# Singular control of SPDEs and backward stochastic partial differential equations with reflection

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# Abstract

We study singular control problems for stochastic partial differential equations. We establish sufficient and necessary maximum principles for an optimal control of such systems. The associated adjoint processes satisfy a kind of backward stochastic partial differential equation (BSPDE) with reflection. Existence and uniqueness of BSPDEs with reflection are obtained.

# The control problem

Let $D$ be a given bounded domain in $\mathbb{R}^d$. We consider a general system where the state $Y(t,x)$ at time $t$ and at the point $x \in D \subset \mathbb{R}$ is given by a stochastic partial differential equation (SPDE) as follows:

$$
\begin{aligned}
dY(t,x) &= \{AY(t,x) + b(t,x,Y(t,x))\}dt + \sigma(t,x,Y(t,x))dB(t) \\
&\quad + \lambda(t,x,Y(t,x))\xi(dt,x) \; ; \; (t,x) \in [0,T] \times D \\
Y(0,x) &= y_0(x) \; ; \; x \in D \\
Y(t,x) &= 0 \; ; \; (t,x) \in (0,T) \times \partial D. \qquad (1)
\end{aligned}
$$

Here $A$ is a given linear second order partial differential operator.

# The control problem

We assume that the coefficients

$$b(t,x,y) : [0,T] \times D \times \mathbb{R} \to R,$$

$$\sigma(t,x,y) : [0,T] \times D \times \mathbb{R}\times \to R,$$

and

$$\lambda(t,x,y) : [0,T] \times D \times \mathbb{R} \to R$$

are $C^1$ functions with respect to $y$.

The set of possible controls, $\mathcal{A}$, is a given family of adapted processes $\xi(t,x)$, which are non-decreasing and left-continuous w.r.t. $t$ for all $x$, $\xi(0,x) = 0$. The performance functional has the form

$$J(\xi) = E\left[\int_D \int_0^T f(t,x,Y(t,x))dtdx + \int_D g(x,Y(T,x))dx + \int_D \int_0^T h(t,x,Y(t,x))\xi(dt,x)\right], \tag{2}$$

# The control problem

where $f(t,x,y), g(x,y)$ and $h(t,x,y)$ are bounded measurable functions which are differentiable in the argument $y$ and continuous w.r.t. $t$. We want to maximize $J(\xi)$ over all $\xi \in \mathcal{A}$, where $\mathcal{A}$ is the set of admissible singular controls. Thus we want to find $\xi^* \in \mathcal{A}$ (called an optimal control) such that

$$\sup_{\xi \in \mathcal{A}} J(\xi) = J(\xi^*)$$

# Sufficient maximum principle

Define the *Hamiltonian* $H$ by

$$H(t,x,y,p,q)(dt,\xi(dt,x)) = \{f(t,x,y) + b(t,x,y)p + \sigma(t,x,y)q\}dt + \{\lambda(t,x,y)p + h(t,x,y)\}\xi(dt,x). \qquad (3)$$

To this Hamiltonian we associate the following *backward* SPDE (BSPDE) in the unknown process $(p(t,x), q(t,x))$:

$$dp(t,x) = -\Big\{A^* p(t,x)dt + \frac{\partial H}{\partial y}(t,x,Y(t,x),p(t,x), q(t,x))(dt,\xi(dt,x))\Big\} + q(t,x)dB(t)\ ;\ (t,x)\in(0,T)\times D \qquad (4)$$

with boundary/terminal values

$$p(T,x) = \frac{\partial g}{\partial y}(x,Y(T,x))\ ;\ x\in D \qquad (5)$$

$$p(t,x) = 0\ ;\ (t,x)\in(0,T)\times\partial D. \qquad (6)$$

Here $A^*$ denotes the adjoint of the operator $A$.

