An introduction to mean-field games: the finite state space case

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Outline

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Problem set-up

- $N+1$ indistinguishable players;
- players can be in a finite number of states $i = 1, \ldots d$;
- at any time each player knows only its state $\mathbf{i}(t)$ and the number of players $\mathbf{n}_j(t)$ in state j;
- each player can only control its switching rate α from one state to another;
- players follow (independent) controlled Markov chains with transition rate β_{jk}.

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Optimization criterion

- Each player chooses the switching rate in order to minimize an expected payoff;
- This payoff has a running cost $c(i, \frac{n}{N}, \alpha)$, where α is the switching rate
- and a terminal cost $\psi^i\left(\frac{n}{N}\right)$;

more precisely

$$\text{cost} = E\int_t^T c(\mathbf{i}(s), \frac{\mathbf{n}(s)}{N}, \alpha(s))ds + \psi^{\mathbf{i}(T)}\left(\frac{\mathbf{n}(T)}{N}\right).$$

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Technical hypothesis

- $c(i, \theta, \alpha)$ is uniformly convex and superlinear in α
- $c(i, \theta, \alpha)$ and $\psi^i(\theta)$ are smooth in θ.

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- Consider the case where $N >> 1$;
- We suppose the **mean-field hypothesis holds**, i.e. the fraction of players in each state j is given by a deterministic function $\theta^j(t)$;
- if all players use the same Markovian control $\beta = \beta_{ji}(t)$, the evolution of θ is determined by

$$\frac{d\theta^i}{dt} = \sum_j \theta^j \beta_{ji}.$$

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If θ is given, the objective of each player is to minimize

$$E\left[\int_t^T c(\mathbf{i}(s), \theta(s), \alpha(\mathbf{i}(s), s))ds + \psi^{\mathbf{i}(T)}(\theta(T))\right],$$

where α is the switching rate.

Notation

- Let $\Delta_i : \mathbb{R}^d \to \mathbb{R}^d$ be
$$\Delta_i z = (z^1 - z^i, ..., z^d - z^i).$$
- The infinitesimal generator for finite state continuous time Markov chain, with transition rate ν_{ij}, is
$$A_i^\nu(\varphi) = \sum_j \nu_{ij}(\varphi^j - \varphi^i) = \nu_{i\cdot} \cdot \Delta_i \varphi.$$
- We define the generalized Legendre transform of c is
$$h(z, \theta, i) = \min_{\mu \in (\mathbb{R}_0^+)^d} c(i, \theta, \mu) + \sum_j \mu_j \Delta_i z$$

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Hamiltonian ODE

The value function is the unique solution to the following Hamilton-Jacobi ordinary differential equation:

$$\begin{cases} -\frac{du^i}{dt} = h(\Delta_i u, \theta, i), \\ u^i(T) = \psi^i(\theta(T)). \end{cases}$$

Furthermore, the optimal control is given by

$$\alpha_j^* = h_{z_j}(\Delta_i u, \theta, i).$$

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Mean-field equations

The mean-field equilibrium arises when all players use the same optimal switching rate.

This gives rise to the system

$$\begin{cases} \frac{d}{dt}\theta^i = \sum_j \theta^j \alpha_i^*(\Delta_j u, \theta, j) \\ -\frac{d}{dt}u^i = h(\Delta_i u, \theta, i). \end{cases}$$

together with the initial-terminal conditions

$$\theta(0) = \theta_0 \qquad u^i(T) = \psi^i(\theta(T)).$$

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- existence of solution is by no means obvious;
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Example

Set

$$c(i,\theta,\alpha) = \sum_j \frac{\alpha_j^2}{2} + f^i(\theta)$$

Then

$$h(\Delta_i u,\theta,i) = f^i(\theta) - \frac{1}{2}\sum_j [(u^i - u^j)^+]^2,$$

and, for $j \neq i$,

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Existence of mean-field equilibria

