ϵ-Nash Mean Field Game Theory for Nonlinear Stochastic Dynamical Systems with Major and Minor Agents

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Parts II and III

This is joint work with Mojtaba Nourian

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

- In this work we extend the Minyi Huang's linear quadratic Gaussian (LQG) model [Huang'10, Nguyen-Huang'11] for major and minor (MM) agents with uniform parameters to the case of a nonlinear stochastic dynamic game formulation of controlled McKean-Vlasov (MV) type [HMC'06].

- We consider a large population dynamic game involving nonlinear stochastic dynamical systems with agents of the following mixed types: (i) a major agent, and (ii) a large N population of minor agents.

- The MM agents are coupled via both: (i) their individual nonlinear stochastic dynamics, and (ii) their individual finite time horizon nonlinear cost functions.

- Applications: Social opinion models with a finite number of leaders, Power markets involving large consumers/utilities and domestic consumers (smart meters and small scale generating units).

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

- Key Notion of the MM MFG Theory [H'10, NH'11]: Even asymptotically (as $N \to \infty$) the noise process of the major agent causes random fluctuation of the mean field behaviour of the minor agents.

- The overall asymptotic ($N \to \infty$) MFG problem is decomposed into:
  (i) two non-standard stochastic optimal control problems (SOCPs) with random coefficient processes, and
  (ii) two stochastic (coefficient) McKean-Vlasov (SMV) equations which characterize the state distribution measure of the major agent and the measure determining the mean field behaviour of the minor agents.

- Feedback coupling: The forward adapted stochastic best response control processes determined from the solution of the (backward in time) stochastic Hamilton-Jacobi-Bellman (SHJB) equations in (i) depend upon the state distribution measures generated by the SMV equations in (ii) which in turn depend upon (i).

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Problem Formulation:

- Subscript 0 for the major agent $\mathcal{A}_0$ and an integer valued subscript for minor agents $\{\mathcal{A}_i : 1 \leq i \leq N\}$.
- The states $\mathcal{A}_0$ and $\mathcal{A}_i$ are denoted by $z_0^N(t)$ and $z_i^N(t)$.

Dynamics of the Major and Minor Agents:

$$dz_0^N(t) = \frac{1}{N}\sum_{j=1}^{N} f_0[t, z_0^N(t), u_0^N(t), z_j^N(t)]dt$$

$$+ \frac{1}{N}\sum_{j=1}^{N} \sigma_0[t, z_0^N(t), z_j^N(t)]dw_0(t), \quad z_0^N(0) = z_0(0), \quad 0 \leq t \leq T,$$

$$dz_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} f[t, z_i^N(t), u_i^N(t), z_0^N(t), z_j^N(t)]dt$$

$$+ \frac{1}{N}\sum_{j=1}^{N} \sigma[t, z_i^N(t), z_0^N(t), z_j^N(t)]dw_i(t), \quad z_i^N(0) = z_i(0), \quad 1 \leq i \leq N.$$

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Cost Functions for the Major and Minor Agents: The objective of each agent is to minimize its finite time horizon nonlinear cost function given by

$$J_0^N(u_0^N; u_{-0}^N) := E \int_0^T \Big((1/N) \sum_{j=1}^N L_0[t, z_0^N(t), u_0^N(t), z_j^N(t)] \Big) dt,$$

$$J_i^N(u_i^N; u_{-i}^N) := E \int_0^T \Big((1/N) \sum_{j=1}^N L[t, z_i^N(t), u_i^N(t), z_0^N(t), z_j^N(t)] \Big) dt.$$

- The major agent has non-negligible influence on the mean field (mass) behaviour of the minor agents due to presence of z_0^N in the dynamics and cost function of each minor agent.
- Note that the coupling terms in the dynamics and the costs of the MM agents may be written as functionals of the empirical distribution of the minor agents $\delta_t^N := (1/N) \sum_{i=1}^N \delta_{z_i^N(t)}$, $0 \leq t \leq T$.

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Assumptions: Let the empirical distribution of N minor agents' initial states be defined by $F_N(x) := (1/N)\sum_{i=1}^{N} 1_{\{Ez_i(0)<x\}}$.

(A1) The initial states $\{z_j(0) : 0 \leq j \leq N\}$ are $\mathcal{F}_0$-adapted random variables mutually independent and independent of all Brownian motions, and there exists a constant k independent of N such that $\sup_{0\leq j\leq N} E|z_j(0)|^2 \leq k < \infty$.

(A2) $\{F_N : N \geq 1\}$ converges weakly to the probability distribution F.

(A3) U_0 and U are compact metric spaces.

(A4) The functions $f_0[t, x, u, y]$, $\sigma_0[t, x, y]$, $f[t, x, u, y, z]$ and $\sigma[t, x, y, z]$ are continuous and bounded with respect to all their parameters, and Lipschitz continuous in (x, y, z). In addition, their first and second order derivatives (w.r.t. x) are all uniformly continuous and bounded with respect to all their parameters, and Lipschitz continuous in (y, z).

(A5) $f_0[t, x, u, y]$ and $f[t, x, u, y, z]$ are Lipschitz continuous in u.

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Assumptions (cnt):

(**A6**) $L_0[t,x,u,y]$ and $L[t,x,u,y,z]$ are continuous and bounded with respect to all their parameters, and Lipschitz continuous in (x,y,z). In addition, their first and second order derivatives (w.r.t. x) are all uniformly continuous and bounded with respect to all their parameters, and Lipschitz continuous in (y,z).

(**A7**) (Non-degeneracy Assumption) There exists a positive constant α such that

$$\sigma_0[t,x,y]\sigma_0^T[t,x,y] \geq \alpha I,\ \ \sigma[t,x,y,z]\sigma^T(t,x,y,z) \geq \alpha I, \quad \forall (t,x,y,z).$$

Notation: Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space. We denote:

- $\mathcal{F}_t := \sigma\{z_j(0), w_j(s) : 0 \leq j \leq N, 0 \leq s \leq t\}$.
- $\mathcal{F}_t^{w_0} := \sigma\{z_0(0), w_0(s) : 0 \leq s \leq t\}$.

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A Preliminary Nonlinear SOCP with Random Coefficients: Let $(W(t))_{t\geq 0}$ and $(B(t))_{t\geq 0}$ be mutually independent standard Brownian motions in $\mathbb{R}^m$. Denote

$$\mathcal{F}_t^{W,B} := \sigma\{W(s), B(s) : s \leq t\}, \quad \mathcal{F}_t^{W} := \sigma\{W(s) : s \leq t\}.$$

Dynamics and cost function for a "single agent":

$$dz(t) = f[t,\omega,z,u]dt + \sigma[t,\omega,z]dW(t) + \varsigma[t,\omega,z]dB(t), \quad 0 \leq t \leq T,$$

$$\inf_{u\in\mathcal{U}} J(u) := \inf_{u\in\mathcal{U}} E\Big[\int_0^T L[t,\omega,z(t),u(t)]dt\Big],$$

where the coefficients f, σ, ς and L are are $\mathcal{F}_t^{W}$-adapted stochastic processes.