**Theorem**[1. Sufficient maximum principle] Let $\hat{\xi} \in \mathcal{A}$ with corresponding solutions $\hat{Y}(t,x)$, $\hat{p}(t,x)$, $\hat{q}(t,x)$. Assume that

$$y \to h(x,y) \text{ is concave} \tag{7}$$

and

$$(y,\xi) \to H(t,x,y,\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,x)) \\ \text{is concave.} \tag{8}$$

Assume that

$$E[\int_D (\int_0^T \{(Y^{\xi}(t,x)-\hat{Y}(t,x))^2\hat{q}^2(t,x)+\hat{p}^2(t,x) \\ \times(\sigma(t,x,Y^{\xi}(t,x))-\sigma(t,x,\hat{Y}(t,x))^2\}dt)dx]<\infty, \tag{9}$$

# Sufficient maximum principle

for all $\xi \in \mathcal{A}$. Moreover, assume that the following maximum condition holds:

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x))\}\xi(dt,x) \\ \leq \{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) \text{ for all } \xi \in \mathcal{A}. \tag{10}$$

Then $\hat{\xi}$ is an optimal singular control.

# Sufficient maximum principle

**Theorem**[2.Sufficient maximum principle II] Suppose the conditions of the above Theorem hold. Suppose $\xi \in \mathcal{A}$, and that $\xi$ together with its corresponding processes $Y^{\xi}(t,x), p^{\xi}(t,x), q^{\xi}(t,x)$ solve the *coupled SPDE-RBSPDE system* consisting of the SPDE (1) together with the *reflected backward* SPDE (RBSPDE) given by

$$
\begin{aligned}
& dp^{\xi}(t,x) \\
& = -\Big\{ A^* p^{\xi}(t,x) + \frac{\partial f}{\partial y}(t,x,Y^{\xi}(t,x)) + \frac{\partial b}{\partial y}(t,x,Y^{\xi}(t,x))p^{\xi}(t,x) \\
& \qquad + \frac{\partial \sigma}{\partial y}(t,x,Y^{\xi}(t,x))q^{\xi}(t,x) \Big\} dt \\
& - \Big\{ \frac{\partial \lambda}{\partial y}(t,x,Y^{\xi}(t,x))p^{\xi}(t,x) + \frac{\partial h}{\partial y}(t,x,Y^{\xi}(t,x)) \Big\} \xi(dt,x) \,; \\
& (t,x) \in [0,T] \times D
\end{aligned}
$$

# Sufficient maximum principle

$$\lambda(t,x,Y^{\xi}(t,x))p^{\xi}(t,x)+h(t,x,Y^{\xi}(t,x))\leq 0\ ;\ \text{for all } t,x,a.s.$$

$$\{\lambda(t,x,Y^{\xi}(t,x))p^{\xi}(t,x)+h(t,x,Y^{\xi}(t,x))\}\xi(dt,x)=0\ ;$$

$$\text{for all}\quad t,x,a.s.$$

$$p(T,x)=\frac{\partial g}{\partial y}(x,Y^{\xi}(T,x))\ ;\ x\in D$$

$$p(t,x)=0\ ;\ (t,x)\in(0,T)\times\partial D.$$

Then $\xi$ maximizes the performance functional $J(\xi)$.

# A necessary maximum principle

A weakness of the sufficient maximum principle obtained in the previous section are the rather restrictive concavity conditions, which do not always hold in applications. Therefore it is of interest to obtain a maximum principle which does not need these conditions.

**Theorem**[3.Necessary maximum principle]
(i) Suppose $\xi^* \in \mathcal{A}$ is optimal, i.e.

$$\max_{\xi \in \mathcal{A}} J(\xi) = J(\xi^*). \tag{11}$$

Let $Y^*, (p^*, q^*)$ be the corresponding solution associated with $\xi^*$. Then

$$\lambda(t, x, Y^*(t, x)) p^*(t, x) + h(t, x, Y^*(t, x)) \leq 0 \tag{12}$$

for all $t, x \in [0, T] \times D$, a.s.

and

$$\{\lambda(t, x, Y^*(t, x))p^*(t, x) + h(t, x, Y^*(t, x))\}\xi^*(dt, x) = 0 \quad (13)$$

for all $t, x \in [0, T] \times D$, a.s.