- Fix θ and consider the map $\mathcal{U}(\theta)$ to be the solution of

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- given u consider the map $\Theta(u)$ to be the solution to

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Monotonicity hypothesis

We assume:

$$\sum_i (\theta^i - \tilde{\theta}^i)(\psi^i(\theta) - \psi^i(\tilde{\theta})) \geq 0$$

and

$$\theta \cdot (h(z, \tilde{\theta}) - h(z, \theta)) + \tilde{\theta} \cdot (h(\tilde{z}, \theta) - h(\tilde{z}, \tilde{\theta})) \leq -\gamma \|\theta - \tilde{\theta}\|^2.$$

Furthermore define $\|v\|_\sharp = \inf_{\lambda \in \mathbb{R}} \|v + \lambda \mathbf{1}\|$. Then we suppose that uniformly on $\|z\|_\sharp \leq M$ there exists $\gamma_i > 0$ such that

$$h(z, \theta, i) - h(w, \theta, i) - \alpha^*(w, \theta, i) \cdot \Delta_i(z - w) \leq -\gamma_i \|\Delta_i(z - w)\|^2.$$

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The last three hypothesis will be satisfied if h can be written as

$$h(\Delta_i z, \theta, i) = \tilde{h}(\Delta_i z, i) + f^i(\theta),$$

with $\tilde{h}$ (locally) uniformly concave and f satisfying the monotonicity hypothesis

$$(f(\tilde{\theta}) - f(\theta)) \cdot (\theta - \tilde{\theta}) \leq -\gamma|\theta - \tilde{\theta}|^2.$$

The previous property holds, for instance, if f is the gradient of a convex function $f(\theta) = \nabla\Phi(\theta)$.

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Theorem

Under the monotonicity hypothesis, the mean-field equations have a unique solution (θ, u).

Lemma

Fix $T > 0$ and suppose that (θ, u) and $(\tilde{\theta}, \tilde{u})$ are solutions with $\|u\|_\sharp, \|\tilde{u}\|_\sharp \leq C$ on $[-T, T]$
Then there exists a constant C independent of T such that, for all $0 < \tau < T$, we have

$$\int_{-\tau}^{\tau} \|(\theta - \tilde{\theta})(s)\|^2 + \|(u - \tilde{u})(s)\|_\sharp^2 ds$$
$$\leq C \int_{-\tau}^{\tau} \frac{d}{dt}\left[(\theta - \tilde{\theta}) \cdot (u - \tilde{u})\right]$$
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- The proof of the lemma follows the Lions-Lasry monotonicity method. The inequality in the lemma is obtained by applying the monotonicity hypothesis to

$$\frac{d}{dt}\left[(\theta-\tilde{\theta})\cdot(u-\tilde{u})\right].$$

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Contractive mean-field games

Let $\langle u\rangle = \dfrac{1}{d}\sum_j u^j$.

We say that h is contractive if there exists $M>0$ such that, if $\|u\|_\sharp > M$, then

$$(\Delta_i u)^j \le 0 \;\forall\, j \text{ implies } h(\Delta_i u,\theta,i) - \langle h(u,\theta,\cdot)\rangle < 0\,,$$

and

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$$(\Delta_i u)^j \ge 0 \; \forall \, j \text{ implies } h(\Delta_i u, \theta, i) - \langle h(u, \theta, \cdot) \rangle > 0 \,.$$

These conditions are natural if one observes that

$$(\Delta_{i_1} u)^j \leq 0 \ \forall j \ \text{ and } \ (\Delta_{i_2} u)^j \geq 0 \ \forall j$$

implies

$$2\|u\|_\sharp = u^{i_1} - u^{i_2} .$$

So, if u is a smooth solution and $\|u(t)\|_\sharp$ is differentiable with $\|u(t)\|_\sharp > M$ then

$$\frac{d}{dt}\|u\|_\sharp > 0 .$$

This implies the flow is backwards contractive with respect to the $\|\cdot\|_\sharp$ norm of the u component.

