^* M_\Sigma = I_{X \oplus U}, \qquad M_\Sigma M_\Sigma^* = I_{X \oplus Y}.$$

Passive Scattering Systems

Transfer function $\theta_\Sigma(z)$ of passive scattering system is called **scattering matrix**.

Restriction of scattering matrix of arbitrary passive scattering system on open unit disk $\mathbb{D}$ belongs to the class $S(U, Y)$ of holomorphic in $\mathbb{D}$ functions $s(z)$ with values from $\mathbb{B}(U, Y)$ that have

$$b(z)^* b(z) \leq I_U, \qquad b(z) b(z)^* \leq I_Y, \qquad z \in \mathbb{D}.$$

We denote $S_{in}(U, Y)$ the subclass of functions $b \in S(U, Y)$ that are bi-inner, i.e.

$$b(\zeta)^* b(\zeta) = I_U, \qquad b(\zeta) b(\zeta)^* = I_Y, \qquad \text{a.e. } |\zeta| = 1.$$

Passive Scattering Systems

Arbitrary function $\theta(z)$ from $S(U, Y)$ is the restriction on $\mathbb{D}$ of scattering matrix of some simple conservative scattering system which can be defined by θ up to unitary similarity.

Simple conservative scattering system is bi-stable if and only if the restriction of its scattering matrix on $\mathbb{D}$ is in the class $S_{in}(U, Y)$.

Passive Scattering Systems

- **Arov D.Z.** *Stable dissipative linear time invariant dynamical scattering systems*, J. Operator Theory, 2 (1979), 95-126.
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Passive Impedance Systems

If passive linear system $\Sigma = (A, B, C, D; X, U, Y)$ has the same input and output spaces $Y = U$ and the "power" operator Φ has the form

$$\Phi = \begin{bmatrix} 0 & I_U \\ I_U & 0 \end{bmatrix}$$

then Σ is said to be **passive impedance system**. For system of this type condition of passivity takes the following form

$$\|x(t+1)\|^2 - \|x(t)\|^2 \leq 2\Re(u(t), y(t))_U. \tag{3}$$

For passive impedance systems input and output data are interpreted as voltages and currents, respectively.

Passive Impedance Systems

Condition (3) is equivalent to the next inequality for coefficients of system $\Sigma = (A, B, C, D; X, U)$

$$\begin{bmatrix} I - A^*A & C^* - A^*B \\ C - B^*A & 2\Re D - B^*B \end{bmatrix} \geq 0. \quad (4)$$

System $\Sigma = (A, B, C, D; X, U)$ is passive impedance system if and only if the adjoint system $\Sigma^* = (A^*, C^*, B^*, D^*; X, U)$ is passive impedance system, i.e.

$$\begin{bmatrix} I - AA^* & B - AC^* \\ B^* - CA^* & 2\Re D - CC^* \end{bmatrix} \geq 0. \quad (5)$$

Passive Impedance Systems

Transfer function

$$\theta_\Sigma(z) = D + zC(I - zA)^{-1}B$$

of passive impedance system is called **impedance matrix**.

Restriction on $\mathbb{D}$ of impedance matrix belongs to the class $\ell(U)$ of analytic in $\mathbb{D}$ functions $c(z)$ with the values from $\mathbb{B}(U)$ and with $\Re c(z) \geq 0$ in $\mathbb{D}$.

An arbitrary function $c \in \ell(U)$ is the restriction on $\mathbb{D}$ of impedance matrix of certain simple conservative impedance system that can be defined by $c(z)$ up to unitary similarity.

Passive impedance system $\Sigma_o = (A_o, B_o, C_o, D_o; X_o, U)$ with impedance matrix $\theta_{\Sigma_o}(z)$ is said to be **optimal** if for any other passive impedance system $\Sigma = (A, B, C, D; X, U)$ with impedance matrix $\theta_{\Sigma}(z) \equiv \theta_{\Sigma_o}(z)$ in $\mathbb{D}$ for any input data

$$\|x_o(t)\|^2 \leq \|x(t)\|^2.$$

Observable passive impedance system $\Sigma_1 = (A_1, B_1, C_1, D_1; X_1, U)$ is said to be ***-optimal** if for any other observable passive impedance system $\Sigma = (A, B, C, D; X, U)$ with the same impedance matrix in $\mathbb{D}$

$$\|x(t)\|^2 \leq \|x_1(t)\|^2.$$

Let $\tilde{U}$ and $\tilde{Y}$ be some Hilbert spaces, $J_1 \in \mathbb{B}(\tilde{U})$ and $J_2 \in \mathbb{B}(\tilde{Y})$ are signature operators, i.e.

