Random Mean Field Approximations in LQG Games with Mixed Players

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From Mean Field Control to Weak KAM Dynamics
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Research on Mean Field Decision Problems
- Related literature
- A look at the mean field LQG game

Mean Field LQG Games with a Major Player: Finite Classes of Minor Players
- A matter of "sufficient statistics"
- State space augmentation method

Continuum Parametrized Minor Players: Non-responsive Case
- The Gaussian mean field approximation
- Decentralized strategies

Major-Minor Players: Responsive Mean Field
- The anticipative variational calculations
- The limiting LQG control problems

Related literature

Mean field game models with peers (i.e. comparably small players):

- J.M. Lasry and P.L. Lions (2006a,b, JJM'07): Mean field equilibrium; O. Gueant (JMPA'09)
- G.Y. Weintraub et. el. (NIPS'05, Econometrica'08): Oblivious equilibria for Markov perfect industry dynamics; S. Adlakha, R. Johari, G. Weibtraub, A. Goldsmith (CDC'08): further generalizations with OEs
- M. Huang, P.E. Caines and R.P. Malhame (CDC'03, 04, CIS'06, IEEE TAC'07, SICON'10): Decentralized ε-Nash equilibrium in mean field dynamic games; HCM (Allerton'09, Preprint'11), Asymptotic social optima; M. Nourian, P.E. Caines, et. al. (Preprint'11): mean field consensus model
- T. Li and J.-F. Zhang (IEEE TAC'08): Mean field LQG games with long run average cost; M. Bardi (preprint'11) LQG

Related literature (ctn)

- H. Yin, P.G. Mehta, S.P. Meyn, U.V. Shanbhag (IEEE TAC'12): Nonlinear oscillator games and phase transition; Yang et. al. (ACC'11); Pequito, Aguiar, Sinopoli, Gomes (NetGCOOP'11): application to filtering/estimation
- H. Tembine et. al. (GameNets'09): Mean field MDP and team; H. Tembine, Q. Zhu, T. Basar (IFAC'11): Risk sensitive mean field games
- D. Gomes, J. Mohr, Q. Souza (JMPA'10): Finite state space models
- V. Kolokoltsov, W. Yang, J. Li (preprint'11): Nonlinear markov processes and mean field games
- Z. Ma, D. Callaway, I. Hiskens (IEEE CST'12): recharging control of large populations of electric vehicles
- Y. Achdou and I. Capuzzo-Dolcetta (SIAM Numer.'11): Numerical solutions to mean field game equations (coupled PDEs)
- R. Buckdahn, P. Cardaliaguet, M. Quincampoix (DGA'11): Survey

Related literature (ctn)

Mean field optimal control:

- D. Andersson and B. Djehiche (AMO'11): Stochastic maximum principle
- J. Yong (Preprint'11): control of mean field Volterra integral equations

Remark: This introduces a different conceptual framework; a single controller strongly affecting the mean; In contrast, a given agent within a mean field game has little impact on the mean field

Mean field models with a major player:

- H. (SICON'10): LQG models with minor players parameterized by a finite parameter set; develop state augmentation
- S. Nguyen and H. (CDC'11, preprint'12): LQG models with continuum parametrization, Gaussian mean field approximation; anticipative variational calculations
- M. Nourian and P.E. Caines (preprint'12): Nonlinear models; B.-C. Wang and J.-F. Zhang (preprint'11): Markov jump models

The mean field LQG game

- Individual dynamics (CDC'03, 04):

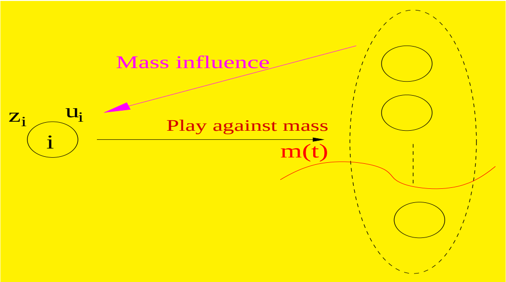
$$dz_i = (a_i z_i + bu_i)dt + \alpha z^{(N)} dt + \sigma_i dw_i, \quad 1 \le i \le N.$$

- Individual costs:

$$J_i = E \int_0^\infty e^{-\rho t}[(z_i - \Phi(z^{(N)}))^2 + ru_i^2]dt.$$

  - z_i: state of agent i; u_i: control; w_i: noise
    a_i: dynamic parameter; $r > 0$; N: population size
    For simplicity: Take the same control gain b for all agents.
  - $z^{(N)} = (1/N)\sum_{i=1}^N z_i$, Φ: nonlinear function
  - Feature: all agents are comparably small