# A necessary maximum principle

(ii) Conversely, suppose that there exists $\hat{\xi} \in \mathcal{A}$ such that the corresponding solutions $\hat{Y}(t,x), (\hat{p}(t,x), \hat{q}(t,x))$ of (1) and (4)-(5), respectively, satisfy

$$\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x)+h(t,x,\hat{Y}(t,x)) \leq 0 \quad \text{for all } t,x \in [0,T]\times D, a.s. \tag{14}$$

and

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) = 0 \tag{15}$$

for all $t, x \in [0,T] \times D$, a.s. Then $\hat{\xi}$ is a directional sub-stationary point for $J(\cdot)$, in the sense that

$$\lim_{y \to 0^+} \frac{1}{y}(J(\hat{\xi} + y\zeta) - J(\hat{\xi})) \leq 0 \quad \text{for all } \zeta \in \mathcal{V}(\hat{\xi}). \tag{16}$$

# Existence and uniqueness for BSPDEs with reflection

Next , I will present the existence and uniqueness result for reflected backward stochastic partial differential equations. For notational simplicity, we choose the operator $A$ to be the Laplacian operator $\Delta$. However, our methods work equally well for general second order differential operators like

$$A = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j}),$$

where $a = (a_{ij}(x)) : D \to \mathbb{R}^{d\times d}$ is a measurable, symmetric matrix-valued function which satisfies the uniform elliptic condition

$$\lambda |z|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x) z_i z_j \leq \Lambda |z|^2, \ \forall\, z \in \mathbb{R}^d \ \text{ and } \ x \in D$$

for some constant $\lambda$, $\Lambda > 0$

First we will establish a comparison theorem for BSPDEs, which is of independent interest. Consider two backward SPDEs:

$$\begin{aligned} du_1(t,x) &= -\Delta u_1(t)dt - b_1(t,u_1(t,x),Z_1(t,x))dt + Z_1(t,x)dB_t, \\ u_1(T,x) &= \phi_1(x) \quad a.s. \end{aligned} \tag{17}$$

$$\begin{aligned} du_2(t,x) &= -\Delta u_2(t)dt - b_2(t,u_2(t,x),Z_2(t,x))dt + Z_2(t,x)dB_t, \\ u_2(T,x) &= \phi_2(x) \quad a.s. \end{aligned} \tag{18}$$

From now on, if $u(t,x)$ is a function of $(t,x)$, we write $u(t)$ for the function $u(t,\cdot)$.

# A comparison theorem

The following result is a comparison theorem for backward stochastic partial differential equations.

**Theorem**[4. Comparison theorem for BSPDEs] Suppose $\phi_1(x) \leq \phi_2(x)$ and $b_1(t, u, z) \leq b_2(t, u, z)$. Then we have $u_1(t, x) \leq u_2(t, x), x \in D$, a.e. for every $t \in [0, T]$.

**Steps of the proof**. For $n \geq 1$, define functions $\psi_n(z)$, $f_n(x)$ as follows (see [DP1]).

$$\psi_n(z) = \begin{cases} 0 & \text{if } z \leq 0, \\ 2nz & \text{if } 0 \leq z \leq \frac{1}{n}, \\ 2 & \text{if } z > \frac{1}{n}. \end{cases} \tag{19}$$

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \int_0^x dy \int_0^y \psi_n(z) dz & \text{if } x > 0. \end{cases} \tag{20}$$

We have

$$f_n'(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ nx^2 & \text{if } x \leq \frac{1}{n}, \\ 2x - \frac{1}{n} & \text{if } x > \frac{1}{n}. \end{cases} \tag{21}$$