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Stationary solutions

A triplet $(\bar{\theta}, \bar{u}, \kappa)$ is called a stationary solution if

$$\begin{cases} \sum_j \bar{\theta}^j \alpha_i^*(\Delta_j \bar{u}, \bar{\theta}, j) = 0\,, \\ h(\Delta_i \bar{u}, \bar{\theta}, i) = \kappa\,. \end{cases}$$

If $(\bar{\theta}, \bar{u}, \kappa)$ is a stationary solution for the MFG equations, then $(\bar{\theta}, \bar{u} - \kappa t)$ solves the time dependent problem with appropriate initial-terminal conditions.

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Existence of stationary solutions

Theorem

Suppose h is contractive. Then

(a) *For M large enough, the set $\{(u, \theta), \|u\|_\sharp < M\}$ is invariant backwards in time by the flow of the mean-field equations.*

(b) *There exist a stationary solution.*

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Theorem

Suppose that monotonicity and contractivity hold.

(a) *Suppose $\|u(T)\|_\sharp \leq M$, where u is a solution, and M is large enough. Then $\|u(t)\|_\sharp \leq M \, \forall t \in [0, T]$.*

(b) *The stationary solution $(\bar\theta, \bar u, \kappa)$ is unique (up to the addition of a constant to $\bar u$).*

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$$f_T(s) := \|(\theta^T - \tilde{\theta}^T)(s)\|^2 + \|(u^T - \tilde{u}^T)(s)\|_\sharp^2,$$

and, for $0 < \tau < T$,

$$F_T(\tau) := \int_{-\tau}^{\tau} f_T(s)ds.$$

Then

$$F_T(\tau) \leq \frac{1}{\tilde{\gamma}}(f_T(\tau) + f_T(-\tau)).$$

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it follows $\frac{d}{dt} \ln F_T(\tau) \ge \tilde{\gamma}$. Therefore

$$\ln F_T(\tau) - \ln F_T(1) \ge (\tau - 1)\tilde{\gamma},$$

for all $0 < \tau < T$. From this we get

$$\int_{-1}^{1} f_T(s)ds = F_T(1) \le \frac{F_T(T)}{e^{(T-1)\tilde{\gamma}}} \to 0 \qquad \text{when } T \to \infty,$$

because F has sub-exponential growth in T.

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Potential mean-field games

We say the mean-field game is potential if h has the form

$$h(z,\theta,i) = \tilde{h}(z,i) + f^i(\theta) \tag{1}$$

where $f(\cdot,\theta) = \nabla_\theta F(\theta)$ for some convex function $F(\theta)$.

Let $H : \mathbb{R}^{2d} \to \mathbb{R}$ be given by

$$\begin{aligned} H(u,\theta) &= \sum_i \theta^i \tilde{h}(\Delta_i u, i) + F(\theta) \\ &= \theta \cdot \tilde{h}(\Delta . u, \cdot) + F(\theta) \end{aligned}$$

A direct computation shows the mean-field equations can be written as

$$\begin{cases} \dfrac{\partial H}{\partial u^j} = \dot{\theta}^j \,, \\ \\ \dfrac{\partial H}{\partial \theta^j} = -\dot{u}^j \,. \end{cases}$$

This means the flow associated to the mean-field game is Hamiltonian.

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Given a convex function $G(p)$ we define the Legendre transform as

$$G^*(q) = \sup_p -q \cdot p - G(p).$$

If G is strictly convex and the previous supremum is achieved, then $q = -\nabla G(p)$, or equivalently $p = -\nabla G^*(q)$.