The Nash certainty equivalence (NCE) methodology



Consistent mean field approximation –

- In the infinite population limit, individual strategies are optimal responses to the mean field $m(t)$;
- Closed-loop behaviour of all agents further replicates the same $m(t)$

The new modeling

Remarks:

- Game theory has a long history of modeling players with vastly different strengths
- The past research is mostly in the setting of cooperative games (static models, coalition, Shapley values, etc.) (Hart'73, Galil'74, ...)
- The agents are called mixed players
- We introduce dynamic modeling (interesting informational issues will appear) (Can model a general exogenous process affect everyone, treated as a passive player)

Dynamics with a major player

The LQG game with mean field coupling:

$$dx_0(t) = \left[A_0 x_0(t) + B_0 u_0(t) + F_0 x^{(N)}(t)\right] dt + D_0 dW_0(t), \quad t \geq 0,$$
$$dx_i(t) = \left[A(\theta_i) x_i(t) + B u_i(t) + F x^{(N)}(t) + G x_0(t)\right] dt + D dW_i(t),$$

$x^{(N)} = \frac{1}{N}\sum_{i=1}^N x_i$ mean field term (average state of minor players).

- Major player $\mathcal{A}_0$ with state $x_0(t)$, minor player $\mathcal{A}_i$ with state $x_i(t)$.
- W_0, W_i are independent standard Brownian motions, $1 \leq i \leq N$.
- All constant matrices have compatible dimensions.
- Underlying filtered probability space: $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.

We introduce the following assumption:
(**A1**) θ_i takes its value from a finite set $\Theta = \{1, \ldots, K\}$ with an empirical distribution $F^{(N)}$, which converges when $N \to \infty$.

Individual costs

The cost for $\mathcal{A}_0$:

$$J_0(u_0, ..., u_N) = E \int_0^\infty e^{-\rho t} \Big\{ \big| x_0 - \Phi(x^{(N)}) \big|_{Q_0}^2 + u_0^T R_0 u_0 \Big\} dt,$$

$\Phi(x^{(N)}) = H_0 x^{(N)} + \eta_0$: cost coupling term; $z^T M z = |z|_M^2$ for $M \geq 0$

The cost for $\mathcal{A}_i$, $1 \leq i \leq N$:

$$J_i(u_0, ..., u_N) = E \int_0^\infty e^{-\rho t} \Big\{ \big| x_i - \Psi(x_0, x^{(N)}) \big|_Q^2 + u_i^T R u_i \Big\} dt,$$

$\Psi(x_0, x^{(N)}) = H x_0 + \hat{H} x^{(N)} + \eta$: cost coupling term.

- The presence of x_0 in the dynamics and cost of $\mathcal{A}_i$ shows the strong influence of the major player $\mathcal{A}_0$.
- All deterministic constant matrices or vectors H_0, H, $\hat{H}$, $Q_0 \geq 0$, $Q \geq 0$, $R_0 > 0$, $R > 0$, η_0 and η have compatible dimensions.

A matter of “sufficient statistics”

One might intuitively conjecture asymptotic Nash equilibrium strategies of the form:

- $u_0(t)$ for the major player: A function of $(t, x_0(t))$.
- In other words:

  $x_0(t)$ *would be a sufficient statistic for* $\mathcal{A}_0$*'s decision;*
  $(x_0(t), x_i(t))$ *would be sufficient statistics for* $\mathcal{A}_i$*'s decision.*

Facts:

- The above conjecture fails!

Build the sufficient statistic

State space augmentation method

Approximate $x^{(N)} = \frac{1}{N}\sum_{i=1}^{N} x_i$ by a process $\bar{z}(t)$. The mean field process is assumed to be governed by ($\bar{x}_0$ is the infinite population version of x_0)

$$d\bar{z}(t) = \overline{A}\bar{z}(t)dt + \overline{G}\bar{x}_0(t)dt + \overline{m}(t)dt,$$

where $\bar{z}(0) = 0$, $\overline{A} \in \mathbb{R}^{nK\times nK}$ and $\overline{G} \in \mathbb{R}^{nK\times n}$ are constant matrices, and $\overline{m}(t)$ is a continuous $\mathbb{R}^{nK}$ function on $[0, \infty)$.