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The value function [Peng'92]:

$$\phi(t,x) := \inf_{u\in\mathcal{U}} E_{\mathcal{F}_t^W}\Big[\int_t^T L[s,\omega,z(s),u(s)]ds \Big| z(t)=x\Big],$$

which is a $\mathcal{F}_t^W$-adapted process for any fixed x.

A semimartingale representation for $\phi(t,x)$ [Peng'92]: Following Peng we assume that the continuous semimartingale $\phi(t,x)$ has the representation

$$\phi(t,x) = \int_t^T \Gamma(s,x)ds - \int_t^T \psi^T(s,x)dW(s), \quad (t,x)\in[0,T]\times\mathbb{R}^n,$$

where, for each x, $\phi(s,x)$, $\Gamma(s,x)$ and $\psi(s,x)$ are $\mathcal{F}_s^W$-adapted stochastic processes.

Question: What are $\Gamma(t,x)$ and $\psi(t,x)$ processes?

Theorem (Itô-Kunita formula (Peng'92))

Let $F(t,x)$ be a stochastic process continuous in (t,x) almost surely (a.s.), such that (i) for each t, $F(t,\cdot)$ is a $C^2(\mathbb{R}^n)$ map a.s., (ii) for each x, $F(\cdot,x)$ is a continuous semimartingale represented as

$$F(t,x) = F(0,x) + \sum_{j=1}^{m}\int_0^t f_j(s,x)dY_s^j,$$

where Y_s^j, $1 \le j \le m$, are continuous semimartingales, $f_j(s,x)$, $1 \le j \le m$, are stochastic processes that are continuous in (s,x) a.s., such that (i) for each s, $f_j(s,\cdot)$ is a $C^1(\mathbb{R}^n)$ map a.s., (ii) for each x, $f_j(\cdot,x)$ is an adapted process. Let $X_t = (X_t^1,\cdots,X_t^n)$ be continuous semimartingale. Then we have

$$F(t,X_t) = F(0,X_0) + \sum_{j=1}^{m}\int_0^t f_j(s,X_s)dY_s^j + \sum_{i=1}^{n}\int_0^t \partial_{x_i}F(s,X_s)dX_s^i$$

$$+ \sum_{j=1}^{m}\sum_{i=1}^{n}\int_0^t \partial_{x_i}f_j(s,X_s)d < Y^j, X^i >_s + \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t \partial^2_{x_i x_j}F(s,X_s)d < X^i, X^j >_s,$$

where $< \cdot,\cdot >_s$ stands for the quadratic variation of semimartingales. □

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A stochastic Hamilton-Jacobi-Bellman (SHJB) equation for the nonlinear SOCP with random coefficients [Peng'92]:

Using the Itô-Kunita formula and the Principle of Optimality, Peng showed if $\phi(t,x)$, $\Gamma(t,x)$ and $\psi(t,x)$ are a.s. continuous in (x,t) and are smooth enough with respect to x, then the pair $\big(\phi(s,x),\psi(s,x)\big)$ satisfies the following backward in time SHJB equation:

$$-d\phi(t,\omega,x) = \Big[H[t,\omega,x,D_x\phi(t,\omega,x)] + \big\langle \sigma[t,\omega,x], D_x\psi(t,\omega,x)\big\rangle + \frac{1}{2}\mathrm{Tr}\big(a[t,\omega,x]D_{xx}^2\phi(t,\omega,x)\big)\Big]dt - \psi^T(t,\omega,x)dW(t,\omega), \quad \phi(T,x)=0,$$

in $[0,T]\times\mathbb{R}^n$, where $a[t,\omega,x] := \sigma[t,\omega,x]\sigma^T[t,\omega,x] + \varsigma[t,\omega,x]\varsigma^T(t,\omega,x)$, and the stochastic Hamiltonian H is given by

$$H[t,\omega,z,p] := \inf_{u\in\mathcal{U}} \big\{\big\langle f[t,\omega,z,u],p\big\rangle + L[t,\omega,z,u]\big\}.$$

The solution of the backward in time SHJB equation is a unique forward in time $\mathcal{F}_t^W$-adapted pair $(\phi,\psi)(t,x) \equiv \big(\phi(t,\omega,x),\psi(t,\omega,x)\big)$

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Assumptions:

(**H1**) $f[t,x,u]$ and $L[t,x,u]$ are a.s. continuous in (x,u) for each t, a.s. continuous in t for each (x,u), $f[t,0,0] \in L^2_{\mathcal{F}_t}([0,T];\mathbb{R}^n)$ and $L[t,0,0] \in L^2_{\mathcal{F}_t}([0,T];\mathbb{R}_+)$. In addition, they an all their first derivatives (w.r.t. x) are a.s. continuous and bounded.

(**H2**) $\sigma[t,x]$ and $\varsigma[t,x]$ are a.s. continuous in x for each t, a.s. continuous in t for each x and $\sigma[t,0]$, $\varsigma[t,0] \in L^2_{\mathcal{F}_t}([0,T];\mathbb{R}^{n\times m})$. In addition, they and all their first derivatives (w.r.t. x) are a.s. continuous and bounded.

(**H3**) (Non-degeneracy Assumption) There exist non-negative constants α_1 and α_2 such that

$$\sigma[t,\omega,x]\sigma^T[t,\omega,x] \geq \alpha_1 I, \quad \varsigma[t,\omega,x]\varsigma^T(t,\omega,x) \geq \alpha_2 I, \quad a.s., \quad \forall(t,\omega,x),$$

where α_1 or α_2 (but not both) can be zero.

Theorem (Peng'92)

*Assume (**H1**)-(**H3**) hold. Then the SHJB equation has a unique solution $(\phi(t,x),\psi(t,x))$ in $\left(L^2_{\mathcal{F}_t}([0,T];\mathbb{R}), L^2_{\mathcal{F}_t}([0,T];\mathbb{R}^m)\right)$.* □

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The optimal control process [Peng'92]:

$$u^o(t,\omega,x) := \arg\inf_{u\in U} H^u[t,\omega,x,D_x\phi(t,\omega,x),u]$$
$$= \arg\inf_{u\in U} \big\{\big\langle f[t,\omega,x,u], D_x\phi(t,\omega,x)\big\rangle + L[t,\omega,x,u]\big\}.$$

which is a forward in time $\mathcal{F}_t^W$-adapted process for any fixed x.

- By a verification theorem approach, Peng showed that if a unique solution $(\phi,\psi)(t,x)$ to the SHJB equation exists, and if it satisfies:

  (i) for each t, $(\phi,\psi)(t,x)$ is a $C^2(\mathbb{R}^n)$ map,

  (ii) for each x, $(\phi,\psi)(t,x)$ and $(D_x\phi, D_{xx}^2\phi, D_x\psi)(t,x)$ are continuous F_t^W-adapted stochastic processes,

  then $\phi(x,t)$ coincides with the value function of the SOCP.