$$J_i^* = J_i, \quad i = 1, 2; \qquad J_1^2 = I_{\tilde{U}}, \quad J_2^2 = I_{\tilde{Y}}.$$

These operators define indefinite metrics on $\tilde{U}$ and $\tilde{Y}$ such that

$$\langle \tilde{u}, \tilde{u}' \rangle = \left(J_1 \tilde{u}, \tilde{u}' \right), \quad \langle \tilde{y}, \tilde{y}' \rangle = \left(J_2 \tilde{y}, \tilde{y}' \right) \quad \tilde{u}, \tilde{u}' \in \tilde{U}, \quad \tilde{y}, \tilde{y}' \in \tilde{Y}.$$

Conservative Transmission Systems

System $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; \tilde{X}, \tilde{U}, \tilde{Y})$ is said to be **conservative transmission system** if for any initial state and for any input data $\{\tilde{u}(t)\}$ the following condition holds

$$\|\tilde{x}(t+1)\|^2 - \|\tilde{x}(t)\|^2 = \left(\Phi_{J_1,J_2} \begin{bmatrix} \tilde{u}(t) \\ \tilde{y}(t) \end{bmatrix}, \begin{bmatrix} \tilde{u}(t) \\ \tilde{y}(t) \end{bmatrix} \right)_{\tilde{U}\oplus\tilde{Y}},$$

$$\Phi_{J_1,J_2} = \begin{bmatrix} J_1 & 0 \\ 0 & -J_2 \end{bmatrix},$$

and dual equality holds for the adjoint system $\tilde{\Sigma}^* = (\tilde{A}^*, \tilde{C}^*, \tilde{B}^*, \tilde{D}^*; \tilde{X}, \tilde{Y}, \tilde{U})$ with the power operator

$$\Phi_{J_2,J_1} = \begin{bmatrix} J_2 & 0 \\ 0 & -J_1 \end{bmatrix}.$$

The fact that $\tilde{\Sigma}$ is conservative transmission system means that operator

$$M_{\tilde{\Sigma}} = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \in \mathbb{B}\left(\tilde{X} \oplus \tilde{U}, \tilde{X} \oplus \tilde{Y}\right)$$

is $(\tilde{J}_1, \tilde{J}_2)$-unitary, i.e.

$$M_{\tilde{\Sigma}}^* \tilde{J}_2 M_{\tilde{\Sigma}} = \tilde{J}_1, \qquad M_{\tilde{\Sigma}} \tilde{J}_1 M_{\tilde{\Sigma}}^* = \tilde{J}_2, \tag{6}$$

where

$$\tilde{J}_i = \begin{bmatrix} I_X & 0 \\ 0 & J_i \end{bmatrix}, \; i = 1, 2.$$

Conservative Transmission SI-Systems

It was shown in **[Arov, Rozhenko 2008]** that an arbitrary passive bi-stable impedance system

$$\Sigma = (A, B, C, D; X, U)$$

is the part of some conservative transmission system $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; X, \tilde{U}, \tilde{Y})$ with outer spaces $\tilde{U} = U_1 \oplus U \oplus U$ and $\tilde{Y} = Y_1 \oplus U \oplus U$ and with corresponding operators J_1, J_2 such that

$$J_1 = \begin{bmatrix} I_{U_1} & 0 & 0 \\ 0 & 0 & -I_U \\ 0 & -I_U & 0 \end{bmatrix}, \qquad J_2 = \begin{bmatrix} I_{Y_1} & 0 & 0 \\ 0 & 0 & -I_U \\ 0 & -I_U & 0 \end{bmatrix}. \quad (7)$$

In this case operators of $\tilde{\Sigma}$ have special block structure:

$$\tilde{A} = A, \quad \tilde{B} = \begin{bmatrix} K & B & 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} M \\ C \\ 0 \end{bmatrix}, \\ \tilde{D} = \begin{bmatrix} S & N & 0 \\ L & D & I_U \\ 0 & I_U & 0 \end{bmatrix}. \tag{8}$$