- But so far, none of $\overline{A}$, $\overline{G}$ and $\overline{m}(t)$ is known a priori.
  - The difficulty is overcome by consistent mean field approximations (here do parameter matching)
  - Each agent solves a local limiting control problem; in the end their closed-loop replicates $\overline{A}$, $\overline{G}$ and $\overline{m}(t)$

Asymptotic Nash equilibrium

Algebraic conditions may be obtained for solubility to the above procedure.

The control strategies of $\mathcal{A}_0$ and $\mathcal{A}_i$, $1 \le i \le N$:

$$\hat{u}_0 = -R_0^{-1}\mathbb{B}_0^T[P_0(x_0^T, \bar{z}^T)^T + s_0],$$
$$\hat{u}_i = -R^{-1}\mathbb{B}^T\left[P(x_i^T, x_0^T, \bar{z}^T)^T + s\right], \quad 1 \le i \le N,$$

where (the stochastic ODE of $\bar{z}$ will be driven by x_0)

- $\mathbb{B}_0$, $\mathbb{B}$: determined from coefficients in the original SDE.
- P_0, P: obtained from coupled Algebraic Riccati Equations (ARE).
- s_0, s: obtained from a set of linear ODE.

Theorem Under some technical conditions, the set of decentralized strategies is an ε-Nash equilibrium as $N \to \infty$.

A numerical example

- The dynamics of the major/minor players are given by

$$dx_0 = 2x_0 dt + u_0 dt + 0.2x^{(N)} dt + dW_0,$$
$$dx_i = 3x_i dt + x_0 dt + u_i dt + 0.3x^{(N)} dt + dW_i, \quad 1 \le i \le N,$$

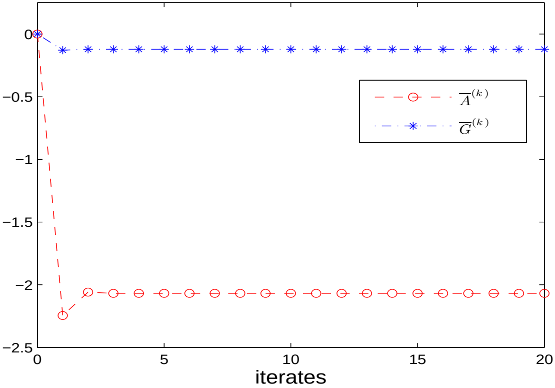
- The parameters in the costs are: discount factor $\rho = 1$,

$$[Q_0, R_0, H_0, \eta_0] = [1, 1, 0.3, 1.5], \qquad [Q, R, H, \hat{H}, \eta] = [1, 3, 0.4, 0.3, 1].$$

- The dynamics for the mean field: $dz = \overline{A}\bar{z}dt + \overline{G}x_0 dt + \overline{m}dt$
- By an iteration algorithm (for NCE approach), we obtain

$$\overline{A} = -2.06819117030469, \quad \overline{G} = -0.12205345839681.$$

Iterations



The dynamics for the mean field: $d\bar{z} = \overline{A}\bar{z}dt + \overline{G}x_0dt + \overline{m}dt$

- Further modeling of heterogeity of minor players

Continuum parametrized minor players

Dynamics and costs:

$$dx_0(t) = \left[A_0 x_0(t) + B_0 u_0(t) + F_0 x^{(N)}(t)\right] dt + D_0 dW_0(t),$$
$$dx_i(t) = \left[A(\theta_i) x_i(t) + B(\theta_i) u_i(t) + F(\theta_i) x^{(N)}(t)\right] dt + D(\theta_i) dW_i(t),$$

We consider the case where x_0 <u>does not</u> appear in the equation of x_i.

$$J_0(u_0, u_{-0}) = E \int_0^T \left[\left|x_0(t) - \chi_0(x^{(N)}(t))\right|^2_{Q_0} + u_0^T(t) R_0 u_0(t)\right] dt,$$

where $\chi_0(x^{(N)}(t)) = H_0 x^{(N)}(t) + \eta_0$, $Q_0 \geq 0$ and $R_0 > 0$.

$$J_i(u_i, u_{-i}) = E \int_0^T \left[\left|x_i(t) - \chi(x_0(t), x^{(N)}(t))\right|^2_{Q} + u_i^T(t) R u_i(t)\right] dt,$$

where $\chi(x_0(t), x^{(N)}(t)) = H x_0(t) + \hat{H} x^{(N)}(t) + \eta$, $Q \geq 0$ and $R > 0$.