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If the function F is strictly convex in θ then the Hamiltonian H is strictly convex in θ. This allow us to consider the Legendre transform

$$L(u, \dot{u}) = \sup_\theta -\dot{u} \cdot \theta - H(u, \theta)$$
$$= \sup_\theta -(\dot{u} + \tilde{h}) \cdot \theta - F(\theta) = F^*(\dot{u} + \tilde{h}(\Delta . u, \cdot)).$$

From this we conclude that any solution is a critical point of the functional

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This variational problem has to be complemented by suitable boundary conditions. The initial-terminal value problem corresponds to

$$\theta_0 = -\nabla F^*(\dot{u}(0) + \tilde{h}(\Delta_\cdot u(0), \cdot)),$$
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Another important boundary condition arises in planning problems. In this case the objective is to find a terminal cost $u(T)$ which steers a initial probability distribution θ_0 into a terminal probability distribution θ^T. Hence we have the following

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The master equation

Let

$$g(u,\theta,i)=\sum_j \theta^j \alpha_i^*(\Delta_j u,\theta,j).$$

Consider the PDE, called the master equation,

$$-\frac{\partial U^i}{\partial t}(\theta,t)=h(U,\theta,i)+\sum_k g(U,\theta,k)\frac{\partial U^i}{\partial \theta^k}(\theta,t),$$

together with the terminal condition

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Theorem

Suppose U is a solution. Let θ and u be such that

1. *the first equation of the mean-field game is satisfied, i.e.* $\frac{d}{dt}\theta^i = g(U^i(\theta(t), t), \theta, i)$;
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Then u satisfies the second equation of the mean-field game, i.e. $-\frac{d}{dt}u^i = h(\Delta_i u, \theta, i)$ *as well as the terminal condition* $u^i(T) = \psi^i(\theta(T))$. *Therefore, u is the value function associated to* θ, *and so it determines a Nash equilibria for the MFG.*

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A Hamilton Jacobi equation for potential MFG

For potential mean field games the master equation can be further simplified if we suppose that the terminal condition is given by a gradient

$$U^i(\theta, T) = \nabla_{\theta^i} \Psi_T(i, \theta) .$$

In this case let Ψ be a solution of the PDE

$$\begin{cases} -\frac{\partial \Psi}{\partial t} = H\left(\nabla_\theta \Psi, \theta\right), \\ \Psi(\theta, T) = \Psi_T(\theta) . \end{cases}$$

Then a direct calculation can show that $U^i(\theta, t) = \nabla_{\theta^i} \Psi(\theta, t)$ is a solution of the master equation.

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Outline

Mean field problems - continuous space and time

In continuous space and time a wide class of mean field equations has the form

$$N(u) = f(\theta)$$
$$L^*(\theta) = 0,$$

where N is a nonlinear operator and L^* is the adjoint of the linearization of N, and f is a monotone increasing function of θ. The function $u : \Omega \times [0, T] \to \mathbb{R}$ is supposed to be sufficiently regular, and θ is a (probability) measure.

An important example is

$$N(u) = -u_t + H(D_x u, x) - \frac{1}{2}\Delta u$$

to which corresponds

$$L^*(\theta) = \theta_t - \mathrm{div}(D_p H\theta) - \frac{1}{2}\Delta\theta,$$

and $f(\theta) = \ln\theta$.

Controlled diffusions

Suppose we know a the distribution of players in $\mathbb{R}^n$ given by a probability measure $\theta(t, \cdot)$. The objective of an individual reference player is to minimize

$$V(x, t) = E \int_t^T \left(L(\mathbf{x}, \mathbf{v}, \theta) ds + \psi(\mathbf{x}(T)) \right).$$

among all diffusions

$$d\mathbf{x} = \mathbf{v} dt + dW_t.$$

Then $V(x,t)$ solves

$$-V_t + H(D_x V, x, \theta) = \frac{1}{2}\Delta V,$$

and the optimal drift v is

$$v = -D_p H(D_x V, x, \theta).$$

If all the population acts according to the optimal strategy then

$$\theta_t - \operatorname{div}(D_p H \theta) = \frac{1}{2}\Delta\theta.$$

- Suppose that $f(z) = g'^{-1}(z)$, for some convex increasing function g.
- We consider the variational problem

$$\int_0^T \int_\Omega g(N(u)).$$

- A simple computation shows that sufficiently smooth critical points of this functional are indeed solutions of the mean field equations, for