Conservative Transmission SI-Systems

The operators

$$M \in \mathbb{B}(X, Y_1), \quad K \in \mathbb{B}(U_1, X), \quad S \in \mathbb{B}(U_1, Y_1),$$

$$N \in \mathbb{B}(U, Y_1), \qquad L \in \mathbb{B}(U_1, U)$$

are such that

$$\begin{bmatrix} I - A^*A & C^* - A^*B \\ C - B^*A & 2\Re D - B^*B \end{bmatrix} = \begin{bmatrix} M^* \\ N^* \end{bmatrix} \begin{bmatrix} M & N \end{bmatrix},$$

$$\begin{bmatrix} I - AA^* & B - AC^* \\ B^* - CA^* & 2\Re D - CC^* \end{bmatrix} = \begin{bmatrix} K \\ L \end{bmatrix} \begin{bmatrix} K^* & L^* \end{bmatrix},$$

$$L = B^*K + N^*S, \qquad N = MC^* + SL^*;$$

and the operator $$V = \begin{bmatrix} A & K \\ M & S \end{bmatrix} \in \mathbb{B}(X \oplus U_1, X \oplus Y_1) \quad (9)$$

is unitary. All these conditions are equivalent to (6).

Conservative Transmission SI-Systems

The inverse statement is also true.

If $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; X, \tilde{U}, \tilde{Y})$ is conservative transmission system with special block structure (8) of operators $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ and operators J_1, J_2 that are defined in (7) and with $\tilde{A} \in C_{00}$, then the part of it – system $\Sigma = (A, B, C, D; X, U)$ is passive impedance bi-stable system.

Conservative transmission system $\tilde{\Sigma}$ with operators J_1, J_2 of the form (7) that have corresponding block structures of the coefficients (8) is called **conservative transmission SI-systems** (scattering-impedance).

Conservative Transmission SI-Systems

Restriction on $\mathbb{D}$ of transfer function $\theta(z)$ of conservative transmission SI-system $\tilde{\Sigma} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}; X, \tilde{U}, \tilde{Y})$ is analytic bi-(J_1, J_2)-inner in $\mathbb{D}$ function, i.e. such that

$$\theta(z)^* J_2 \theta(z) \leq J_1, \qquad \theta(z) J_1 \theta(z)^* \leq J_2, \qquad z \in \mathbb{D},$$

$$\theta(\zeta)^* J_2 \theta(\zeta) = J_1, \qquad \theta(\zeta) J_1 \theta(\zeta)^* = J_2, \quad \text{a.e.} \quad |\zeta| = 1,$$

with special block structure

$$\theta(z) = \begin{bmatrix} \alpha(z) & \beta(z) & 0 \\ \gamma(z) & c(z) & I_U \\ 0 & I_U & 0 \end{bmatrix}, \qquad z \in \mathbb{D}, \tag{10}$$

where operators J_1, J_2 are defined in (7).

Conservative Transmission SI-Systems

Blocks of $\theta(z)$ have the following forms for $z \in \mathbb{D}$

$$\alpha(z) = S + zM(I - zA)^{-1}K, \qquad \beta(z) = N + zM(I - zA)^{-1}B,$$
$$\gamma(z) = L + zC(I - zA)^{-1}K, \qquad c(z) = D + zC(I - zA)^{-1}B.$$

Moreover, arbitrary function θ with block structure (10) with marked properties is the restriction on $\mathbb{D}$ of transmission matrix of certain simple conservative transmission SI-system which can be defined by θ up to unitary similarity.

Such a function $\theta(z)$ with given block $c(z)$ in $\mathbb{D}$ is said to be **the dilation** of $c(z)$.

Theorem.

Function $c(z)$ that maps from $\mathbb{D}$ to $\mathbb{B}(U)$ is the restriction on $\mathbb{D}$ of impedance matrix of some passive bi-stable impedance system $\Sigma = (A, B, C, D; X, U)$

$$\Longleftrightarrow$$

- *$c(z) \in \ell(U)$ and has absolutely continuous spectral function,*
- *there exists the dilation θ of c with values from $\mathbb{B}(U_1 \oplus U \oplus U, Y_1 \oplus U \oplus U)$ with special block structure (10), where U_1 and Y_1 are Hilbert spaces and J_1, J_2 are operators of the form (7).*

[Arov, Rozhenko 2008-2009]

Conservative Transmission SI-Systems

Blocks of $\theta(z)$ have the following properties:

- $\beta \in H^2(U, Y_1)$ and $\gamma^{\sim} \in H^2(U, U_1)$ are the solutions of

$$\beta(\zeta)^*\beta(\zeta) = 2\Re c(\zeta), \ \gamma(\zeta)\gamma(\zeta)^* = 2\Re c(\zeta), \ \text{a.e.}\ |\zeta| = 1, \tag{11}$$

- function $\alpha(z)$ is bi-inner scattering matrix of the conservative scattering system $\Sigma_{scat} = (A, K, M, S; X, U_1, Y_1)$, where operators $K \in \mathbb{B}(U_1, X)$, $M \in \mathbb{B}(X, Y_1)$ and $S \in \mathbb{B}(U_1, Y_1)$ appear as blocks of unitary operator V in (9).

Passive Impedance Systems

- **Arov D.Z., Rozhenko N.A.** *$J_{p,m}$-inner dilations of matrix-valued functions that belong to the Caratheodory class and admit pseudocontinuation*, St. Petersburg Mathematical Journal, 2008, 19:3, 375–395.
- **Arov D.Z., Rozhenko N.A.** *Passive impedance systems with losses of scattering channels*, Ukr. Math. J. 59, 2007, No. 5, 678-707.
- **Arov D.Z., Rozhenko N.A.** *To the theory of passive systems of resistance with losses of scattering channels* Journal of Mathematical Sciences, 2009, Vol. 156, No. 5, 742–760.
- **Arov D.Z., Rozhenko N.A.** *On connection between Darlington representations of Caratheodory matrix functions and their $J_{p,r}$-inner SI-dilations*, Mathematical Notes, Vol. 90, 6, 2011, 821–832.

Let $y(t) = \{y_k(t)\}_{k=1}^{p}$ be a stationary in a wide sense regular stochastic vector process with spectral density $\rho(\mu)$ of rank $m \leq p$ and with corresponding Hilbert space $H(y)$ of it's values. In **[A.Lindquist, G.Picci 1981–2008]** the forward (Σ_f) and backward (Σ_b) realizations of $y(t)$ as output data of systems

$$(\Sigma_f) \quad \begin{cases} x_f(t+1) = Ax_f(t) + Kw_f(t), \\ y(t) = Cx_f(t) + Lw_f(t), \end{cases} \tag{12}$$

$$(\Sigma_b) \quad \begin{cases} x_b(t-1) = \tilde{A}x_b(t) + \tilde{K}w_b(t), \\ y(t) = \tilde{C}x_b(t) + \tilde{L}w_b(t), \end{cases} \tag{13}$$

were considered.

Stochastic Realizations of Stationary Processes

System Σ_f develops forward in time $t \in \mathbb{Z}$, system Σ_b develops backward in time $t \in \mathbb{Z}$. In the equations (12), (13) w_f and w_b are vector white noises of order m that have properties

$$H(w_f) = H(w_b) := \mathfrak{H}, \qquad H(y) \subseteq \mathfrak{H}; \tag{14}$$

x_f and x_b are processes of internal state such that

$$\begin{gathered} \lim_{t\to-\infty} x_f(t) = \lim_{t\to+\infty} x_b(t) = 0, \\ H(x_f) \subset \mathfrak{H}, \qquad H(x_b) \subset \mathfrak{H}; \end{gathered} \tag{15}$$

coefficients $A, K, C, L, \tilde{A}, \tilde{K}, \tilde{C}, \tilde{L}$ are linear bounded operators in $\mathfrak{H}$.

Stochastic Realizations of Stationary Processes

It was shown in **[A.Lindquist, G.Picci 1981–2008]** that the coefficients $A, K, C, L, \tilde{A}, \tilde{K}, \tilde{C}, \tilde{L}$ of systems Σ_f and Σ_b completely defined via given process $y(t)$ and are such that

$$A \in C_{00}, \quad \tilde{A} = A^*, \quad I = AA^* + KK^* = A^*A + \tilde{K}\tilde{K}^*,$$

$$\tilde{C} = CA^* + LK^*, \quad C = \tilde{C}A + \tilde{L}\tilde{K}^*,$$

$$E\{y(0)y(0)^*\} = CC^* + LL^* = \tilde{C}\tilde{C}^* + \tilde{L}\tilde{L}^*.$$

- **A. Lindquist, G. Picci** *On a condition for minimality of Markovian splitting subspace* Systems Control Lett. 1, 1981–1982, 4, 264–269.
- **A. Lindquist, M. Pavon** *On the structure of state-space models for discrete time stochastic vector processes*, IEEE Trans. Auto. Control, 1984, AC-29, 418–432.
- **A. Lindquist, G. Picci** *Realization theory for multivariate stationary Gaussian processes*, SIAM J. Control and Optimization, 1985, vol 23, 809–857.
- **A. Lindquist, G. Picci** *Linear Stochastic systems: a geometric approach to modeling, estimation and identification*, in preparation.