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Major and Minor Mean Field Convergence Theorem:

- A probabilistic approach to show a "decoupling effect" result such that a generic minor agent's statistical properties can effectively approximate the distribution produced by all minor agents as the number of minor agents N goes to infinity (based on the HMC'06).

Let $\varphi_0(\omega, t, x)$ and $\varphi(\omega, t, x)$ be two arbitrary $\mathcal{F}_t^{w_0}$-measurable stochastic processes. We introduce the following assumption:

(H4) $\varphi_0(\omega, t, x)$ and $\varphi(\omega, t, x)$ are Lipschitz continuous in x, and $\varphi_0(\omega, t, 0) \in L^2_{\mathcal{F}_t^{w_0}}([0, T]; U_0)$ and $\varphi(\omega, t, 0) \in L^2_{\mathcal{F}_t^{w_0}}([0, T]; U)$.

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Assume that $\varphi_0(t,x) \equiv \varphi_0(\omega,t,x)$ and $\varphi(t,x) \equiv \varphi(\omega,t,x)$ are respectively used by the Major and Minor agents as their control laws. Then we have the following closed-loop equations with random coefficients:

$$
\begin{aligned}
d\hat{z}_0^N(t) &= \frac{1}{N}\sum_{j=1}^{N} f_0[t,\hat{z}_0^N(t),\varphi_0(t,\hat{z}_0^N(t)),\hat{z}_j^N(t)]dt \\
&\quad + \frac{1}{N}\sum_{j=1}^{N}\sigma_0[t,\hat{z}_0^N(t),\hat{z}_j^N(t)]dw_0(t), \quad \hat{z}_0^N(0)=z_0(0), \quad 0\le t\le T, \\
d\hat{z}_i^N(t) &= \frac{1}{N}\sum_{j=1}^{N} f[t,\hat{z}_i^N(t),\varphi(t,\hat{z}_i^N(t)),\hat{z}_0^N(t),\hat{z}_j^N(t)]dt \\
&\quad + \frac{1}{N}\sum_{j=1}^{N}\sigma[t,\hat{z}_i^N(t),\hat{z}_0^N(t),\hat{z}_j^N(t)]dw_i(t), \quad \hat{z}_i^N(0)=z_i(0), \quad 1\le i\le N.
\end{aligned}
$$

Under **(A4)**-**(A5)** and **(H4)** there exists a unique solution $(\hat{z}_0^N,\cdots,\hat{z}_N^N)$ to the above system.

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We now introduce the McKean-Vlasov (MV) SDE system

$$d\bar{z}_0(t) = f_0[t, \bar{z}_0(t), \varphi_0(t, \bar{z}_0(t)), \mu_t]dt + \sigma_0[t, \bar{z}_0(t), \mu_t]dw_0(t), \quad 0 \le t \le T,$$
$$d\bar{z}(t) = f[t, \bar{z}(t), \varphi(t, \bar{z}(t)), \mu_t^0, \mu_t]dt + \sigma[t, \bar{z}(t), \mu_t^0, \mu_t]dw(t),$$

with initial condition $(\bar{z}_0(0), \bar{z}(0))$, where for an arbitrary function g and probability distributions μ_t and μ_t^0 in $\mathbb{R}^n$ denote

$$g[t, z, \mu_t] := \int g(t, z, x)\mu_t(dx), \quad g[t, z, \mu_t^0] := \int g(t, z, x)\mu_t^0(dx).$$

- In the above MV system $\left(\bar{z}_0, \bar{z}, \mu^0, \mu\right)$ is a consistent solution if $(\bar{z}_0, \bar{z})$ is a solution to the above SDE system, and μ_t^0 and μ_t are the corresponding distributions (laws of the processes $\bar{z}_0$ and $\bar{z}$) at time t (HMC'06).
- Under (**A4**)-(**A5**) and (**H4**) there exists a unique solution $\left(\bar{z}_0, \bar{z}, \mu^0, \mu\right)$ to the above MV SDE system.

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We also introduce the equations

$$d\bar{z}_0(t) = f_0[t, \bar{z}_0(t), \varphi_0(t, \bar{z}_0(t)), \mu_t]dt + \sigma_0[t, \bar{z}_0(t), \mu_t]dw_0(t), \quad 0 \le t \le T,$$
$$d\bar{z}_i(t) = f[t, \bar{z}_i(t), \varphi(t, \bar{z}_i(t)), \mu_t^0, \mu_t]dt + \sigma[t, \bar{z}_i(t), \mu_t^0, \mu_t]dw_i(t), \quad 1 \le i \le N,$$

with initial conditions $\bar{z}_j(0) = z_j(0)$ for $0 \le j \le N$, which can be viewed as N independent samples of the MV SDE system above.

- We develop the decoupling result such that each $\hat{z}_i^N$, $1 \le i \le N$, has the natural limit $\bar{z}_i$ in the infinite population limit (HMC'06).

Theorem

*Assume (**A1**), (**A3**)-(**A5**) and (**H4**) hold. Then we have*

$$\sup_{0 \le j \le N} \sup_{0 \le t \le T} E|\hat{z}_j^N(t) - \bar{z}_j(t)| = O(1/\sqrt{N}),$$

where the right hand side may depend upon the terminal time T. □

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The Stochastic Mean Field (SMF) System

- The noise process of the major agent w_0 causes random fluctuation of the mean-field behaviour of the minor agents $\Longrightarrow$ the mean field behaviour of the minor agents is stochastic [H'10, NH'11].

The Major Agent' SMF System: We construct the major agent's SMF system in the following steps.

Step I (Major Agent's Stochastic Hamilton-Jacobi-Bellman (SHJB) Equation): By the decoupling result we shall approximate the empirical distribution of minor agents $\delta^N_{(\cdot)}$ with a stochastic probability measure $\mu_{(\cdot)}$.

Let $\mu_t(\omega)$, $0 \leq t \leq T$, be a given exogenous stochastic process. Then we define the following SOCP with $\mathcal{F}^{w_0}_t$-adapted random coefficients from the major agent's model in the infinite population limit:

$$dz_0(t) = f_0[t, z_0(t), u_0(t), \mu_t(\omega)]dt + \sigma_0[t, z_0(t), \mu_t(\omega)]dw_0(t, \omega), \quad z_0(0),$$

$$\inf_{u_0 \in \mathcal{U}_0} J_0(u_0) := \inf_{u_0 \in \mathcal{U}_0} E\Big[\int_0^T L_0[t, z_0(t), u_0(t), \mu_t(\omega)]dt\Big],$$

where we explicitly indicate the dependence of random measure $\mu_{(\cdot)}$ on the sample point $\omega \in \Omega$.

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The value function [based on Peng'92]:

$$\phi_0(t,x) := \inf_{u_0 \in \mathcal{U}_0} E_{\mathcal{F}_t^{w_0}} \Big[\int_t^T L_0[s, z_0(s), u_0(s), \mu_s(\omega)] ds \Big| z_0(t) = x \Big],$$

which is a $\mathcal{F}_t^{w_0}$-adapted process for any fixed x.