Assumptions (with θ being from a continuum set)

(**A1**) The initial states $\{x_j(0), 0 \le j \le N\}$, are independent, and there exists a constant C independent of N such that $\sup_{0\le j\le N} E|x_j(0)|^2 \le C$.
(**A2**) There exists a distribution function $\mathbf{F}(\theta, x)$ on $\mathbb{R}^{d+n}$ such that the sequence of empirical distribution functions $\mathbf{F}_N(\theta, x) = \frac{1}{N}\sum_{i=1}^N 1_{\{\theta_i\le\theta, Ex_i(0)\le x\}}$, $N \ge 1$, where each inequality holds componentwise, converges to $\mathbf{F}(\theta, x)$ weakly, i.e., for any bounded and continuous function $h(\theta, x)$ on $\mathbb{R}^{d+n}$,

$$\lim_{N\to\infty}\int_{\mathbb{R}^{d+n}} h(\theta, x)d\mathbf{F}_N(\theta, x) = \int_{\mathbb{R}^{d+n}} h(\theta, x)d\mathbf{F}(\theta, x).$$

(**A3**) $A(\cdot), B(\cdot), F(\cdot)$ and $D(\cdot)$ are continuous matrix-valued functions of $\theta \in \Theta$, where Θ is a compact subset of $\mathbb{R}^d$.

The Gaussian mean field approximation

- The previous finite dimensional sub-mean field approximations are not applicable
- We consider Gaussian mean field approximations and use a kernel representation:

$$(x^{(N)} \approx) z(t) = f_1(t) + f_2(t) x_0(0) + \int_0^t g(t,s) dW_0(s),$$

  where f_1, f_2, g are continuous vector/matrix functions of t
- Individual agents solve limiting control problems with random coefficient processes by a linear BSDE approach (Bismut'73, 76)
- Consistency condition is imposed for mean field approximations
- The resulting decentralized strategies are not Markovian
- ε-Nash equilibrium can be established (Nguyen and H., CDC'11)

Problem (I) – optimal control of the major player

We approximate $x^{(N)} = \frac{1}{N}\sum_{i=1}^{N} x_i$ by a process $z(t)$. The mean field process is assumed to be governed by

$$z(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s)$$

where $f_1 \in C([0,T],\mathbb{R}^n)$, $f_2 \in C([0,T],\mathbb{R}^{n\times n})$, $g \in C(\Delta,\mathbb{R}^{n\times n_2})$ will be determined where $\Delta = \{(t,s) : 0 \le s \le t \le T\}$.

Approximate the dynamics and the cost by:

$$dx_0(t) = [A_0x_0(t) + B_0u_0(t) + F_0\mathbf{z(t)}]dt + D_0dW_0(t),$$

$$\bar{J}_0(u_0) = E\int_0^T \left[|x_0 - H_0\mathbf{z} - \eta_0|^2_{Q_0} + u_0^T R_0 u_0\right] dt.$$

z replaces $x^{(N)}$ in the finite population model.

The optimal control law

Following Bismut (SIAM, 1976), we can solve this optimal control problem.
Let the pair $(\bar{x}_0, \bar{u}_0)$ be the optimal solution to Problem (I). Then

$$\bar{u}_0(t) = R_0^{-1} B_0^T(-P_0(t)\bar{x}_0(t) + \nu_0(t)),$$

where $P_0(t) \geq 0$ is the unique solution of the Riccati equation

$$\begin{cases} \dot{P}_0 + P_0 A_0 + A_0^T P_0 - P_0 B_0 R_0^{-1} B_0^T P_0 + Q_0 = 0, \\ P(T) = 0, \end{cases}$$

and ν_0 is the unique solution to a forward-backward SDE.

An explicit solution of the optimal state process

We intend to find a representation of $\bar{x}_0$ (closed-loop state) in the form:

$$\bar{x}_0(t) = f_{\bar{x}_0,1}(t) + f_{\bar{x}_0,2}(t)x_0(0) + \int_0^t g_{\bar{x}_0}(t,s)dW_0(s),$$

$f_{\bar{x}_0,1} \in C([0,T],\mathbb{R}^n)$, $f_{\bar{x}_0,2} \in C([0,T],\mathbb{R}^{n\times n})$, $g_{\bar{x}_0} \in C(\Delta,\mathbb{R}^{n\times n_2})$ are to be determined.

$$
\begin{aligned}
& f_{\bar{x}_0,1}(t) \\
& = \int_0^t \int_{s_1}^T \Phi_0(t,s_1) B_0 R_0^{-1} B_0^T \Phi_0^T(s_2,s_1) \Big\{ \big[Q_0 H_0 - P_0(s_2) F_0 \big] \mathbf{f_1(s_2)} \\
& \qquad + Q_0 \eta_0 \Big\} ds_2 ds_1 + \int_0^t \Phi_0(t,s_1) F_0 \mathbf{f_1(s_1)} ds_1 \\
& =: [\Gamma_{0,1} \mathbf{f_1}](t),
\end{aligned}
$$

where $\Phi_0(t,s)$ is the unique solution to an ODE system.