Also $f_n(x) \uparrow (x^+)^2$ as $n \to \infty$. For $h \in K := L^2(D)$, set

$$F_n(h) = \int_D f_n(h(x))dx.$$

Applying Ito's formula we get

$$\begin{aligned} & F_n(u_1(t) - u_2(t)) \\ = \quad & F_n(\phi_1 - \phi_2) + \int_t^T F_n'(u_1(s) - u_2(s))(\Delta(u_1(s) - u_2(s)))ds \end{aligned}$$

# A comparison theorem

$$
\begin{aligned}
& + \int_t^T F_n'(u_1(s) - u_2(s))(b_1(s, u_1(s), Z_1(s)) - b_2(s, u_2(s), \\
& \quad Z_2(s)))ds \\
& - \int_t^T F_n'(u_1(s) - u_2(s))(Z_1(s) - Z_2(s))dB_s \\
& - \frac{1}{2} \int_t^T F_n''(u_1(s) - u_2(s))(Z_1(s) - Z_2(s), Z_1(s) - Z_2(s))ds \\
=: \quad & I_n^1 + I_n^2 + I_n^3 + I_n^4 + I_n^5, \qquad (22)
\end{aligned}
$$

# A comparison theorem

After carefully analyzing every term on the right and after cancelation of terms, we can show that

$$\begin{aligned} & F_n(u_1(t)-u_2(t)) \\ \leq\ & F_n(\phi_1-\phi_2)+C\int_t^T\int_D((u_1(s,x)-u_2(s,x))^+)^2dxds \\ & -\int_t^T F_n'(u_1(s)-u_2(s))(Z_1(s)-Z_2(s))dB_s \end{aligned} \qquad (23)$$

Take expectation and let $n\to\infty$ to get

$$E[\int_D((u_1(t,x)-u_2(t,x))^+)^2dx]\leq\int_t^T dsE[\int_D((u_1(s,x)-u_2(s,x))^+)^2dx] \qquad (24)$$

Gronwall's inequality yields that

$$E[\int_D((u_1(t,x)-u_2(t,x))^+)^2dx]=0, \qquad (25)$$

which completes the proof of the theorem

# Existence and uniqueness for BSPDEs with reflection

Let $V = W_0^{1,2}(D)$ be the Sobolev space of order one with the usual norm $||\cdot||$. Consider the reflected backward stochastic partial differential equation:

$$
\begin{aligned}
& du(t) = -\Delta u(t)dt - b(t, u(t,x), Z(t,x))dt + Z(t,x)dB_t \\
& \qquad -\eta(dt,x), t \in (0,T), \qquad (26) \\
& u(t,x) \geq L(t,x), \\
& \int_0^T \int_D (u(t,x) - L(t,x))\eta(dt,x)dx = 0, \\
& u(T,x) = \phi(x) \quad a.s. \qquad (27)
\end{aligned}
$$

**Theorem**[5. Existence and Uniqueness]
*Assume that* $E[|\phi|_K^2] < \infty$. *and that*

$$|b(s, u_1, z_1) - b(s, u_1, z_1)| \leq C(|u_1 - u_2| + |z_1 - z_2|).$$

*Let* $L(t, x)$ *be a measurable function which is differentiable in* $t$ *and twicely differentiable in* $x$ *such that* $\phi(x) \geq L(T, x)$ and

$$\int_0^T \int_D L'(t, x)^2 dxdt < \infty, \int_0^T \int_D |\Delta L(t, x)|^2 dxdt < \infty.$$

*Then there exists a unique* $K \times L^2(D, \mathbb{R}^m) \times K$*-valued progressively measurable process* $(u(t, x), Z(t, x), \eta(t, x))$ *such that*

