On optimal control of stochastic differential equations associated with Lévy generators

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Based on joint works with Jonathan Bennett

[1] Explicit construction of SDEs associated with polar-decomposed Lévy measures and application to stochastic optimization, Frontiers of Mathematics in China **2** (2007), 539–558.

[2] An optimal control problem associated with SDEs driven by Lévy-type processes, Stochastic Analysis and Applications, **26** (2008), 471–494.

[3] Stochastic control of SDEs associated with Lévy generators and application to financial optimization, Frontiers of Mathematics in China **5** (2010), 89–102.

and [4] Harry Zheng and JLW, On an optimal portfolio-comsumption problem associated with Lévy-type generators, in preparation.

A fairly large class of Markov processes on $\mathbb{R}^d$ are governed by Lévy generator, either via martingale problem (cf e.g. D W Stroock, "Markov Processes from K. Itô's Perspectives", Princeton Univ Press 2003 or V.N. Kolokoltsov, "Markov Processes, Semigroups and Generators", de Gruyter, 2011) or via Dirichlet form (cf e.g. N Jacob,"Pseudo-Differential Operators and Markov Processes III" Imperial College Press, 2005)

$$\begin{aligned} Lf(t,x) &:= \frac{1}{2}a^{i,j}(t,x)\partial_i\partial_j f(t,x) + b^i(t,x)\partial_i f(t,x) \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}} \{f(t,x+z) - f(t,x) \\ &\qquad - \frac{z1_{\{|z|<1\}}\cdot\nabla f(t,x)}{1+|z|^2}\}\nu(t,x,dz) \end{aligned}$$

where $a(t,x) = (a^{i,j}(t,x))_{d\times d}$ is non-negative definite symmetric and $\nu(t,x,dz)$ is a Lévy kernel, i.e.,

$\forall (t,x) \in [0,\infty) \times \mathbb{R}^d$, $\nu(t,x,\cdot)$ is a σ-finite measure on $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ such that

$$\int_{\mathbb{R}^d \setminus \{0\}} \frac{|z|^2}{1+|z|^2} \nu(t,x,dz) < \infty.$$

For such L, in order to get rid of variable dependence on ν, N El Karoui and J P Lepeltier (Z. Wahr. verw. Geb. 39 (1977)) construct a bimeasurable bijection

$$c : [0,\infty) \times \mathbb{R}^d \times U \to \mathbb{R}^d \setminus \{0\}$$

such that

$$\int_U 1_A(c(t,x,y))\lambda(dy) = \int_{\mathbb{R}^d \setminus \{0\}} 1_A(z)\nu(t,x,dz), \quad \forall (t,x)$$

for $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. Where $(U, \mathcal{B}(U))$ is a Lusin space and λ is a σ-finite measure on it. Actually, we can construct c explicitly in case ν has a polar decomposition (with the stable-like case as a concrete example). It is well-known (cf e.g. Theorem I.8.1 in N Ikeda and S Watanabe's book): $\exists$ a Poisson random measure

$$N : \mathcal{B}([0,\infty)) \times \mathcal{B}(U) \times \Omega \to \mathbb{N} \cup \{0\} \cap \{\infty\}$$

on any given probability set-up $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\geq 0})$ with $\mathbf{E}(N(dt, dy, \cdot)) = dt\lambda(dy)$, and

$$\tilde{N}(dt, dy, \omega) := N(dt, dy, \omega) - dt\lambda(dy)$$

being the associated compensating $\{\mathcal{F}_t\}_{t\geq 0}$-martingale measure.

We then can formulate a jump SDE associated with L

$$dS_t = b(t, S_t)dt + \sigma(t, S_t)dW_t + \int_U c(t, S_{t-}, y)\tilde{N}(dt, dy)$$

where $\sigma(t, x)$ is a $d \times m$-matrix such that

$$\sigma(t, x)\sigma^T(t, x) = a(t, x)$$

and $\{W_t\}_{t\in[0,\infty)}$ is an m-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion. We shall consider such equation in the following general formulation

$$\begin{aligned} dS_t &= b(t,S_t)dt + \sigma(t,S_t)dW_t \\ &\qquad + \int_{U\setminus U_0} c_1(t,S_{t-},z)\tilde{N}(dt,dz) \\ &\qquad + \int_{U_0} c_2(t,S_{t-},z)N(dt,dz) \end{aligned}$$

where $U_0 \in \mathcal{B}(U)$ with $\lambda(U_0) < \infty$ is arbitrarily fixed.