Similarly,

$$
f_{\bar{x}_0,2}(t) = [\Gamma_{0,2} f_2](t), \quad g_{\bar{x}_0}(t,s) = [\Lambda_0 g](t,s).
$$

Problem (II) – optimal control of the minor player.

After solving Problem (I), we may represent the state x_0 by $x_0(0)$ and W_0, and denote the state process by $\bar{x}_0$. We introduce the equation system

$$\begin{cases} z(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s), \\ \bar{x}_0(t) = f_{\bar{x}_0,1}(t) + f_{\bar{x}_0,2}(t)x_0(0) + \int_0^t g_{\bar{x}_0}(t,s)dW_0(s), \\ dx_i(t) = \big[A(\theta_i)x_i(t) + B(\theta_i)u_i(t) + F(\theta_i)\mathbf{z(t)}\big]dt + D(\theta_i)dW_i(t), \end{cases}$$

$(f_{\bar{x}_0,1}, f_{\bar{x}_0,2}, g_{\bar{x}_0})$ is determined from the solution of Problem (I).

The cost function is given by

$$\bar{J}_i(u_i) = E\int_0^T \big[\big|x_i - H\bar{\mathbf{x}}_\mathbf{0} - \hat{H}\mathbf{z} - \eta\big|_Q^2 + u_i^T R u_i\big]dt.$$

The optimal control law

The optimal control law is given in the form

$$\bar{u}_i(t) = R^{-1}B^T(\theta_i)(-P_{\theta_i}(t)\bar{x}_i(t) + \nu_{\theta_i}(t)),$$

where $P_{\theta_i}(t) \geq 0$ is the unique solution of the Riccati equation

$$\begin{cases} \dot{P}_{\theta_i} + P_{\theta_i}A(\theta_i) + A^T(\theta_i)P_{\theta_i} - P_{\theta_i}B(\theta_i)R_0^{-1}B^T(\theta_i)P_{\theta_i} + Q = 0, \\ P_{\theta_i}(T) = 0. \end{cases}$$

and ν_{θ_i} is the unique solution to a forward-backward SDE.

An explicit solution of the optimal state process

We can represent $\bar{x}_i(t)$ in the form

$$\begin{aligned}\bar{x}_i(t) =& f_{\bar{x}_i,1}(t) + f_{\bar{x}_i,2}(t)x_0(0) + f_{\bar{x}_i,3}(t)x_i(0)\\ &+ \int_0^t g_{\bar{x}_i}(t,s)dW_0(s) + \int_0^t h_{\bar{x}_i}(t,s)dW_i(s),\end{aligned}$$

where $f_{\bar{x}_i,3}(t) = \Phi_{\theta_i}(t,0), \quad h_{\bar{x}_i}(t,s) = \Phi_{\theta_i}(t,s)D(\theta_i),$

$$f_{\bar{x}_i,1}(t) = [\Gamma_{\theta_i,1}f_1](t), \quad f_{\bar{x}_i,2}(t) = [\Gamma_{\theta_i,2}f_2](t), \quad g_{\bar{x}_i}(t,s) = [\Lambda_{\theta_i}g](t,s).$$

(Integral equations)

- $\Phi_{\theta_i}(t,s)$ is a solution to an ODE system.
- $f_{\bar{x}_i,1} \in C([0,T],\mathbb{R}^n)$, $f_{\bar{x}_i,2}$, $f_{\bar{x}_i,3} \in C([0,T],\mathbb{R}^{n\times n})$, and $g_{\bar{x}_i}, h_{\bar{x}_i} \in C(\Delta,\mathbb{R}^{n\times n_2})$.