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For instance, in the example we have the variational problem

$$\int_0^T \int_\Omega e^{-u_t + H(Du,x) - \frac{1}{2}\Delta u} dxdt. \tag{2}$$

- For convex nonlinear operators N this variational formulation yields in many instances uniqueness results for smooth solutions to the mean-field equations.
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The Evans-Aronsson variational problem is

$$\min \int_{\mathbb{T}^d} e^{H(Du,x)-\frac{1}{2}\Delta u} dx.$$

The lack of coercivity of this functional is the key technical problem.

Key results

Existence and uniqueness of smooth solutions for:

- $H(p,x) = \frac{|P+p|^2}{2} + V(x)$, through explicit solutions;
- a wide class of Hamiltonians if $d = 2$, through a-priori bounds.

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- a wide class of Hamiltonians if $d = 2$, through a-priori bounds.

Hopf-Cole type transform

Theorem

Let u and v be periodic solutions to

$$\begin{cases} \frac{1}{2}\Delta u + \frac{|P+Du|^2}{2} + V(x) & = u - v \\ -\frac{1}{2}\Delta v + \frac{|P+Dv|^2}{2} + V(x) & = u - v. \end{cases}$$

Then u is a minimizer.

Note such solutions do exist and are smooth as we have the a-priori bound:

Theorem

$$\sup |Du| + |Dv| \leq C.$$

General case, dimension independent bounds

Set $m = e^{\frac{1}{2}\Delta u + H(x,Du) - \lambda}$. Here λ is such that m is a probability measure.

Theorem

$$\int_{\mathbb{T}^d} |D \ln m|^2 \leq C,$$

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$$\int_{\mathbb{T}^d} H(x, Du) m \leq C.$$

General case, dimension independent bounds

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Theorem

$$\int_{\mathbb{T}^d} |\Delta Du|^2 m \leq C$$

$$\|\sqrt{m}\|_{H^1} \leq C$$

$$\left(\int m^{\frac{2^*}{2}}\right)^{\frac{2}{2^*}} \leq C$$

$$\int |D^2 u|^2 m \leq C,$$

and

$$\int H^2 m \leq C.$$

In dimension 2 the previous bounds yield

Theorem

$$\int_{\mathbb{T}^d} |D^2 u|^2 \leq C.$$

which it is then enough to prove existence of smooth solutions by the continuation method.

Outline

Problem set-up - review

- $N+1$ indistinguishable players;
- players can be in a finite number of states $i = 1, \ldots d$;
- at any time each player knows only its state $\mathbf{i}(t)$ and the number of players $\mathbf{n}_j(t)$ in state j;
- each player can only control its switching rate α from one state to another;
- players follow (independent) controlled Markov chains with transition rate β_{jk}.

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Optimization criterion - review

- Each player chooses the switching rate in order to minimize an expected payoff;
- This payoff has a running cost $c(i, \frac{n}{N}, \alpha)$, where α is the switching rate
- and a terminal cost $\psi^i\left(\frac{n}{N}\right)$;

more precisely

$$\text{cost} = E \int_t^T c(\mathbf{i}(s), \frac{\mathbf{n}(s)}{N}, \alpha(s))ds + \psi^{\mathbf{i}(T)}\left(\frac{\mathbf{n}(T)}{N}\right).$$

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- The reference player switches from state i to state j according to a switching Markovian rate $\alpha_{ij}(n, t)$
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Let e_k be the $k-th$ vector of the canonical basis of $\mathbb{R}^d$, and let $e_{jk} = e_j - e_k$.