A semimartingale representation for $\phi_0(t,x)$ [based on Peng'92]:

$$\phi_0(t,x) = \int_t^T \Gamma_0(s,x) ds - \int_t^T \psi_0^T(s,x) dw_0(s), \quad (t,x) \in [0,T] \times \mathbb{R}^n,$$

where $\phi_0(s,x)$, $\Gamma_0(s,x)$ and $\psi_0(s,x)$ are $\mathcal{F}_s^{w_0}$-adapted stochastic processes.

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SHJB equation for the Major agent: If $\phi_0(t,x)$, $\Gamma_0(t,x)$ and $\psi_0(t,x)$ are a.s. continuous in (x,t) and are smooth enough with respect to x, then the pair $\big(\phi_0(s,x), \psi_0(s,x)\big)$ satisfies the backward in time SHJB equation

$$-\,d\phi_0(t,\omega,x) = \Big[H_0[t,\omega,x,D_x\phi_0(t,\omega,x)] + \big\langle \sigma_0[t,x,\mu_t(\omega)], D_x\psi_0(t,\omega,x)\big\rangle$$
$$+\,\frac{1}{2}\mathrm{Tr}\big(a_0[t,\omega,x]D^2_{xx}\phi_0(t,\omega,x)\big)\Big]dt - \psi_0^T(t,\omega,x)dw_0(t,\omega), \quad \phi_0(T,x) = 0,$$

in $[0,T]\times\mathbb{R}^n$, where $a_0[t,\omega,x] := \sigma_0[t,x,\mu_t(\omega)]\sigma_0^T[t,x,\mu_t(\omega)]$, and the stochastic Hamiltonian H_0 is given by

$$H_0[t,\omega,x,p] := \inf_{u\in\mathcal{U}_0}\big\{\big\langle f_0[t,x,u,\mu_t(\omega)], p\big\rangle + L_0[t,x,u,\mu_t(\omega)]\big\}.$$

The solution of the backward in time SHJB equation is a forward in time $\mathcal{F}_t^{w_0}$-adapted pair $\big(\phi_0(t,x),\psi_0(t,x)\big) \equiv \big(\phi_0(t,\omega,x),\psi_0(t,\omega,x)\big)$.

- Note that the appearance of the term $\big\langle \sigma_0[t,x,\mu_t(\omega)], D_x\psi_0(t,\omega,x)\big\rangle$ is due to the quadratic variation of the major agent's Brownian motion w_0 in the Itô-Kunita formula for the composition of $\mathcal{F}_t^{w_0}$-adapted stochastic processes $\phi_0(t,\omega,x)$ and $z_0(t,\omega)$.

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The best response control process of the major agent:

$$u_0^o(t,\omega,x) \equiv u_0^o(t,x|\{\mu_s(\omega)\}_{0\le s\le T}) := \arg\inf_{u_0\in U_0} H_0^{w_0}[t,\omega,x,u_0,D_x\phi_0(t,\omega,x)]$$
$$\equiv \arg\inf_{u_0\in U_0} \big\{\big\langle f_0[t,x,u_0,\mu_t(\omega)], D_x\phi_0(t,\omega,x)\big\rangle + L_0[t,x,u_0,\mu_t(\omega)]\big\}.$$

is a forward in time $\mathcal{F}_t^{w_0}$-adapted process which depends on the Brownian motion w_0 via the stochastic measure $\mu_t(\omega)$, $0\le t\le T$.

Step II (Major Agent's Stochastic Coefficients McKean-Vlasov (SMV) and Stochastic Fokker-Planck-Kolmogorov (SFPK) Equations): By substituting u_0^o into the major agent's dynamics we get the SMV dynamics with random coefficients:

$$dz_0^o(t,\omega) = f_0[t,z_0^o(t,\omega),u_0^o(t,\omega,z_0),\mu_t(\omega)]dt + \sigma_0[t,z_0^o(t,\omega),\mu_t(\omega)]dw_0(t,\omega),$$

where f_0 and σ_0 are random processes via the stochastic measure μ and u_0^o.

- We denote the corresponding random probability measure (law) of the major agent $z_0^o(t,\omega)$ by $\mu_t^0(\omega)$, $0\le t\le T$.

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Stochastic Fokker-Planck-Kolmogorov (SFPK) equation: An equivalent method to characterize the SMV equation. Let $p^0(t,\omega,x) := p^0(t,\omega,x)\big|\mathcal{F}_t^{w_0}$:

$$dp^0(t,\omega,x) = \Big(-\big\langle D_x, f_0[t,x,u_0^o(t,\omega,x),\mu_t(\omega)]p^0(t,\omega,x)\big\rangle + \frac{1}{2}\mathrm{Tr}\big\langle D_{xx}^2, a_0[t,\omega,x]p^0(t,\omega,x)\big\rangle\Big)dt - \big\langle D_x, \sigma_0[t,x,\mu_t(\omega)]p^0(t,\omega,x)dw_0(t,\omega)\big\rangle, \quad p^0(t,0) = p_0^0.$$

The density function $p^0(t,\omega,x)$ generates the random measure of the major agent $\mu_t^0(\omega)$ such that $\mu^0(t,\omega,dx) = p^0(t,\omega,x)dx$ (a.s.), $0 \le t \le T$.

The weak solution to the SFPK equation is

$$\big(g(t,\omega,x), p^0(t,\omega,x)\big) = \big(g(0,\omega,x), p^0(0,x)\big) + \int_0^t \big(A_0(s,\omega,x)g(s,\omega,x), p^0(s,\omega,x)\big)ds + \int_0^t \big(\sigma_0[s,x,\mu_t(\omega)]^T D_x g(s,\omega,x), p^0(s,\omega,x)\big)dw_0(s,\omega),$$

where $\big(h(t,\omega,x), p^0(t,\omega,x)\big) := \int h(t,\omega,x)p^0(t,\omega,x)dx$, and A_0 is

$$A_0(t,\omega,x)h(x) := \big\langle f_0[t,x,u_0^o(t,\omega,x),\mu_t(\omega)], D_x h(x)\big\rangle + \frac{1}{2}\mathrm{Tr}\big(a_0[t,\omega,x]D_{xx}^2 h(x)\big).$$

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

- We note that the major agent's SOCP may be written with respect to the random mean field density of minor agents $p(t,\omega,x)$ instead of since $\mu(t,\omega,dx)$ by the fact that $\mu(t,\omega,dx) = p(t,\omega,x)dx$ (a.s.), $0 \le t \le T$.