# Existence and uniqueness for BSPDEs with reflection

$$
\begin{array}{ll}
\text{(i)} & E[\int_0^T ||u(t)||_V^2 dt] < \infty, \quad E[\int_0^T |Z(t)|^2_{L^2(D,R^m)} dt] < \infty. \\
\text{(ii)} & \eta \quad \text{is a} \quad \text{K-valued continuous process, non-negative} \\
& \text{and nondecreasing in } t \quad \text{and} \quad \eta(0,x) = 0. \\
\text{(iii)} & u(t,x) = \phi(x) + \int_t^T \Delta u(t,x) ds + \int_t^T b(s, u(s,x), Z(s,x)) ds \\
& - \int_t^T Z(s,x) dB_s + \eta(T,x) - \eta(t,x); \quad 0 \leq t \leq T, \\
\text{(iv)} & u(t,x) \geq L(t,x) \quad a.e. \quad x \in D, \forall t \in [0,T]. \\
\text{(v)} & \int_0^T \int_D (u(t,x) - L(t,x)) \eta(dt,x) dx = 0
\end{array}
\tag{28}
$$

where $u(t)$ stands for the $K$-valued continuous process $u(t,\cdot)$ and (iii) is understood as an equation in the dual space $V^*$ of $V$.

# The proof of existence and uniqueness

I will indicate how we prove the theorem. we introduce the penalized BSPDEs:

$$
\begin{aligned}
du^n(t) &= -\Delta u^n(t)dt - b(t, u^n(t,x), Z^n(t,x))dt + Z^n(t,x)dB_t \\
&\quad -n(u^n(t,x) - L(t,x))^- dt, \quad t \in (0,T) \qquad (29) \\
u^n(T,x) &= \phi(x) \quad a.s. \qquad (30)
\end{aligned}
$$

According to [ØPZ], the solution $(u^n, Z^n)$ of the above equation exists and is unique. We are going to show that the sequence $(u^n, Z^n)$ has a limit, which will be a solution of the equation (28).

First we need some a priori estimates.

**Lemma**[1]

Let $(u^n, Z^n)$ be the solution of equation (29). We have

$$\sup_n E[\sup_t |u^n(t)|_K^2] < \infty, \tag{31}$$

$$\sup_n E[\int_0^T ||u^n(t)||_V^2] < \infty, \tag{32}$$

$$\sup_n E[\int_0^T |Z^n(t)|^2_{L^2(D,R^m)}] < \infty. \tag{33}$$

We also need the following crucial estimates.
**Lemma**[2] Suppose the conditions in Theorem 5 hold. Then there is a constant $C$ such that

$$E[\int_0^T \int_D ((u^n(t,x) - L(t,x))^-)^2 dxdt] \leq \frac{C}{n^2}. \qquad (34)$$

**Main ideas of the proof**. Let $f_m$ be defined as in the proof of Theorem 4. Then $f_m(x) \uparrow (x^+)^2$ and $f_m'(x) \uparrow 2x^+$ as $m \to \infty$. For $h \in K$, set

$$G_m(h) = \int_D f_m(-h(x))dx.$$

The idea is to apply Ito's formula to the process $u^n(t) - L(t)$ and for the functional $G_m(\cdot)$.

**Lemma**[3].

Let $(u^n, Z^n)$ be the solution of equation (29). We have

$$\lim_{n,m\to\infty} E[\sup_{0\le t\le T} |u^n(t) - u^m(t)|_K^2] = 0, \qquad (35)$$

$$\lim_{n,m\to\infty} E[\int_0^T ||u^n(t) - u^m(t)||_V^2 dt] = 0. \qquad (36)$$

$$\lim_{n,m\to\infty} E[\int_0^T |Z^n(t) - Z^m(t)|^2_{L^2(D,\mathbb{R}^m)} dt] = 0. \qquad (37)$$