Sufficient Maximum Principle

Framstad,Øksendal, Sulem (J Optim Theory Appl 121 (2004))

Øksendal, Sulem ("Applied Stochastic Control of Jump-Diffusions", Springer, 2005); Math Finance 19 (2009); SIAM J Control Optim 2010; Commun Stoch Anal 4 (2010)

Start with a controlled jump Markov process

$$S_t = S_t^{(u)}, \quad t \in [0, T]$$

for any arbitrarily fixed $T \in (0, \infty)$, by the following

$$\begin{aligned} dS_t &= b(t, S_t, u_t)dt + \sigma(t, S_t, u_t)dW_t \\ &\quad + \int_{U \setminus U_0} c_1(t, S_{t-}, u_{t-}, z)\tilde{N}(dt, dz) \\ &\quad + \int_{U_0} c_2(t, S_{t-}, u_{t-}, z)N(dt, dz) \end{aligned} \tag{1}$$

where the control process $u_t = u(t, \omega)$, taking values in a given Borel set $\mathcal{U} \in \mathcal{B}(\mathbb{R}^d)$, is assumed to be $\{\mathcal{F}_t\}$-predictable and cádlág.

The performance criterion is

$$J(u) := \mathbf{E}\left(\int_0^T f(t, S_t, u_t)dt + g(S_T)\right), \quad u \in \mathcal{A}$$

for $\mathcal{A}$ the totality of all admissible controls, and for

$$f : [0, T] \times \mathbb{R}^d \times \mathcal{U} \to \mathbb{R}$$

being continuous, and for $g : \mathbb{R}^d \to \mathbb{R}$ being concave. The objective is to achieve the following

$$J(\hat{u}) = \sup_{u \in \mathcal{A}} J(u)$$

referring $\hat{u}$ being the optimal control of the system. Moreover, if $\hat{S}_t = S_t^{(\hat{u})}$ is the solution to the jump type SDE (1) corresponding to $\hat{u}$, then the pair $(\hat{S}, \hat{u})$ is called *the optimal pair*.

Now the Hamiltonian is defined

$$H : [0, T] \times \mathbb{R}^d \times \mathcal{U} \times \mathbb{R}^d \times \mathbb{R}^{d\otimes m} \times \mathcal{R} \to \mathbb{R}$$

via

$$\begin{aligned} & H(t, r, u, p, q, n^{(1)}, n^{(2)}) \\ = \ & f(t, r, u) + \mu(t, r, u)p + \frac{1}{2}\sigma^T(t, r, u)q \\ & + \int_{U\setminus U_0} n^{(1)}(t, z)c_1(t, r, u, z)\lambda(dz) \\ & + \int_{U_0} [n^{(2)}(t, z)c_2(t, r, u, z) + c_2(t, r, u, z)p]\lambda(dz) \end{aligned}$$

where $\mathcal{R}$ is the collection of all $\mathbb{R}^{d\otimes d}$-valued processes $n : [0, \infty) \times \Omega \to \mathbb{R}^{d\otimes d}$ such that the two integrals in the above formulation converge absolutely.

It is known that the adjoint equation corresponding to an admissible pair (S, u) is the BSDE

$$\begin{aligned} dp(t) &= -\nabla_r H(t, S_t, u_t, p(t), q(t), n^{(1)}(t,\cdot), n^{(2)}(t,\cdot))dt \\ &\quad +q(t)dW_t + \int_{U\setminus U_0} n^{(1)}(t-, z)\tilde{N}(dt, dz) \\ &\quad + \int_{U_0} n^{(2)}(t-, z)N(dt, dz) \end{aligned}$$

with terminal condition

$$p(T) = \triangledown g(S_T)\,.$$

Theorem ([3])

Given an admissible pair $(\hat{S}, \hat{u})$. Suppose $\exists$ an $\{\mathcal{F}_t\}$-adapted solution $(\hat{p}(t), \hat{q}(t), \hat{n}(t,z))$ to the BSDE s.t. for $u \in \mathcal{A}$

$$\mathbf{E}[\int_0^T (\hat{S}_t - S_t^{(u)})^T \{\hat{q}(t)\hat{q}(t)^T + \int_{U_0} [tr(\hat{n}(t,z)\hat{n}(t,z)^T)\lambda(dz)]\} \times (\hat{S}_t - S^{(u)}(t))dt] < \infty\,,$$

$$\mathbf{E}[\int_0^T \hat{p}^T(t)\{\int_{U_0} [tr(c(t,S_{t-},u_t,z)c^T(t,S_{t-},u_t,z))\lambda(dz)] + \sigma(t,S_t,u_t)\sigma^T(t,S_t,u_t)\}\hat{p}(t)dt] < \infty,$$

Theorem (cont'd)

and $\forall t \in [0, T]$

$$H(t, \hat{S}_t, \hat{u}_t, \hat{p}(t), \hat{q}(t), \hat{n}(t, .)) = \sup_{u \in \mathcal{A}} H(t, \hat{S}_t, u_t, \hat{p}(t), \hat{q}(t), \hat{n}(t, .)). \tag{2}$$

If $\hat{H}(r) := max_{u \in \mathcal{A}} H(t, r, u, \hat{p}(t), \hat{q}(t), \hat{n}(t, \cdot))$ exists and is a concave function of r, then $(\hat{S}, \hat{u})$ is an optimal pair.