The Major Agent's Stochastic Mean Field (SMF) System:

$$
\begin{aligned}
-\,d\phi_0(t,\omega,x) &= \Big[H_0[t,\omega,x,D_x\phi_0(t,\omega,x)] \\
&\quad + \big\langle \sigma_0[t,x,\mu_t(\omega)], D_x\psi_0(t,\omega,x)\big\rangle + \frac{1}{2}\mathrm{Tr}\big(a_0[t,\omega,x]D^2_{xx}\phi_0(t,\omega,x)\big)\Big]dt \\
&\quad - \psi_0^T(t,\omega,x)dw_0(t,\omega), \quad \phi_0(T,x) = 0, \qquad \text{[MF-SHJB]} \\
u_0^o(t,\omega,x) &\equiv u_0^o(t,x|\{\mu_s(\omega)\}_{0\le s\le T}) \qquad \text{[MF-SBR]} \\
&:= \arg\inf_{u_0\in U_0}\big\{\big\langle f_0[t,x,u_0,\mu_t(\omega)], D_x\phi_0(t,\omega,x)\big\rangle + L_0[t,x,u_0,\mu_t(\omega)]\big\}, \\
dz_0^o(t,\omega) &= f_0[t,z_0^o(t,\omega),u_0^o(t,\omega,z_0),\mu_t(\omega)]dt \\
&\quad + \sigma_0[t,z_0^o(t,\omega),\mu_t(\omega)]dw_0(t,\omega), \quad z_0^o(0) = z_0(0), \qquad \text{[MF-SMV]}
\end{aligned}
$$

- The solution of the SMF system above consists of 4-tuple $\mathcal{F}_t^{w_0}$-adapted random processes $\big(\phi_0(t,\omega,x),\psi_0(t,\omega,x),u_0^o(t,\omega,x),z_0^o(t,\omega)\big)$, for a given exogenous stochastic process $\mu_t(\omega)$, where $z_0^o(t,\omega)$ generates the random measure $\mu_t^0(\omega)$.

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

The Minor Agents' Stochastic Mean Field (SMF) System: We construct the SMF stochastic mean field (SMF) system for a "generic" minor agent i in the following steps.

Step I (Minor Agent's Stochastic Hamilton-Jacobi-Bellman (SHJB) Equation):

- By the decoupling result we may approximate the empirical distribution of minor agents $\delta^N_{(\cdot)}$ with a stochastic probability measure $\mu_{(\cdot)}$.
- As in major player's case let μ_t, $0 \le t \le T$, be the exogenous stochastic process approximating δ^N_t in the infinite population limit. We let $\mu^0_t(\omega)$, $0 \le t \le T$, be the random measure of the major agent obtained from the major agent's SMF system.

We define the following SOCP with $\mathcal{F}^{w_0}_t$-adapted random coefficients from the minor agent's model in the infinite population limit:

$$dz_i(t) = f[t, z_i(t), u_i(t), \mu^0_t(\omega), \mu_t(\omega)]dt + \sigma[t, z_i(t), \mu^0_t(\omega), \mu_t(\omega)]dw_i(t, \omega),$$

$$\inf_{u_i \in \mathcal{U}} J_i(u_i) := \inf_{u_i \in \mathcal{U}} E\Big[\int_0^T L[t, z_i(t), u_i(t), \mu^0_t(\omega), \mu_t(\omega)]dt\Big], \quad z_i(0),$$

where we explicitly indicate the dependence of random measures $\mu^0_{(\cdot)}$ and $\mu_{(\cdot)}$ on the sample point $\omega \in \Omega$.

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

The value function [based on Peng'92]:

$$\phi_i(t,x) := \inf_{u_i \in \mathcal{U}_0} E_{\mathcal{F}_t^{w_0}} \Big[\int_t^T L[s, z_i(s), u_i(s), \mu_s^0(\omega), \mu_s(\omega)] ds \Big| z_i(t) = x \Big],$$

which is a $\mathcal{F}_t^{w_0}$-adapted process for any fixed x.

A semimartingale representation for $\phi_i(t,x)$ [based on Peng'92]:

$$\phi_i(t,x) = \int_t^T \Gamma_i(s,x) ds - \int_t^T \psi_i^T(s,x) dw_0(s), \quad (t,x) \in [0,T] \times \mathbb{R}^n,$$

where $\phi_0(s,x)$, $\Gamma_0(s,x)$ and $\psi_0(s,x)$ are $\mathcal{F}_s^{w_0}$-adapted stochastic processes.

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

SHJB equation for the generic Minor agent: If $\phi_i(t,x)$, $\Gamma_i(t,x)$ and $\psi_i(t,x)$ are a.s. continuous in (x,t) and are smooth enough with respect to x, then the pair $\big(\phi_i(s,x), \psi_i(s,x)\big)$ satisfies the backward in time SHJB equation

$$-\,d\phi_i(t,\omega,x) = \Big[H[t,\omega,x,D_x\phi_i(t,\omega,x)] + \frac{1}{2}\mathrm{Tr}\big(a[t,\omega,x]D^2_{xx}\phi_i(t,\omega,x)\big)\Big]dt \\ -\,\psi_i^T(t,\omega,x)dw_0(t,\omega), \quad \phi_i(T,x) = 0,$$

in $[0,T]\times\mathbb{R}^n$, where $a[t,\omega,x] := \sigma[t,x,\mu_t^0(\omega),\mu_t(\omega)]\sigma^T[t,x,\mu_t^0(\omega),\mu_t(\omega)]$, and the stochastic Hamiltonian H is given by

$$H[t,\omega,x,p] := \inf_{u\in\mathcal{U}}\big\{\big\langle f[t,x,u,\mu_t^0(\omega),\mu_t(\omega)],p\big\rangle + L[t,x,u,\mu_t^0(\omega),\mu_t(\omega)]\big\}.$$

The solution of the backward in time SHJB equation is a forward in time $\mathcal{F}_t^{w_0}$-adapted pair $\big(\phi_i(t,x),\psi_i(t,x)\big) \equiv \big(\phi_i(t,\omega,x),\psi_i(t,\omega,x)\big)$.

- Since the coefficients of the minor agent's SOCP are $\mathcal{F}_t^{w_0}$-adapted random processes we have the major agent's Brownian motion w_0 in the SHJB equation above.
- Unlike the major agent's SHJB equation we do not have the term $\big\langle\sigma[t,x,\mu_t^0(\omega),\mu_t(\omega)]D_x\psi_i(t,\omega,x)\big\rangle$ since the coefficients in the minor agent's SOCP are $\mathcal{F}_t^{w_0}$-adapted random processes depending upon (w_0) which is independent of (w_i) (see the Itô-Kunita formula).

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

The best response control process of the generic minor agent:

$$u_i^o(t,\omega,x) \equiv u_i^o(t,x|\{\mu_s^0(\omega),\mu_s(\omega)\}_{0\le s\le T}) := \arg\inf_{u\in U} H^u[t,\omega,x,u,D_x\phi_i(t,\omega,x)]$$
$$\equiv \arg\inf_{u\in U}\big\{\big\langle f[t,x,u,\mu_t^0(\omega),\mu_t(\omega)], D_x\phi_i(t,\omega,x)\big\rangle + L[t,x,u,\mu_t^0(\omega),\mu_t(\omega)]\big\}.$$

is a forward in time $\mathcal{F}_t^{w_0}$-adapted process which depends on the Brownian motion w_0 via the stochastic measures $\mu_t^0(\omega)$ and $\mu_t(\omega)$, $0\le t\le T$.