The NCE equation system

For $f_1 \in C([0,T], \mathbb{R}^n)$, $f_2 \in C([0,T], \mathbb{R}^{n\times n})$, $g \in C(\Delta, \mathbb{R}^{n\times n_2})$, and $0 \le s \le t \le T$, denote

$$[\Gamma_1 f_1](t) = \int_\Theta [\Gamma_{\theta,1} f_1](t) d\mathbf{F}(\theta) + \int_{\Theta\times\mathbb{R}^n} \Phi_\theta(t,0) x d\mathbf{F}(\theta,x),$$

$$[\Gamma_2 f_2](t) = \int_\Theta [\Gamma_{\theta,2} f_2](t) d\mathbf{F}(\theta), [\Lambda g](t,s) = \int_\Theta [\Lambda_\theta g](t,s) d\mathbf{F}(\theta).$$

A triple (f_1, f_2, g) is called a consistent solution to the Nash certainty equivalence (NCE) equation system if

$$\begin{cases} f_j(t) = [\Gamma_j f_j](t), \quad 0 \le t \le T, \ j = 1,2, \\ g(t,s) = [\Lambda g](t,s), \quad 0 \le s \le t \le T. \end{cases}$$

Theorem (Existence and Uniqueness) *Under mild assumptions, the NCE equation system has a unique solution* (f_1, f_2, g).

Decentralized strategies

Assume that there exists a solution (f_1, f_2, g) to the NCE system.

Let the control laws be given by

$$\hat{u}_0(t) = R_0^{-1} B_0^T \big[- P_0(t) x_0(t) + \nu_0(t) \big],$$
$$\hat{u}_i(t) = R^{-1} B^T(\theta_i) \big[- P_{\theta_i}(t) x_i(t) + \nu_{\theta_i}(t) \big].$$

Each player can use the information from its own states and the major player's Brownian motion.

Limiting equation system for N+1 players

After these control laws are applied,

$$dx_0 = \left[\mathbb{A}_0 x_0 + B_0 R_0^{-1} B_0^T P_0 \nu_0 + F_0 x^{(N)}\right] dt + D_0 dW_0,$$

$$dx_i = \left[\mathbb{A}_{\theta_i} x_i + B(\theta_i) R^{-1} B^T(\theta_i) P_{\theta_i} \nu_{\theta_i} + F(\theta_i) x^{(N)}\right] dt + D(\theta_i) dW_i,$$

where $x^{(N)} = \frac{1}{N}\sum_{i=1}^{N} x_i$, and

$$\mathbb{A}_0(t) = A_0 - B_0 R_0^{-1} B_0^T P_0(t),$$

$$\mathbb{A}_{\theta_i}(t) = A(\theta_i) - B(\theta_i) R^{-1} B^T(\theta_i) P_{\theta_i}(t).$$

Theorem *Assume* (**A1**)-(**A3**). *We have*

$$E\int_0^T \left|z(t) - x^{(N)}(t)\right|^2 dt = O(\epsilon_N^2),$$

where

$$z(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s)$$

and $\epsilon_N \to 0$ *as* $N \to \infty$.

For $0 \le j \le N$, denote $\hat{u}_{-j} = (..., \hat{u}_{j-1}, \hat{u}_{j+1}, ...)$.

Theorem (ε**-Nash Equilibrium Property)** *Assume* (**A1**)-(**A3**). *Then the set of controls* $\hat{u}_j$, $0 \le j \le N$, *for the* $N+1$ *players is an* ϵ*-Nash equilibrium, i.e., for* $0 \le j \le N$,

$$J_j(\hat{u}_j, \hat{u}_{-j}) - \epsilon \le \inf_{u_j} J_j(u_j, \hat{u}_{-j}) \le J_j(\hat{u}_j, \hat{u}_{-j}),$$

where $0 < \epsilon = O(\epsilon_N)$.

An example

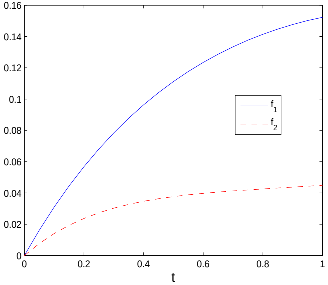
The dynamics and costs:

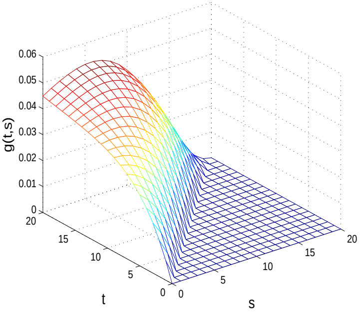
$$dx_0 = [a_0 x_0(t) + b_0 u_0(t)]dt + D_0 dw_0(t), \quad t \geq 0,$$
$$dx_i = [a_i x_i(t) + bu_i(t)]dt + Ddw_i(t), \quad 1 \leq i \leq N,$$

$$J_0(u_0, u_{-0}) = E\int_0^T \{q_0(x_0(t) - h_0 x^{(N)}(t) - \eta_0)^2 + u_0^2(t)\}dt,$$
$$J_i(u_i, u_{-i}) = E\int_0^T \{q(x_i(t) - hx_0(t) - \hat{h}x^{(N)}(t) - \eta)^2 + u_i^2(t)\}dt.$$