Stochastic Realizations of Stationary Processes

Following theorem is the criterium of existence of such realizations for stationary vector stochastic process.

Theorem.

Stationary stochastic process $y(t) = \{y_k(t)\}_{k=1}^{p}$ *of rank m can be presented as an output data of systems* Σ_f *and* Σ_b *if and only if the spectral density of* $y(t)$ *is nontangential boundary value of certain function with bounded Nevanlinna characteristic.*

[Arov, Rozhenko 2011]

Now we present a new approach for realizations of stationary stochastic processes using our model of conservative transmission SI-system.
Let $\rho(e^{i\mu})$ be the spectral density of stationary in a wide sense stochastic vector process $y(t) = \{y_k(t)\}_{k=1}^{p}$ of rank $m \leq p$.
From now on we'll suppose that density

$$\rho(e^{i\mu}) = \sum_{t=-\infty}^{\infty} R(t)e^{it\mu}, \quad \text{where} \quad R(t) = \{Ey_k(t)\overline{y_j(0)}\}_{k,j=1}^{p},$$

satisfies conditions of the previous theorem.

In this case it can be shown that Caratheodory function (corresponding "nonnegative tale" of spectral density)

$$c(z) = \frac{1}{2}R(0) + \sum_{t=1}^{\infty} R(t)z^t,$$

$$\rho(\zeta) = 2\Re c(\zeta) \quad \text{a.e.} \quad |\zeta| = 1$$

will be such that there exists the dilation $\theta(z)$ of $c(z)$ of the form

$$\theta = \begin{bmatrix} \alpha & \beta & 0 \\ \gamma & c & I_p \\ 0 & I_p & 0 \end{bmatrix} \quad \text{with} \quad J_1 = J_2 = \begin{bmatrix} I_m & 0 & 0 \\ 0 & 0 & -I_p \\ 0 & -I_p & 0 \end{bmatrix}.$$

Blocks of matrix function $\theta(z)$ have the following properties:

$$\alpha \in S_{in}^{m\times m}, \qquad \beta \in H_2^{m\times p}\Pi, \qquad \gamma \in H_2^{p\times m}\Pi,$$

$$\beta(z)^*\beta(z) \leq 2\Re c(z), \qquad \gamma(z)\gamma(z)^* \leq 2\Re c(z), \qquad z \in \mathbb{D},$$

$$\beta(\zeta)^*\beta(\zeta) = \rho(\zeta)(= 2\Re c(\zeta)), \quad \text{a.e.} \quad |\zeta| = 1,$$
$$\gamma(\zeta)\gamma(\zeta)^* = \rho(\zeta)(= 2\Re c(\zeta)), \quad \text{a.e.} \quad |\zeta| = 1,$$

$$\alpha(\zeta)^*\beta(\zeta) = \gamma(\zeta)^*, \qquad \text{a.e.} \quad |\zeta| = 1.$$

Using dilation $\theta(z)$ and results of **[Arov, Rozhenko 2008]** it is possible to build up a functional model of corresponding conservative transmission SI-system $\dot{\tilde{\Sigma}} = (\dot{\tilde{A}}, \dot{\tilde{B}}, \dot{\tilde{C}}, \dot{\tilde{D}}; \dot{X}, \tilde{U}, \tilde{Y})$ with transfer function $\tilde{\theta} \equiv \theta$ in $\mathbb{D}$ such that

$$\tilde{U} = \tilde{Y} = \mathbb{C}^m \oplus \mathbb{C}^p \oplus \mathbb{C}^p, \qquad \dot{X} = H_2^m \ominus \alpha H_2^m,$$

$$\begin{bmatrix} \dot{\tilde{A}} & \dot{\tilde{B}} \\ \dot{\tilde{C}} & \dot{\tilde{D}} \end{bmatrix} = \begin{bmatrix} \dot{A} & \dot{K} & \dot{B} & 0 \\ \dot{M} & \dot{S} & \dot{N} & 0 \\ \dot{C} & \dot{L} & \dot{D} & I_p \\ 0 & 0 & I_p & 0 \end{bmatrix} : \quad \dot{X} \oplus \tilde{U} \longrightarrow \dot{X} \oplus \tilde{Y};$$