Step II (The Generic Minor Agent's Stochastic Coefficients McKean-Vlasov (SMV) and Stochastic Fokker-Planck-Kolmogorov (SFPK) Equations): By substituting u_i^o into the minor agent's dynamics we get the SMV dynamics with random coefficients:

$$dz_i^o(t,\omega) = f[t,z_i^o(t,\omega),u_i^o(t,\omega,z_i),\mu_t^0(\omega),\mu_t(\omega)]dt$$
$$+\sigma[t,z_i^o(t,\omega),\mu_t^0(\omega),\mu_t(\omega)]dw_i(t,\omega), \quad z_i^o(0)=z_i(0),$$

where f and σ are random processes via the stochastic measures μ^0 and μ, and the best response control process u_i^o which all depend on the Brownian motion of the major agent w_0.

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

- Based on the decoupling effect the generic minor agent's statistical properties can effectively approximate the empirical distribution produced by all minor agents in a large population system.
- From the minor agent's SMV equation we obtain a new stochastic measure $\hat{\mu}_t(\omega)$ for the mean field behaviour of minor agents from the statistical behaviour of the generic minor agent $z_i^o(t,\omega)$. We characterize $\hat{\mu}_t(\omega)$, $0 \le t \le T$, as the law of $z_i^o(t,\omega)$.

The mean field games (MFG) or Nash certainty equivalence (NCE) consistency [HCM'03, HMC'06, LL'06] is now imposed by letting $\hat{\mu}_t(\omega) = \mu_t(\omega)$ a.s., $0 \le t \le T$.

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

Stochastic Fokker-Planck-Kolmogorov (SFPK) equation: An equivalent method to characterize the SMV equation. Let $\hat{p}(t,\omega,x) := \hat{p}(t,\omega,x)\big|\mathcal{F}_t^{w_0}$:

$$d\hat{p}(t,\omega,x) = \Big(-\big\langle D_x, f[t,x,u_i^o(t,\omega,x),\mu_t^0(\omega),\mu_t(\omega)]\hat{p}(t,\omega,x)\big\rangle + \frac{1}{2}\mathrm{Tr}\big\langle D_{xx}^2, a[t,\omega,x]\hat{p}(t,\omega,x)\big\rangle\Big)dt, \quad \hat{p}(t,0) = p_0,$$

The density function $\hat{p}(t,\omega,x)$ generates the random measure of the minor agents's mean field behaviour $\hat{\mu}_t(\omega)$ such that $\hat{\mu}(t,\omega,dx) = \hat{p}(t,\omega,x)dx$ (a.s.), $0 \le t \le T$.

The weak solution to the SFPK equation is

$$\big(g(t,\omega,x),\hat{p}(t,\omega,x)\big) = \big(g(0,\omega,x),\hat{p}(0,x)\big) + \int_0^t \big(A(s,\omega,x)g(s,\omega,x),\hat{p}(s,\omega,x)\big)ds,$$

where $\big(h(t,\omega,x),\hat{p}(t,\omega,x)\big) := \int h(t,\omega,x)\hat{p}(t,\omega,x)dx$, and A is

$$A(t,\omega,x)h(x) := \big\langle f[t,x,u^o(t,\omega,x),\mu_t^0(\omega),\mu_t(\omega)], D_x h(x)\big\rangle + \frac{1}{2}\mathrm{Tr}\big(a[t,\omega,x]D_{xx}^2 h(x)\big).$$

- The reason that the generic minor agent's SFPK equation does not include the Itô integral term with respect to w_i is due to the fact that the independent Brownian motions of individual minor agents are averaged out in their mean field behaviour.

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

The MFG or NCE consistency is imposed in: (i) the major agent's stochastic mean field (SMF) system together with (ii) the following SMF system for the minor agents below.

The Minor Agents' Stochastic Mean Field (SMF) System:

$$
\begin{aligned}
&-d\phi(t,\omega,x) = \Big[H[t,\omega,x,D_x\phi(t,\omega,x)] + \frac{1}{2}\mathrm{Tr}\big(a[t,\omega,x]D^2_{xx}\phi(t,\omega,x)\big)\Big]dt \\
&\quad -\psi^T(t,\omega,x)dw_0(t,\omega), \quad \phi(T,x)=0, \qquad \text{[MF-SHJB]} \\
&u^o(t,\omega,x) \equiv u^o(t,x|\{\mu^0_s(\omega),\mu_s(\omega)\}_{0\le s\le T}) \qquad \text{[MF-SBR]} \\
&\quad \equiv \arg\inf_{u\in U}\big\{\langle f[t,x,u,\mu^0_t(\omega),\mu_t(\omega)], D_x\phi(t,\omega,x)\rangle + L[t,x,u,\mu^0_t(\omega),\mu_t(\omega)]\big\}, \\
&dz^o(t,\omega) = f[t,z^o(t,\omega),u^o(t,\omega,z),\mu^0_t(\omega),\mu_t(\omega)]dt \\
&\quad +\sigma[t,z^o(t,\omega),\mu^0_t(\omega),\mu_t(\omega)]dw(t,\omega), \qquad \text{[MF-SMV]}
\end{aligned}
$$

in $[0,T]\times\mathbb{R}^n$, where $z^o(0)$ has the measure $\mu_0(dx)=dF(x)$ where F is defined in (**A2**).

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

The Major-Minor Stochastic Mean Field (SMF) System:

- The SMF system is given by the major and minor agents' coupled SMF systems.
- The solution of the major-minor SMF system consists of 8-tuple $\mathcal{F}_t^{w_0}$-adapted random processes

$$\big(\phi_0(t,\omega,x), \psi_0(t,\omega,x), u_0^o(t,\omega,x), z_0^o(t,\omega), \phi(t,\omega,x), \psi(t,\omega,x), u^o(t,\omega,x), z^o(t,\omega)\big),$$

  where $z_0^o(t,\omega)$ and $z^o(t,\omega)$ respectively generate the random measures $\mu_t^0(\omega)$ and $\mu_t(\omega)$.
- The solution to the major-minor SMF system is a public stochastic mean field in contrast to the public deterministic mean field of the standard MFG problems (HCM'03,HMC'06,LL'06).

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

Existence and uniqueness of Solution to the Major and Minor (MM) Agents' Stochastic Mean Field (SMF) System: A fixed point argument with random parameters in the space of stochastic probability measures.