# The proof of existence and uniqueness

**Main ideas of the proof**. Applying Itô's formula, it follows that

$$
\begin{aligned}
& |u^n(t) - u^m(t)|_K^2 \\
= \; & 2\int_t^T < u^n(s) - u^m(s), \Delta(u^n(s) - u^m(s)) > ds \\
& +2\int_t^T < u^n(s) - u^m(s), b(s, u^n(s), Z^n(s)) - b(s, u^m(s), Z^m(s)) > \\
& -2\int_t^T < u^n(s) - u^m(s), Z^n(s) - Z^m(s) > dB_s \\
& +2\int_t^T < u^n(s) - u^m(s), n(u^n(s) - L(s))^- - m(u^m(s) - L(s))^- > \\
& -\int_t^T |Z^n(s) - Z^m(s)|^2_{L^2(D,\mathbb{R}^m)} ds \qquad (
\end{aligned}
$$

Now we estimate each of the terms on the right side.

# The proof of existence and uniqueness

$$\begin{aligned} & 2\int_t^T < u^n(s) - u^m(s), \Delta(u^n(s) - u^m(s)) > ds \\ = & -2\int_t^T ||u^n(s) - u^m(s)||_V^2 ds. \end{aligned} \quad (39)$$

By the Lipschitz continuity of $b$ and the inequality $ab \leq \varepsilon a^2 + C_\varepsilon b^2$, one has

$$\begin{aligned} & 2\int_t^T < u^n(s) - u^m(s), b(s, u^n(s), Z^n(s)) - b(s, u^m(s), Z^m(s)) > c \\ \leq & C\int_t^T |u^n(s) - u^m(s)|_K^2 ds + \frac{1}{2}\int_t^T |Z^n(s) - Z^m(s)|_{L^2(D,\mathbb{R}^m)}^2 ds. \quad (40 \end{aligned}$$

# The proof of existence and uniqueness

In view of (34), we can show that

$$
\begin{aligned}
& 2E[\int_t^T < u^n(s) - u^m(s), n(u^n(s) - L(s))^- - m(u^m(s) - L(s))^- > \\
\leq \; & 2m(E[\int_t^T \int_D ((u^n(s,x) - L(s,x))^-)^2 dxds])^{\frac{1}{2}} \\
& \times (E[\int_t^T \int_D ((u^m(s,x) - L(s,x))^-)^2 dxds])^{\frac{1}{2}} \\
& + 2n(E[\int_t^T \int_D ((u^n(s,x) - L(s,x))^-)^2 dxds])^{\frac{1}{2}} \\
& \times (E[\int_t^T \int_D ((u^m(s,x) - L(s,x))^-)^2 dxds])^{\frac{1}{2}} \\
\leq \; & C'(\frac{1}{n} + \frac{1}{m}). \qquad (
\end{aligned}
$$

# The proof of existence and uniqueness

It follows from (38) and (39) that

$$\begin{aligned}
& E[|u^n(t)-u^m(t)|_K^2]+\frac{1}{2}E[\int_t^T |Z^n(s)-Z^m(s)|^2_{L^2(D,\mathbb{R}^m)}ds] \\
& +E[\int_t^T ||u^n(s)-u^m(s)||_V^2 ds] \\
\leq \ & C\int_t^T E[|u^n(s)-u^m(s)|_K^2]ds + C'(\frac{1}{n}+\frac{1}{m}). \qquad (42)
\end{aligned}$$

Application of the Gronwall inequality yields

$$\lim_{n,m\to\infty}\{E[|u^n(t)-u^m(t)|_K^2]+\frac{1}{2}E[\int_t^T |Z^n(s)-Z^m(s)|^2_{L^2(D,\mathbb{R}^m)}ds]\}=0, \qquad (43)$$

# The proof of existence and uniqueness

$$\lim_{n,m\to\infty} E[\int_t^T ||u^n(s) - u^m(s)||_V^2 ds] = 0. \qquad (44)$$

By (43) and the Burkholder inequality we can further show that

$$\lim_{n,m\to\infty} E[\sup_{0\leq t\leq T} |u^n(t) - u^m(t)|_K^2] = 0. \qquad (45)$$