$[a_0, b_0, D_0, q_0, h_0, \eta_0] = [0.5, 1, 1, 1, 0.6, 1.5]$, $[\underline{a}, \bar{a}, b, D, q, h, \hat{h}, \eta] = [0.1, 0.4, 1, 1, 1.2, 0.5, 0.4, 0.5]$. The empirical distribution of $\{a_i, i \geq 1\}$ converges to uniform distribution on $[\underline{a}, \bar{a}]$.

Numerical solution





$$z(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s).$$

Dynamics (responsive mean field)

The LQG game with mean field coupling:

$$dx_0(t) = \left[A_0 x_0(t) + B_0 u_0(t) + F_0 x^{(N)}(t)\right] dt + D_0 dW_0(t), \quad t \geq 0,$$

$$dx_i(t) = \left[A x_i(t) + B u_i(t) + F x^{(N)}(t) + G x_0(t)\right] dt + D dW_i(t),$$

$x^{(N)} = \frac{1}{N}\sum_{i=1}^{N} x_i$ mean field term

- The state of the major player $x_0(t)$ appears in the dynamics of each minor player.
- The matrices A, B, F, G and D do not depend on parameters.

Individual costs

The cost for $\mathcal{A}_0$:

$$J_0(u) = E\int_0^T \Big\{ \big|x_0 - \Psi_0(x^{(N)})\big|^2_{Q_0} + u_0^T R_0 u_0 \Big\} dt.$$

The cost for $\mathcal{A}_i$, $1 \leq i \leq N$:

$$J_i(u) = E\int_0^T \Big\{ \big|x_i - \Psi(x_0, x^{(N)})\big|^2_{Q} + u_i^T R u_i \Big\} dt,$$

where

- $\Psi_0(x^{(N)}) = H_0 x^{(N)} + \eta_0$: cost coupling term,
- $\Psi(x_0, x^{(N)}) = H x_0 + \hat{H} x^{(N)} + \eta$: cost coupling term,
- $|z|^2_M = z^T M z$ for a positive semi-definite matrix M.

We introduce the following assumption:

(**A**) The initial states $x_j(0)$, $0 \leq j \leq N$, are independent; $\sup_{0\leq j\leq N} E|x_j(0)|^2 \leq C$ for some C independent of N.

Features of the problem:

- Finite horizon cost
- Mean field responsive to the major player
- Look for decentralized strategies
- Use Gaussian mean field approximation (i.e., conditioned on the initial states of the major player, the mean field is a Gaussian process).
- State space augmentation may be used for uniform minor players; but the approach to be introduced has its advantage

The method of anticipative variational calculations

- When u_0 changes to $u_0 + \delta u_0$, it causes a state variation δx_0 for the major player.
- This change generates a state variation δx_i for the minor player i.
- Subsequently, a large number of minor players contribute to a variation $\delta x^{(N)}$ for the mean field $x^{(N)}$.

Let $N \to \infty$, approximate $x^{(N)}$ by z. Equations for the variations:

$$\begin{aligned} d\delta x_0 &= (A_0 \delta x_0 + B_0 \delta u_0 + F_0 \delta z) dt, \\ d\delta z &= \big((A - BR^{-1}B^T P + F)\delta z + G\delta x_0\big) dt, \\ \delta x_0(0) &= \delta z(0) = 0, \end{aligned}$$

and $P(t)$ is the solution of the Riccati equation

$$\dot{P} + PA + A^T P - PBR_0^{-1} B^T P + Q = 0, \quad P(T) = 0.$$

Definition

Let $z^* \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ be given. We say $(\bar{x}_0, \bar{u}_0)$ is an equilibrium solution with respect to z^* if $(\bar{x}_0, \bar{u}_0, z^*)$ satisfies

$$d\bar{x}_0(t) = \big(A_0\bar{x}_0(t) + B_0\bar{u}_0(t) + F_0 z^*(t)\big)dt + D_0 dW_0(t),$$
$$\bar{J}_0(\bar{x}_0, \bar{u}_0; z^*) \le \bar{J}_0(\bar{x}_0 + \delta x_0, \bar{u}_0 + \delta u_0; z^* + \delta z)$$

for all $\delta u_0 \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})$, where

$$\begin{aligned} d\delta x_0 &= (A_0\delta x_0 + B_0\delta u_0 + F_0\delta z)dt, \\ d\delta z &= \big((A - BR^{-1}B^T P + F)\delta z + G\delta x_0\big)dt, \\ \delta x_0(0) &= \delta z(0) = 0, \end{aligned}$$

$$\bar{J}_0(x_0, u_0; z) = E\int_0^T \left\{ \left|x_0 - \Psi_0(z)\right|^2_{Q_0} + u_0^T R_0 u_0 \right\} dt.$$