- On the Banach space $C([0,T];\mathbb{R}^n)$ we define the metric $\rho_T(x,y) :=_{0\leq t\leq T} |x(t)-y(t)| \wedge 1$ where $\wedge$ denotes minimum.
- $C_\rho := \big(C([0,T];\mathbb{R}^n), \rho_T\big)$ forms a separable complete metric space.
- Let $\mathcal{M}(C_\rho)$ be the space of all Borel probability measures μ on $C([0,T];\mathbb{R}^n)$ such that $\int |x| d\mu(x) < \infty$.
- We also denote $\mathcal{M}(C_\rho \times C_\rho)$ as the space of probability measures on the product space $C([0,T];\mathbb{R}^n) \times C([0,T];\mathbb{R}^n)$.
- Let the canonical process x be a random process with the sample space $C([0,T];\mathbb{R}^n)$, i.e., $x(t,\omega) = \omega(t)$ for $\omega \in C([0,T];\mathbb{R}^n)$ [HMC'06].
- $C_{\mathrm{Lip}(x)}$: the class of a.s. continuous functions which are a.s. Lipschitz continuous in x

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

Based on the metric ρ, we introduce the Wasserstein (or Vasershtein) metric on $\mathcal{M}(C_\rho)$:

$$D_T^\rho(\mu,\nu) := \inf_{\gamma\in\Pi(\mu,\nu)} \Big[\int_{C_\rho\times C_\rho} \rho_T(x(\omega_1),x(\omega_2))d\gamma(\omega_1,\omega_2)\Big],$$

where $\Pi(\mu,\nu) \subset \mathcal{M}(C_\rho \times C_\rho)$ is the set of Borel probability measures γ such that $\gamma(A \times C([0,T];\mathbb{R}^n)) = \mu(A)$ and $\gamma(C([0,T];\mathbb{R}^n) \times A) = \nu(A)$ for any Borel set $A \in C([0,T];\mathbb{R}^n)$. The metric space $\mathcal{M}_\rho := \big(\mathcal{M}(C_\rho), D_T^\rho\big)$ is a separable and complete metric space since $C_\rho \equiv \big(C([0,T];\mathbb{R}^n), \rho_T\big)$ is a separable complete metric space.

Definition

A stochastic probability measure $\mu_t(\omega)$, $0 \le t \le T$, in the space $\mathcal{M}_\rho$ is in $\mathcal{M}_\rho^\beta$ if μ is a.s. uniformly Hölder continuous with exponent $0 < \beta < 1/2$, i.e., there exists $\beta \in (0,1/2)$ and constant c such that for any bounded and Lipschitz continuos function ϕ on $\mathbb{R}^n$,

$$\Big|\int_{\mathbb{R}^n} \phi(x)\mu_t(\omega,dx) - \int_{\mathbb{R}^n} \phi(x)\mu_s(\omega,dx)\Big| \le c(\omega)|t-s|^\beta, \quad a.s.,$$

for all $0 \le s < t \le T$, where c may depend upon the Lipschitz constant of ϕ and the sample point $\omega \in \Omega$.

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

Assumption:

(**A8**) For any $p \in \mathbb{R}^n$ and $\mu, \mu^0 \in \mathcal{M}_\rho^\beta$, the sets

$$S_0(t,\omega,x,p) := \arg \inf_{u_0 \in U_0} H_0^{u_0}[t,\omega,x,u_0,p],$$
$$S(t,\omega,x,p) := \arg \inf_{u \in U} H^u[t,\omega,x,u,p],$$

are singletons and the resulting u and u_0 as functions of $[t,\omega,x,p]$ are a.s. continuous in t, Lipschitz continuous in (x,p), uniformly with respect to t and $\mu, \mu^0 \in \mathcal{M}_\rho^\beta$. In addition, $u_0[t,\omega,0,0]$ and $u[t,\omega,0,0]$ are in the space $L^2_{\mathcal{F}_t}([0,T];\mathbb{R}^n)$.

The Analysis of the Major Agent's SMF System: Assume (**A3**)-(**A8**) holds. Then we have the following well-defined maps:

$$\Gamma_0^{\mathrm{SHJB}} : M_\rho^\beta \longrightarrow C_{\mathrm{Lip}(x)}([0,T]\times\Omega\times\mathbb{R}^n;U_0), \qquad 0<\beta<1/2,$$
$$\Gamma_0^{\mathrm{SHJB}}\big(\mu_{(\cdot)}(\omega)\big) = u_0^o(t,\omega,x) \equiv u_0^o(t,x|\{\mu_s(\omega)\}_{0\le s\le T}).$$

$$\Gamma_0^{\mathrm{SMV}} : C_{\mathrm{Lip}(x)}([0,T]\times\Omega\times\mathbb{R}^n;U_0) \longrightarrow M_\rho^\beta, \qquad 0<\beta<1/2,$$
$$\Gamma_0^{\mathrm{SMV}}\big(u_0^o(t,\omega,x)\big) = \mu_{(\cdot)}^0(\omega),$$

which together give

$$\Gamma_0 : M_\rho^\beta \longrightarrow M_\rho^\beta, \qquad 0<\beta<1/2,$$
$$\Gamma_0\big(\mu_{(\cdot)}(\omega)\big) := \Gamma_0^{\mathrm{SMV}}\Big(\Gamma_0^{\mathrm{SHJB}}\big(\mu_{(\cdot)}(\omega)\big)\Big) = \mu_{(\cdot)}^0(\omega).$$

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

The Analysis of the Generic Minor Agent's SMF System: Assume (**A3**)-(**A8**) holds. Then we have the following well-defined maps:

$$\Gamma_i^{\text{SHJB}} : M_\rho^\beta \times M_\rho^\beta \longrightarrow C_{\text{Lip}(x)}([0,T] \times \Omega \times \mathbb{R}^n; U), \quad 0 < \beta < 1/2,$$
$$\Gamma_i^{\text{SHJB}}\big(\mu_{(\cdot)}(\omega), \mu^0_{(\cdot)}(\omega)\big) = u_i^o(t,\omega,x) \equiv u_i^o(t,x|\{\mu_s^0(\omega), \mu_s(\omega)\}_{0\le s\le T}).$$

$$\Gamma_i^{\text{SMV}} : M_\rho^\beta \times C_{\text{Lip}(x)}([0,T] \times \Omega \times \mathbb{R}^n; U) \longrightarrow M_\rho^\beta,\ 0 < \beta < 1/2,$$
$$\Gamma_i^{\text{SMV}}\big(\mu^0_{(\cdot)}(\omega), u_i^o(t,\omega,x)\big) = \mu_{(\cdot)}(\omega).$$

□

Hence, we obtain the following well-defined map:

$$\Gamma : M_\rho^\beta \longrightarrow M_\rho^\beta, \qquad 0 < \beta < 1/2,$$
$$\Gamma\big(\mu_{(\cdot)}(\omega)\big) = \Gamma_i^{\text{SMV}}\Big(\Gamma_0\big(\mu_{(\cdot)}(\omega)\big), \Gamma_i^{\text{SHJB}}\big(\mu_{(\cdot)}(\omega)\big), \Gamma_0\big(\mu_{(\cdot)}(\omega)\big)\big)\Big).$$

Subsequently, the problem of existence and uniqueness of solution to the SMF system is translated into a fixed point problem with random parameters for the map Γ on the separable complete metric space $\mathcal{M}_\rho^\beta$, $0 < \beta < 1/2$.