Stochastic Realizations of Stationary Processes

In this model system $\dot{\Sigma}_{imp} = (\dot{A}, \dot{B}, \dot{C}, \dot{D}; \dot{X}, \mathbb{C}^p)$ is passive bi-stable impedance system with impedance matrix

$$c(z) = \dot{D} + z\dot{C}(I_m - z\dot{A})^{-1}\dot{B}, \qquad z \in \mathbb{D}. \tag{16}$$

System $\dot{\Sigma}_{scat} = (\dot{A}, \dot{K}, \dot{M}, \dot{S}; \dot{X}, \mathbb{C}^m, \mathbb{C}^m)$ is simple conservative scattering system with bi-inner scattering matrix

$$\alpha(z) = \dot{S} + z\dot{M}(I_m - z\dot{A})^{-1}\dot{K}, \qquad z \in \mathbb{D}, \tag{17}$$

Operators of conservative transmission SI-system $\dot{\tilde{\Sigma}}$ are connected in the following way

$$I_m - \dot{A}^*\dot{A} = \dot{M}^*\dot{M}, \qquad \dot{C}^* - \dot{A}^*\dot{B} = \dot{M}^*\dot{N},$$
$$2\Re\dot{D} - \dot{B}^*\dot{B} = \dot{N}^*\dot{N},$$

$$I_m - \dot{A}\dot{A}^* = \dot{K}\dot{K}^*, \qquad \dot{B} - \dot{A}\dot{C}^* = \dot{K}\dot{L}^*,$$
$$2\Re\dot{D} - \dot{C}\dot{C}^* = \dot{L}\dot{L}^*,$$

$$\dot{L} = \dot{B}^*\dot{K} + \dot{N}^*\dot{S}, \qquad \dot{N} = \dot{M}\dot{C}^* + \dot{S}\dot{L}^*,$$

$$\dot{A}^*\dot{K} = -\dot{M}^*\dot{S}, \qquad I_m - \dot{S}^*\dot{S} = \dot{K}^*\dot{K},$$
$$\dot{A}\dot{M}^* = -\dot{K}\dot{S}^*, \qquad I_m - \dot{S}\dot{S}^* = \dot{M}\dot{M}^*.$$

Using matrix functions γ and β we can construct white noises $w_f(t)$ and $w_b(t)$ of size m respectively. Then

$$H(w_f) = H(w_b) = H(y),$$

and it can be shown that the system $\dot{\Sigma}_f = (\dot{A}, \dot{K}, \dot{C}, \dot{L}; \dot{X}, \mathbb{C}^m, \mathbb{C}^p)$ is forward realization of stochastic process $y(t)$:

$$(\dot{\Sigma}_f) \quad \begin{cases} x_f(t+1) = \dot{A}x_f(t) + \dot{K}w_f(t), \\ y(t) = \dot{C}x_f(t) + \dot{L}w_f(t). \end{cases} \tag{18}$$

Restriction on $\mathbb{D}$ of transfer function of system $\dot{\Sigma}_f$ coincides with block $\gamma(z)$ of the dilation $\theta(z)$:

$$\gamma(z) = \dot{L} + z\dot{C}(I_m - z\dot{A})^{-1}\dot{K}, \qquad z \in \mathbb{D}, \tag{19}$$

and it is the full rank spectral factor of density ρ, i.e.

$$\gamma(\zeta)\gamma(\zeta)^* = \rho(\zeta) \qquad \text{a.e. } |\zeta| = 1.$$

Stochastic Realizations of Stationary Processes

System $\dot{\Sigma}_b = (\dot{A}^*, \dot{M}^*, \dot{B}^*, \dot{N}^*; \dot{X}, \mathbb{C}^m, \mathbb{C}^p)$ is backward realization of process $y(t)$. Values of $y(t)$ are outputs of this system:

$$(\dot{\Sigma}_b) \quad \begin{cases} x_b(t-1) = \dot{A}^* x_b(t) + \dot{M}^* w_b(t), \\ y(t) = \dot{B}^* x_b(t) + \dot{N}^* w_b(t). \end{cases} \qquad (20)$$