From the symmetry and independence of transitions assumption, for $k \neq j$, we have

$$\mathbb{P}\left(\mathbf{n}_{t+h} = n + e_{jk} \| \mathbf{n}_t = n, \mathbf{i}_t = i\right) = \gamma^{n,i}_{\beta,kj}(t).h + o(h),$$

where

$$\gamma^{n,i}_{\beta,kj}(t) = n_k \beta_{kj}(n + e_{ik}, t).$$

The term $n + e_{ik}$ instead of n, follows from the fact that from the point of view of a player which is in state k, and is distinct from the reference player, the number of other players in any state is given by $\mathbf{n} + e_{\mathbf{i}} - e_k = \mathbf{n} + e_{\mathbf{i}k}$.

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Control problem from a player's point of view

- Fix a reference player, and suppose the remaining N players use a transition rate β;
- Then the process $\mathbf{n}(t)$ is a Markov process with rate $\gamma^{n,i}_{\beta,kj}(t)$
- The reference player wants to

$$u^i_n(t,\beta,\alpha) = \mathbb{E}^{\beta,\alpha}\left[\int_t^T c\left(\mathbf{i}_s, \frac{\mathbf{n}_s}{N}, \alpha(s)\right) ds + \psi^{\mathbf{i}_T}\left(\frac{\mathbf{n}_T}{N}\right)\right],$$

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where the expectation is conditioned on $\mathbf{i}_t = i, \mathbf{n}_t = n$.

Hamilton-Jacobi ODE

Fix an admissible control β. Consider the system of ODE´s indexed by i and n given by

$$-\frac{d\varphi_n^i}{dt}(t) = \sum_{k,j} \gamma_{\beta,kj}^{n,i}(t)\left(\varphi_{n+e_{jk}}^i(t) - \varphi_n^i(t)\right) + h\left(\Delta_i\varphi_n(t), \frac{n}{N}, i\right),$$

where

$$h(p,\theta,i) = \min_{\alpha \geq 0}\left[c(i,\theta,\alpha) + \alpha p\right],$$

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A verification theorem

Theorem

The previous terminal value problem for φ_n^i has a unique solution. This solution is the value function for the reference player, and

$$\tilde{\alpha}(\beta)(i,n,t) \equiv \alpha^*\left(\Delta_i\varphi_n(t), \frac{n}{N}, i\right)$$

is an optimal strategy.
Furthermore φ_n^i is bounded uniformly in β.

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Furthermore φ_n^i is bounded uniformly in β.

Note that α^* depends on β. We say that β is a symmetric Nash equilibrium if $\alpha^*_\beta = \beta$.

Theorem

There exists a unique Nash equilibrium $\bar{\beta}$.

A necessary condition for a control $\bar{\beta}$ to be a Nash equilibrium is

$$\bar{\beta}_{kj}(n,t) = \alpha_j^* \left(\Delta_k u_n(t;\bar{\beta}), \frac{n}{N}, k \right).$$

Hence this gives rise to the system of nonlinear differential equations

$$-\frac{du_n^i}{dt} = \sum_{k,j} \gamma_{kj}^{n,i}(u_{n+e_{jk}}^i - u_n^i) + h\left(\Delta_i u_n, \frac{n}{N}, i\right) ,$$

with terminal condition

$$u_n^i(T) = \psi^i\left(\frac{n}{N}\right),$$

where $\gamma_{kj}^{n,i}$ are given by

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Outline

INSTITUTO SUPERIOR TÉCNICO

Mean-field requations - review

$$\begin{cases} \frac{d}{dt}\theta^i = \sum_j \theta^j \alpha_i^*(\Delta_j u, \theta, j) \\ -\frac{d}{dt}u^i = h(\Delta_i u, \theta, i). \end{cases}$$