Remark For (2), it suffices that the function

$$(r, u) \to H(t, r, u, \hat{p}(t), \hat{q}(t), \hat{n}(t, \cdot))$$

is concave, $\forall t \in [0, T]$.

Optimal control problem

Hindy, Huang (Econometrica 61 (1993))
Bank (Ann Appl Prob (2001))
Benth, Karlsen, Reikvam (Finance Stoch 5 (2001); Stochastics Stochastics Rep 74(2002))
Ishikawa (Appl Math Optim 50 (2004))
Jakobsen, Karlsen (JDE 212 (2005); NoDEA 13 (2006))

Start with a Lévy type process

$$Z_t = \mu t + \int_0^t \theta(s) dW_s + \int_0^t \int_{U \setminus U_0} c_1(z) \tilde{N}(ds, dz) + \int_0^t \int_{U_0} c_2(z) N(ds, dz)$$

where μ is a constant, $\theta : [0, T] \to \mathbb{R}$ and $c_1, c_2 : U \to \mathbb{R}$ are measurable. Here assume that

$$\int_{U_0} (e^{c_2(z)} - 1) \lambda(dz) < \infty.$$

We are concerned with the following 1-dimensional linear SDE

$$\begin{aligned} dS_t &= b(t)S_t dt + \frac{1}{2}\sigma(t)^2 S_t dt + \sigma(t)S_t dW_t \\ &+ S_t \int_U (e^{c_1(z)} - 1 - c_1(z)1_{\{U \setminus U_0\}}(z))\lambda(dz)dt \\ &+ S_{t-} \int_U (e^{c_1(z)} - 1)\tilde{N}(dt, dz). \end{aligned}$$

Based on the driving processes Z_t and S_t, we construct two processes X_t and Y_t with $X_0 = x$, $Y_0 = y$, via

$$
\begin{aligned}
X_t = x - G_t + \int_0^t \sigma(s)\pi_s X_s dW_s + L_t \\
+ \int_0^t (r + ([b(s) + \frac{1}{2}\sigma(s)^2 + \int_{U\setminus U_0} (e^{c_1(z)} \\
-1 - c_1(z))\lambda(dz)] - r)\pi_s) X_s ds \\
+ \int_0^t \pi_{s-} X_{s-} \int_{U\setminus U_0} (e^{c_1(z)} - 1)\tilde{N}(ds, dz) \\
+ \int_0^t \pi_{s-} X_{s-} \int_{U_0} (e^{c_2(z)} - 1) N(ds, dz)
\end{aligned}
$$

and

$$
Y_t = ye^{-\beta t} + \beta \int_0^t e^{-\beta(t-s)} dG_s
$$

respectively, where

$$G_t := \int_0^t g_s ds$$

with $(g_t)_{t\geq 0}$ being a nondecreasing $\{\mathcal{F}_t\}$-adapted cádlág process of finite variation such that $0 \leq \sup_{t\geq 0} g_t < \infty$, L_t is a nondecreasing, nonnegative, and $\{\mathcal{F}_t\}$-adapted cádlág process, and $\pi_t \in [0, 1]$ is $\{\mathcal{F}_t\}$-adapted cádlág. The triple (G_t, L_t, π_t) is referred as the parameter process.

Remark The background for X_t being the self-financing investment policy according to the portfolio π_t:

$$\frac{dX_t}{X_{t-}} = (1 - \pi_t)\frac{dB_t}{B_t} + \pi_t \frac{dS_t}{S_{t-}}$$

with B_t standing for the riskless bond $dB_t = rB_t dt$.