The proof is complete

**Proof of Theorem 5**. From Lemma 3 we know that $(u^n, Z^n), n \geq 1$, forms a Cauchy sequence. Denote by $u(t,x)$, $Z(t,x)$ the limit of $u^n$ and $Z^n$. Put

$$\bar{\eta}^n(t,x) = n(u^n(t,x) - L(t,x))^-$$

Lemma 3.4 implies that $\bar{\eta}^n(t,x)$ admits a non-negative weak limit, denoted by $\bar{\eta}(t,x)$, in the following Hilbert space:

$$\begin{aligned} \bar{K} &= \{h; \quad \text{h is a K-valued adapted process such that} \\ & E[\int_0^T |h(s)|_K^2 ds] < \infty\} \end{aligned} \tag{46}$$

with inner product

$$< h_1, h_2 >_{\bar{K}} = E[\int_0^T \int_D h_1(t,x) h_2(t,x) dtdx].$$

# The proof of existence and uniqueness

Set $\eta(t,x) = \int_0^t \bar{\eta}(s,x)ds$. Then $\eta$ is a continuous $K$-valued process which is increasing in $t$ Let $n \to \infty$ in (29) to obtain

$$\begin{aligned} & u(t,x) \\ = \ & \phi(x) + \int_t^T \Delta u(t,x)ds + \int_t^T b(s,u(s,x),Z(s,x))ds \\ & - \int_t^T Z(s,x)dB_s + \eta(T,x) - \eta(t,x); \quad 0 \le t \le T. \end{aligned} \tag{47}$$

Furthermore we can show that $(u,\eta)$ fulfills the required properties.

# The proof of existence and uniqueness

**Uniqueness**. Let $(u_1, Z_1, \eta_1)$, $(u_2, Z_2, \eta_2)$ be two such solutions to equation (28). By Itô's formula, we have

$$
\begin{aligned}
& |u_1(t) - u_2(t)|_K^2 \\
= \; & 2\int_t^T < u_1(s) - u_2(s), \Delta(u_1(s) - u_2(s)) > ds \\
& +2\int_t^T < u_1(s) - u_2(s), b(s, u_1(s), Z_1(s)) - b(s, u_2(s), Z_2(s)) > ds \\
& -2\int_t^T < u_1(s) - u_2(s), Z_1(s) - Z_2(s) > dB_s \\
& +2\int_t^T < u_1(s) - u_2(s), \eta_1(ds) - \eta_2(ds) > \\
& -\int_t^T |Z_1(s) - Z_2(s)|^2_{L^2(D,\mathbb{R}^m)} ds \qquad (48)
\end{aligned}
$$

Note that

$$
\begin{aligned}
& 2E\Big[\int_t^T < u_1(s)-u_2(s), \eta_1(ds)-\eta_2(ds) >\Big] \\
= & \ 2E\Big[\int_t^T \int_D (u_1(s,x)-L(s,x))\eta_1(ds,x)dx\Big] \\
& -2E\Big[\int_t^T \int_D (u_1(s,x)-L(s,x))\eta_2(ds,x)dx\Big] \\
& +2E\Big[\int_t^T \int_D (u_2(s,x)-L(s,x))\eta_2(ds,x)dx\Big] \\
& -2E\Big[\int_t^T \int_D (u_2(s,x)-L(s,x))\eta_1(ds,x)dx\Big] \\
\leq & \ 0 \qquad (49)
\end{aligned}
$$

This observation allows to prove

# The proof of existence and uniqueness

$$
\begin{aligned}
& E[|u_1(t) - u_2(t)|_K^2] + \frac{1}{2}E[\int_t^T |Z_1(s) - Z_2(s)|^2_{L^2(D,\mathbb{R}^m)} ds] \\
\leq \;\; & C \int_t^T E[|u_1(s) - u_2(s)|_K^2] ds. \qquad (50)
\end{aligned}
$$

Appealing to Gronwall inequality, this implies

$$u_1 = u_2, \quad Z_1 = Z_2$$

which further gives $\eta_1 = \eta_2$ from the equation they satisfy.