Master equation - review

Recall the master equation

$$-\frac{\partial U^i}{\partial t}(\theta,t) = h(U,\theta,i) + \sum_k g(U,\theta,k)\frac{\partial U^i}{\partial \theta^k}(\theta,t),$$

where

$$g(u,\theta,i) = \sum_j \theta^j \alpha_i^*(\Delta_j u,\theta,j).$$

together with the terminal condition

$$U^i(\theta,T) = \psi^i(\theta).$$

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Consistency

Let U be a smooth solution to the mean-field master equation.
Set $\tilde{u}_n^i = U^i(\frac{n}{N})$.Then

$$-\frac{d\tilde{u}_n^i}{dt} = \sum_{k,j} \gamma_{kj}^{n,i}(\tilde{u}_{n+e_{jk}}^i - \tilde{u}_n^i) + h\left(\Delta_i \tilde{u}_n, \frac{n}{N}, i\right) + E_N$$

where $E_N \to 0$ as $N \to \infty$.

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where $E_N \to 0$ as $N \to \infty$.

Stability of controls

Lemma

We have

$$\left|\gamma_{kj}^{n+e_{rs},i} - \gamma_{kj}^{n,i}\right| \leq C + CN \max_{rs} \|u^i_{n+e_{rs}}(t) - u^i_n(t)\|_\infty .$$

Gradient estimates

Lemma

Let $u_n^i(t)$ be a solution. Then there exists $C > 0$ and $T^\star > 0$ such that, for $0 < T < T^\star$, we have

$$\max_{rs} \|u^i_{n+e_{rs}}(t) - u^i_n(t)\|_\infty \le \frac{2C}{N},$$

for all $0 \le t \le T$.

Theorem

There exists a constant C, independent of N, for which, if $T < T^*$, satisfies $\rho = TC < 1$, then

$$V_N(t) + W_N(t) \leq \frac{C}{1-\rho}\frac{1}{N} \, ,$$

for all $t \in [0, T]$, where

$$V_N(t) \equiv \mathbb{E}\left[\left\|\frac{\mathbf{n}_t}{N} - \theta(t)\right\|^2\right] \, ,$$

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Lemma

Suppose $T < T^*$. There exists $C_1 > 0$ such that

$$V_N(t) \leq \int_0^t C_1(V_N(s) + W_N(s))ds + \frac{C_1}{N}.$$

Lemma

Suppose $T < T^*$. There exists $C_2 > 0$ such that

$$W_N(t) \leq \int_t^T C_2(V_N(s) + W_N(s))ds + \frac{C_2}{N}.$$

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Adding both inequalities from previous Lemmas:

$$W_N(t) + V_N(t) \leq C \int_0^T (V_N(s) + W_N(s))ds + \frac{C}{N}.$$

Suppose $\rho = TC < 1$.
Set

$$W_N + V_N = \max_{0 \leq t \leq T} W_N(t) + V_N(t),$$

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$$W_N + V_N \leq \rho(W_N + V_N) + \frac{C}{N},$$

which proves the Theorem

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Outline

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- Mean field games original papers:
  - M. Huang, P. E. Caines, and R. P. Malhamé
  - J-M. Lasry and P.L. Lions.
- Mean field games with finite state space
  - D. Gomes, J. Mohr, and R. R. Souza (GMS)
  - O. Guéant
- Other mean-field problems with finite state
  - H. Tembine, J. Y. Le Boudec, R. El-Azouzi, and E. Altman.
- Convergence to equilibrium
  - P. Cardaliaguet, J-M. Lasry, P.-L. Lions, and A. Porretta (see also GMS)
- Convergence as $N \to \infty$
  - V. N. Kolokoltsov, J. Li, and W. Yang (see also GMS)
- Variational mean-field games

- D. Gomes, and H. Sanchez-Morgado
- D. Gomes, G. E. Pires, and H. Sanchez-Morgado

- Numerics
  - Y. Achdou and I. C. Dolcetta
  - J. Salomon, A. Lachapelle and G. Turinici
- Surveys
  - J.-M. Lasry, P.-L. Lions, and O. Guéant
  - P. Cardaliaguet