Control problem 1 (P1)

We introduce a process of the form

$$\bar{z}(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s),$$

where $f_1 \in C([0,T], \mathbb{R}^n)$,$f_2 \in C([0,T], \mathbb{R}^{n\times n})$, $g \in C(\Delta, \mathbb{R}^{n\times n_2})$.

Find an equilibrium solution $(\bar{x}_0, \bar{u}_0)$ with respective to $\bar{z}$.

- The dynamics of the mean field are specified by $(\bar{z}(t), \delta z)$. (Nonstandard optimal control)
- Use BSDE and Riccati technique to solve the equilibrium control

Control problem 2 (P2)

Minimize the cost

$$\bar{J}_i = E\int_0^T \left\{ \left| x_i - \Psi(x_i, \bar{z}) \right|_Q^2 + u_i^T R u_i \right\} dt$$

subject to the system dynamics

$$dx_i = \left(Ax_i + Bu_i + F\bar{z} + G\bar{x}_0 \right) dt + DdW_i,$$
$$d\bar{x}_0 = (A_0\bar{x}_0 + B_0\bar{u}_0 + F_0\bar{z})dt + D_0 dW_0,$$
$$\bar{z}(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s)$$

where $(\bar{x}_0, \bar{u}_0)$ is determined from (P1) as the equilibrium solution with respect to $\bar{z}$.

An explicit solution of the optimal state process

By using FBSDE approach, under suitable conditions, we can represent $\bar{x}_0$ in the form:

$$\bar{x}_0(t) = f_{\bar{x}_0,1}(t) + f_{\bar{x}_0,2}(t)x_0(0) + \int_0^t g_{\bar{x}_0}(t,s)dW_0(s),$$

$f_{\bar{x}_0,1} \in C([0,T],\mathbb{R}^n)$, $\quad f_{\bar{x}_0,2} \in C([0,T],\mathbb{R}^{n\times n})$ and $g_{\bar{x}_0} \in C(\Delta,\mathbb{R}^{n\times n_2})$ are determined by the following integral equations

$$f_{\bar{x}_0,1}(t) = [\Gamma_{0,1}\mathbf{f_1}](t), \quad f_{\bar{x}_0,2}(t) = [\Gamma_{0,2}\mathbf{f_2}](t), \quad g_{\bar{x}_0}(t,s) = [\Lambda_0\mathbf{g}](t,s).$$

Similarly, we can represent $\bar{x}_i$ in the form:

$$\bar{x}_i(t) = f_{\bar{x}_i,1}(t) + f_{\bar{x}_i,2}(t)x_0(0) + f_{\bar{x}_i,3}(t)x_i(0) + \int_0^t g_{\bar{x}_0}(t,s)dW_0(s) + \int_0^t h_{\bar{x}_0}(t,s)dW_i(s),$$

$f_{\bar{x}_0,1} \in C([0,T],\mathbb{R}^n)$, $f_{\bar{x}_0,2}, f_{\bar{x}_0,3} \in C([0,T],\mathbb{R}^{n\times n})$ and $g_{\bar{x}_0}, h_{\bar{x}_0} \in C(\Delta,\mathbb{R}^{n\times n_2})$ are determined by the following integral equations

$$f_{\bar{x}_0,1}(t) = [\Gamma_1 \mathbf{f_1}](t), \quad f_{\bar{x}_0,2}(t) = [\Gamma_2 \mathbf{f_2}](t), \quad g_{\bar{x}_0}(t,s) = [\Lambda \mathbf{g}](t,s).$$

and

$$f_{\bar{x}_i,3}(t) = \Phi(t,0), \quad h_{\bar{x}_i}(t,s) = \Phi(t,s)D,$$

where $\Phi(t,s)$ is the solution to an ODE.

Close the loop

- Let the closed loop of all minor players replicate the mean field
- Obtain the fixed point operator