Restriction on $\mathbb{D}_e$ of transfer function of $\dot{\Sigma}_b$ coincides with β^*:

$$\beta(1/\bar{z})^* = \dot{N}^* + \dot{B}^*(\bar{z} I_m - \dot{A}^*)^{-1} \dot{M}^*, \qquad z \in \mathbb{D}_e, \qquad (21)$$

and it is the full rank spectral factor of density ρ, i.e.

$$\beta(\zeta)^* \beta(\zeta) = \rho(\zeta) \qquad \text{a.e. } |\zeta| = 1.$$

Stochastic Realizations of Stationary Processes

Since $\dot{A} \in C_{00}$, using properties of transmission SI-system $\dot{\hat{\Sigma}}$ it also can be shown that vector processes $\begin{bmatrix} x_f \\ y \end{bmatrix}$ and $\begin{bmatrix} x_b \\ y \end{bmatrix}$ of size $m + p$ are stationary in wide sense, in particular, x_f and y (x_b and y) are stationary connected.

Lemma.

Realizations $\dot{\Sigma}_f$ and $\dot{\Sigma}_b$ of stationary process $y(t)$ are forward-passive systems, i.e. for any initial states and input data next passivity condition is true for both of them

$$\|x(t+1)\|^2 - \|x(t)\|^2 \leq \left(\Phi \begin{bmatrix} w(t) \\ y(t) \end{bmatrix}, \begin{bmatrix} w(t) \\ y(t) \end{bmatrix} \right)_{\mathbb{C}^{m+p}} \tag{22}$$

with the power operator Φ of the form

$$\Phi = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}.$$

Simple conservative scattering system

$$\dot{\Sigma}_{scat} = (\dot{A}, \dot{K}, \dot{M}, \dot{S}; \dot{X}, \mathbb{C}^m, \mathbb{C}^m)$$

connects with each other forward and backward white noises $\{w_f(t)\}$ and $\{w_b(t)\}$ in the following way

$$(\dot{\Sigma}_{scat}) \quad \begin{cases} x_f(t+1) = \dot{A}x_f(t) + \dot{K}w_f(t), \\ w_b(t) = \dot{M}x_f(t) + \dot{S}w_f(t). \end{cases}$$

Adjoint simple conservative scattering system

$$\dot{\Sigma}^*_{scat} = (\dot{A}^*, \dot{M}^*, \dot{K}^*, \dot{S}^*; \dot{X}, \mathbb{C}^m, \mathbb{C}^m)$$

is such that

$$(\Sigma^*_{scat}) \quad \begin{cases} x_b(t-1) = \dot{A}^* x_b(t) + \dot{M}^* w_b(t), \\ w_f(t) = \dot{K}^* x_b(t) + \dot{S}^* w_b(t). \end{cases}$$

Minimal Realizations

Now we demonstrate how it is possible to get minimal realizations of given process using corresponding dilation $\theta(z)$. In **[Arov, Rozhenko 2007]** it was given complete description of the set of all dilations of matrix function $c(z)$ from class $\ell^{p\times p}\Pi$. Dilation θ of matrix-function $c(z)$ is said to be minimal if it can not be presented in the form

$$\theta(z) = \begin{bmatrix} u(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix} \tilde{\theta}(z) \begin{bmatrix} v(z) & 0 & 0 \\ 0 & I_p & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\tilde{\theta}$ is also dilation c, $u, v \in S_{in}^{m\times m}$ and at least one of functions u or v is not constant.

Minimal Realizations

Condition of minimality of the dilation θ of matrix function $c(z)$ can be reformulated as

$$(i) \quad (\alpha, \gamma)_R = I, \qquad (ii) \quad (\alpha, \beta)_L = I. \tag{23}$$

Condition (i) ((ii)) means that matrix functions α and γ (α and β) have no nontrivial bi-inner right (respectively left) common divider.