By Itô formula, the generator A to (X_t, Y_t) is

$$
\begin{aligned}
Av(x,y) &= -\alpha v - \beta y v_y + \sigma(t)\pi x v_{xx} \\
&\quad +\{(r + \pi([b(t) + \frac{1}{2}\sigma(t)^2 \\
&\quad + \int_{U\setminus U_0} (e^{c_1(z)} - 1 - c_1(z))\lambda(dz)] - r))x v_x \\
&\quad + \int_{U\setminus U_0} (v(x + \pi x(e^{c_1(z)} - 1), y) \\
&\qquad - v(x,y) - \pi x v_x(e^{c_1(z)} - 1))\lambda(dz) \\
&\quad + \int_{U_0} (v(x + \pi x(e^{c_2(z)} - 1), y) - v(x,y))\lambda(dz)\} \\
&\quad + u(g) - g(v_x - \beta v_y)
\end{aligned}
$$

for any $v \in C^{2,2}(\mathbb{R} \times \mathbb{R})$ and for $\pi \in [0,1], g \in [0, M_1]$.

Now we define the value function

$$v(x,y) := \sup_{(\pi_\cdot, g_\cdot, L_\cdot)\in\mathcal{A}} \mathbf{E}^{(X^{(\pi_\cdot, g_\cdot, L_\cdot)}, Y^{(\pi_\cdot, g_\cdot, L_\cdot)})}[\int_0^\infty e^{-\alpha s} u(g_s) ds]$$

where the supremum is taken over all admissible controls and u is a utility function, i.e., u is strictly increasing, differential, and concave on $[0,\infty)$ such that

$$u(0) = u'(\infty) = 0, \quad u(\infty) = u'(0) = \infty.$$

We also denote that

$$
\begin{aligned}
k(\gamma,\rho) \quad := \quad & \max_{\pi}\Big\{\gamma(r+\pi([b(t)+\frac{1}{2}\sigma(t)^2 \\
& +\int_{U\setminus U_0}(e^{c_1(z)}-1-c_1(z))\lambda(dz)]-r)) \\
& +\sigma(t)\pi\rho+\int_{U\setminus U_0}[(1+\pi(e^{c_1(z)}-1))^{\gamma} \\
& \qquad -1-\gamma\pi(e^{c_1(z)}-1)) \\
& +\int_{U_0}(1+\pi(e^{c_2(z)}-1))^{\gamma}-1]\lambda(dz)\Big\} .
\end{aligned}
$$

Theorem ([2])

[i](Existence) v is well-defined, i.e., there exists an optimal control $(\pi^*, g^*, L^*) \in \mathcal{A}$ such that

$$v(x,y) = \mathbf{E}^{(X^{(\pi^*,g^*,L^*)},Y^{(\pi^*,g^*,L^*)})}\left[\int_0^\infty e^{-\alpha s} u(g_s^*) ds\right].$$

Furthermore, v is a constrained viscosity solution to the following Hamilton-Jacobi-Bellman integro-variational inequality

$$\max\left\{ v_x 1_{\{x\leq 0\}}, \sup_{(\pi,g)\in\mathcal{A}}\{Av\}, (\beta v_y - v_x)1_{\{x\geq 0\}} \right\} = 0$$

in $D_\beta := \{(x,y) : y > 0, y + \beta x > 0\}$, and

$$v = 0 \quad \text{outside of} \quad D_\beta.$$

Theorem (cont'd)

[ii] (Uniqueness) For $\gamma > 0$ and each $\rho \geq 0$ choose $\alpha > 0$ s.t. $k(\gamma, \rho) < \alpha$. Then the HJB integro-variational inequality admits at most one constrained viscosity solution.

Recently, in comparing with the seminal paper of Hindy and Huang (Econometrica 61 (1993)), and the papers by Bank (Ann Appl Prob (2001)) and by Benth, Karlsen, Reikvam (Finance Stoch 5 (2001), we (with Harry Zheng) propose to investigate the following value function

$$v(x, y) = sup_{(\pi,g,L)\in\mathcal{A}} \mathbf{E}\left[\int_0^\infty e^{-\alpha s} u(Y_s^{\pi,g,L}) ds\right] .$$

Under the assumption that the following dynamic programming principle holds: $\forall t \geq 0$ and for any stopping time τ

$$v(x,y) = sup_{(\pi,g,L)\in\mathcal{A}}\mathbf{E}\left[\int_0^{t\wedge\tau} e^{-\alpha s}u(Y_s^{\pi,g,L})ds + e^{-\alpha(t\wedge\tau)}v(X_{t\wedge\tau}^{(\pi,g,L)}, Y_{t\wedge\tau}^{(\pi,g,L)})\right],$$

one could then derive that v is the unique, constrained (subject to a gradient constraint) viscosity solution of the following integro-differential HJB equation

$$\max\left\{\beta v_y - v_x, \sup_{(\pi,g,L)\in\mathcal{A}}\{Av\}\right\} = 0$$

in $D := \{(x,y) : x > 0, y > 0\}$.