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

Assumptions:

(**A9**) We assume that σ_0 does not contain z_0^N and z_i^N for $1 \le i \le N$.

(**A10**) Feedback Regularity Assumptions:

(i) There exists a constant c_1 such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} \left|u_0(t,\omega,x) - u_0'(t,\omega,x)\right| \le c_1 D_T^\rho\big(\mu(\omega),\nu(\omega)\big), \qquad a.s.,$$

where u_0, u_0' are induced by the map Γ_0^{SHJB} using $\mu_{(.)}(\omega)$ and $\nu_{(.)}(\omega)$.

(ii) There exists a constant c_2 such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} \left|u(t,\omega,x) - u'(t,\omega,x)\right| \le c_2 D_T^\rho\big(\mu(\omega),\nu(\omega)\big), \qquad a.s.,$$

where u, u' are induced by the map Γ_i^{SHJB} using $\mu_{(.)}(\omega)$ and $\nu_{(.)}(\omega)$.

(iii) There exists a constant c_3 such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} \left|u(t,\omega,x) - u'(t,\omega,x)\right| \le c_3 D_T^\rho\big(\mu^0(\omega),\nu^0(\omega)\big), \qquad a.s.,$$

where u, u' are induced by the map Γ_i^{SHJB} using $\mu_{(.)}^0(\omega)$ and $\nu_{(.)}^0(\omega)$.

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

- The feedback regularity assumptions may be shown under some conditions by a sensitivity analysis of the major and minor agents' SHJB equations with respect to the stochastic measures (based on the analysis in [Kolokoltsov, Li, Yang, 2011])

Theorem (Existence and Uniquness of the Solution)

*Assume (**A3**)-(**A10**) hold. Under a contraction gain condition there exists a unique solution for the map Γ, and hence a unique solution to the major and minor agents' SMF system.*

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

ϵ-Nash Equilibria:

Given $\epsilon > 0$, the set of controls $\{u_j^o; 0 \le j \le N\}$ generates an ϵ-Nash equilibrium w.r.t. the costs J_j^N, 1$\le j \le N\}$ if, for each j,

$$J_j^N(u_j^0, u_{-j}^0) - \epsilon \le \inf_{u_j \in \mathcal{U}_j} J_j^N(u_j, u_{-j}^0) \le J_j^N(u_j^0, u_{-j}^0).$$

□

The decentralized admissible control sets:

$$\mathcal{U}_j := \{u_j(\cdot, \omega, x) \in C_{\mathrm{Lip}(x)} : u_j(t, \omega, x) \text{ is adapted to the sigma-field}$$
$$\sigma\{z_j(\tau), \omega_0(\tau) : 0 \le \tau \le t\} \text{ such that } E\int_0^T |u_j(t)|^2 dt < \infty\}.$$

(**A11**) We assume that functions f and σ in the minor agents' "dynamics" do not contain the state of the major agent z_0.

Theorem

*Assume (**A1**)-(**A8**) and (**A11**) hold, and there exists a unique solution to the SMF system such that the MF best response control processes $(u_0^o, \cdots, u_N^o)$ satisfies the Lipschitz condition. Then $\{u_j^o \in \mathcal{U}_j : 0 \le j \le N\}$ generates an ϵ_N-Nash equilibrium where $\epsilon_N = O(1/\sqrt{N})$.*

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

Example: Consider the MM MF-LQG model of (SLN,MYH'11) with uniform parameters

Dynamics:

$$\mathcal{A}_0 : \; dz_0(t) = \big(a_0 z_0(t) + b_0 u_0(t) + c_0 z^{(N)}(t)\big)dt + \sigma_0 dw_0(t),$$

$$\mathcal{A}_i : \; dz_i(t) = \big(a z_i(t) + b u_i(t) + c z^{(N)}(t)\big)dt + \sigma dw_i(t), \; 1 \le i \le N,$$

where $z^{(N)}(\cdot) := (1/N)\sum_{i=1}^{N} z_i(\cdot)$ is the average state of minor agents.

Costs:

$$\mathcal{A}_0 : \; J_0(u_0, u_{-0}) = E\int_0^T \Big[\Big(z_0(t) - \big(\lambda_0 z^{(N)}(t) + \eta_0\big)\Big)^2 + r_0 u_0^2(t)\Big]dt,$$

$$\mathcal{A}_i : \; J_i(u_i, u_{-i}) = E\int_0^T \Big[\Big(z_i(t) - \big(\lambda z^{(N)}(t) + \lambda_1 z_0(t) + \eta\big)\Big)^2 + r u_i^2(t)\Big]dt,$$

where $r_0, r > 0$.

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

Let $z^*(\cdot)$ be the stochastic mean filed of the minor agents

The Major Agent's SMF LQG System:

$$
\begin{aligned}
&[\text{Back. SDE}]: \quad -dp_0(t) = \big(a_0 p_0(t) + \lambda_0 z^*(t) + \mu_0 - z_0^*(t)\big)dt - q_0(t)dw_0(t),\\
&[\text{SBR}]: \quad u_0^*(t) = (b_0/r_0)p_0(t),\\
&[\text{Forw. SDE}]: \quad dz_0^*(t) = \big(a_0 z_0^*(t) + b_0 u_0^*(t) + c_0 z^*(t)\big)dt + \sigma_0 dw_0,
\end{aligned}
$$

with $p_0(T) = 0$ and $z_0^*(0) = z_0(0)$.

The Minor Agent's SMF LQG System:

$$
\begin{aligned}
&[\text{Back. SDE}]: \quad -dp(t) = \big(ap(t) + (\lambda - 1)z^*(t) + \lambda_1 z_0^*(t) + \mu\big)dt - q(t)dw_0(t),\\
&[\text{SBR}]: \quad u^*(t) = (b/r)p(t),\\
&[\text{Forw. SDE}]: \quad dz^*(t) = \big((a+c)z^*(t) + bu^*(t)\big)dt,
\end{aligned}
$$

with $p(T) = 0$ and $z^*(0) = \bar{z}(0)$.

The solution of the above SMF systems of equations consist of $\big(p_0(\cdot), q_0(\cdot), u_0^*(\cdot), z_0^*(\cdot)\big)$ and $\big(p(\cdot), q(\cdot), u^*(\cdot), z^*(\cdot)\big)$

Major-Minor ϵ-Nash Mean Field Game (MM ϵ-NMFG)

In the MM MF-LQG model of (NH'11):

- A Gaussian mean field approximation is used for the average state of minor agents:

$$(z^{(N)}(t) \approx)\ z^*(t) = f_1(t) + f_2(t) z_0(0) + \int_0^t g(t,s) dw_0(s),$$

  where f_1, f_2 and g are continuous functions.

- Consistency conditions are imposed for the mean field approximations