Minimal Realizations

Theorem.

$$y, \rho \quad \to \quad c \quad \to \quad \theta \quad \to \quad \tilde{\Sigma}$$

$$\Downarrow$$

$$\{\Sigma_f, \Sigma_b, \Sigma_{imp}, \Sigma_{scat}\}$$

The following statements are true

- *forward realization Σ_f of process y is minimal if and only if blocks α and β of dilation θ satisfy condition (ii) in (23);*
- *backward realization Σ_b of process y is minimal if and only if blocks α and γ of dilation θ satisfy condition (i) in (23);*
- *dual pare of realizations Σ_f, Σ_b of stationary process y is minimal if and only if corresponding dilation θ is minimal.*

Minimal and Optimal (*-Optimal) Realizations

Dilation θ of matrix function $c \in \ell^{p\times p}\Pi$ is called optimal if it's block $\beta = \varphi_N$, where φ_N is outer solution of the equation

$$\varphi(\zeta)^*\varphi(\zeta) = \rho(\zeta) \qquad \text{a.e. } |\zeta| = 1,$$

i.e. $\bigvee_{n\geq 0} z^n\varphi_N(z) = H_2^m$.

Dilation θ of matrix function $c \in \ell^{p\times p}\Pi$ is called *-optimal if $\gamma = \psi_N$, where ψ_N is *-outer solution of the equation

$$\psi(\zeta)\psi(\zeta)^* = \rho(\zeta) \qquad \text{a.e. } |\zeta| = 1,$$

i.e. $\psi_N(\bar{z})^*$ is an outer function.

Minimal and Optimal (*-Optimal) Realizations

All optimal dilations of $c \in \ell^{p\times p}\Pi$ can be described as

$$\theta_\circ(z) = \begin{bmatrix} \alpha(z) & \varphi_N(z) & 0 \\ \gamma(z) & c(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}.$$

In this case optimal dilation $\theta_\circ$ is minimal if and only if it's blocks α and γ satisfy condition $(\alpha, \gamma)_R = I$.

Such minimal and optimal dilation exists and it is essentially unique.

Minimal and Optimal (*-Optimal) Realizations

Using minimal and optimal dilation $\theta_\circ$ of c we can construct corresponding realization $\{\Sigma_{f\circ}, \Sigma_{b\circ}, \Sigma_{imp\circ}, \Sigma_{scat\circ}\}$ of given process $y(t)$.
In this model passive bi-stable impedance system $\Sigma_{imp\circ} = (A_\circ, B_\circ, C_\circ, D_\circ; X_\circ, \mathbb{C}^p)$ is minimal and optimal.
Matrix function φ_N^* is the transfer function of backward realization

$$(\Sigma_{b\circ}) \quad \begin{cases} x_{b\circ}(t-1) = A_\circ^* x_{b\circ}(t) + M_\circ^* w_{b\circ}(t), \\ y(t) = B_\circ^* x_{b\circ}(t) + N_\circ^* w_{b\circ}(t). \end{cases} \tag{24}$$

that can be interpreted as backward Kalman filter.

Minimal and Optimal (*-Optimal) Realizations

All *-optimal dilations of matrix function $c \in \ell^{p\times p}\Pi$ can be presented in the form

$$\theta_{\bullet}(z) = \begin{bmatrix} \alpha(z) & \beta(z) & 0 \\ \psi_N(z) & c(z) & I_p \\ 0 & I_p & 0 \end{bmatrix}.$$

In this case *-optimal dilation $\theta_{\bullet}$ is minimal if and only if it's blocks α and β satisfy condition $(\alpha, \beta)_L = I$.

Such minimal and *-optimal exists and it is essentially unique.

Minimal and Optimal (*-Optimal) Realizations

Using minimal and *-optimal dilation $\theta_\bullet$ of c we consider the corresponding realization $\{\Sigma_{f\bullet}, \Sigma_{b\bullet}, \Sigma_{imp\bullet}, \Sigma_{scat\bullet}\}$ of given process $y(t)$.
In this model the passive bi-stable impedance system $\Sigma_{imp\bullet} = (A_\bullet, B_\bullet, C_\bullet, D_\bullet; X_\bullet, \mathbb{C}^p)$ is minimal and *-optimal.
The matrix function ψ_N is the transfer function of forward realization

$$(\Sigma_{f\bullet}) \quad \begin{cases} x_{f\bullet}(t+1) = A_\bullet x_{f\bullet}(t) + K_\bullet w_{f\bullet}(t), \\ y(t) = C_\bullet x_{f\bullet}(t) + L_\bullet w_{f\bullet}(t). \end{cases} \qquad (25)$$

that can be interpreted as forward Kalman filter.

This talk is based on the joint work with professor Damir Z. Arov

Thank you very much for your attention!