Further discussion on the properties of v is in progress.

The case of polar-decomposed Lévy measures

Recall the Lévy generator

$$
\begin{aligned}
Lf(t,x) \;:=\;& \frac{1}{2}a^{i,j}(t,x)\partial_i\partial_j f(t,x) + b^i(t,x)\partial_i f(t,x) \\
&+ \int_{\mathbb{R}^d\setminus\{0\}} \{f(t,x+z) - f(t,x) \\
&\qquad - \frac{z 1_{\{|z|<1\}}\cdot\nabla f(t,x)}{1+|z|^2}\}\nu(t,x,dz)
\end{aligned}
$$

and the associated SDE

$$
dS_t = b(t,S_t)dt + \sigma(t,S_t)dW_t + \int_U c(t,S_{t-},y)\tilde{N}(dt,dy)
$$

Here we consider a special case: ν admits a polar-decomposition.

$$(U, \mathcal{B}(U), \lambda) = (\mathrm{S}^{d-1} \times (0, \infty), \lambda)$$

where λ is σ-finite. Now let

m: a finite Borel measure on S^{d-1}

$z : \mathbb{R}^d \times \mathrm{S}^{d-1} \times (0, \infty) \to \mathbb{R}^d \setminus \{0\}$ bimeasurable bijection

$g : \mathbb{R}^d \times \mathrm{S}^{d-1} \times \mathcal{B}((0, \infty)) \to (0, \infty)$ is a positive kernel

Our ν is then taken the form

$$\nu(x, dz) = \int_{\mathrm{S}^{d-1}} \int_0^\infty 1_{dz}(z(x, \theta, r) g(x, \theta, dr)$$

M Tsuchiya, Stoch Stoch Reports 38 (1992)
V. Kolkoltsov, Proc London Maths Soc 80 (2000)
V. Kolkoltsov, *Nonlinear Markov Processes and Kinetic Equations.* (CUP, 2010)

Example (Bass, PTRF (1988); Kolkoltsov) Take

$$z(x,\theta,r) = r\theta \quad \text{and} \quad g(x,\theta,dr) = \frac{dr}{r^{1+\alpha(x)}}$$

then

$$\nu(x,dz) = \frac{dr}{r^{1+\alpha(x)}} m(d\theta)$$

Theorem ([1])

(i) For $d \geq 2$, i.e., for the case that the given σ-finite measure space

$$U = (S^{d-1} \times (0, \infty))$$

the coefficient of the jump term in the SDE associated to $\nu(x, dz)$ is given by $c(t, x, (r, \theta)) = r\theta$;
(ii) For the case when $d = 1$, namely, for the case that the given σ-finite measure space

$$(U, \mathcal{B}(U), \lambda) = ((0, \infty), \mathcal{B}((0, \infty)), \lambda)$$

the coefficient of the jump term in the SDE associated to $\nu(x, dz)$ defined by

$$\nu(x, dz) = \frac{dr}{r^{1+\alpha(x)}} \quad \alpha(x) \in (0, 2), \quad x \in \mathbb{R}$$

is given by $|c(t, x, (r, \theta))| = r$.

As an application, we consider a consumption-portfolio optimzation problem. The wealth process is modelled via

$$
\begin{aligned}
dS(t) \quad = \quad & \{\rho_t S(t) + (b(t) - \rho_t)u(t) - w(t)\}dt \\
& +\sigma(t)w(t)dW(t) \\
& +w(t-)\int_{0<|r|<1} r\theta \tilde{N}(dt, drd\theta) \\
& +w(t-)\int_{|r|\geq 1} r\theta N(dt, drd\theta).
\end{aligned}
$$

Our objective is to solve the following consumption-portfolio optimization problem:

$$\sup_{(w,u)\in\mathcal{A}} \mathbf{E}[\int_0^T \exp(-\int_0^t \delta(s)ds)[\frac{w(t)^\gamma}{\gamma}]dt] \tag{3}$$

subject to the terminal wealth constraint

$$S(T) \geq 0 \quad a.s.$$

where $\mathcal{A}$ is the set of predictable consumption-portfolio pairs (w, u) with the control u being tame and the consumption w being nonnegative, such that the above SDE has a strong solution over $[0, T]$.

Theorem ([1])

An optimal control (u^*, w^*) is given by

$$u^*(t,x) = exp\left\{\int_0^t \frac{\delta(s)}{\gamma - 1} ds\right\} f(t)^{\frac{1}{\gamma-1}} x$$

and

$$w^*(t,x) = \hat{\pi} x$$

with $f(t)$ and $\hat{\pi}$ being explicitly constructed.

Thank You!