# Link to optimal stopping

This part provides a link between the solution of a reflected backward stochastic partial differential equation and an optimal stopping problem. Let $u(t,x)$ be the solution of the following reflected BSPDE.

$$
\begin{aligned}
& u(t,x) \\
= \; & \phi(x) + \int_t^T \frac{1}{2}\Delta u(t,x)ds + \int_t^T k(s,x,u(s,x),Z(s,x))ds \\
& - \int_t^T Z(s,x)dB_s + \eta(T,x) - \eta(t,x); \quad 0 \le t \le T, \\
& u(t,x) \ge L(t,x), \\
& \int_0^T \int_D (u(s,x) - L(s,x))\eta(dt,x)dx = 0 \quad a.s. \qquad (51)
\end{aligned}
$$

# Link to optimal stopping

Let $\mathcal{S}_{t,T}$ be the set of all stopping times $\tau$ satisfying $t \le \tau \le T$.
For $\tau \in \mathcal{S}_{t,T}$, define

$$R_t(\tau, x) = \int_t^\tau P_{s-t} k(s,x) ds + P_{\tau-t} L(\tau, x) \chi_{\{\tau<T\}} + P_{\tau-t}\phi(x)\chi_{\{\tau=T\}},$$

where $k(s,\cdot) = k(s,\cdot,u(s,\cdot),Z(s,\cdot))$ and $P_t$ denotes the heat semigroup generated by the Laplacian operator $\frac{1}{2}\Delta$.

# Link to optimal stopping

Here, and in the following we will use the simplified notation $P_t k(s,x) = (P_t k(s,\cdot))(x)$ etc.

**Theorem**[6. Optimal stopping]

$u(t,x)$ is the value function of the the optimal stopping problem associated with $R_t(\tau,x)$, i.e.,

$$u(t,x) = esssup_{\tau\in\mathcal{S}_{t,T}} E[R_t(\tau,x)|\mathcal{F}_t] \tag{52}$$

# References


- C. Donai-Martin, E. Pardoux: White noise driven SPDEs with reflection. Probab. Theory Rel. Fields 95, 1-24(1993).
- C. Donai-Martin, E. Pardoux: EDPS réfléchies et calcul de Malliavin. (French)[SPDEs with reflection and Malliavin Calculus]. Bull. Sci. Math. 121(5)(1997), 405-422.
- A. Gegout-Petit, E. Pardoux: Equations Différentielles Stochastiques Rétrogrades Réfléchies Dans Un Convexe. Stochastics and Stochastics Reports 57 (1996) 111-128.
- U. G. Haussmann, E. Pardoux: Stochastic variational inequalities of parabolic type. Appl. Math. Optim. 20(1989), 163-192.

# References


- D. Nualart, E. Pardoux: White noise driven by quasilinear SPDEs with reflection. Probab. Theory Rel. Fields 93,77-89(1992).
- B. Øksendal, A. Sulem: Singular stochastic control and optimal stopping with partial information of jump diffusions. Preprint 2010, Oslo.
- B. Øksendal, F. Proske and T. Zhang: Backward stochastic partial differential equations with jumps and application to optimal control of random jump fields. Stochastics 77:5 (2005) 381-399.

# References

E. Pardoux and S. Peng: Adapted solutions of backward stochastic differential equations. System and Control Letters 14 (1990) 55-61.

E. Pardoux and S. Peng: Backward doubly stochastic differntial equations and systems of quasilinear stochastic partial differential equations. Probab. Theory and Rel. Fields 98 (1994) 209-227.

T.Zhang : White noise driven SPDEs with reflection: strong Feller properties and Harnack inequalities. Potential Analysis 33:2 (2010) 137-151.