Arnold Diffusion in Fictitious Play Dynamics

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- Mathematically, the systems of interest are nonsmooth flows on S^3, with global sections whose first return maps are
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- Among other interesting features of this class of nonsmooth dynamics we are particularly interested in different forms of coexistence of quasi-periodic and stochastic behaviour.

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The payoff functions can be represented by a bimatrix (A, B), $A, B \in \mathbb{R}^{n \times n}$:

$$u_1(i, j) = A_{ij} = e_i^T A e_j \quad \text{and} \quad u_2(i, j) = B_{ij} = e_i^T B e_j.$$

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(pure strategy i implicitly identified with e_i, the ith unit vector in $\mathbb{R}^n$)

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- Nash (1950): Every game has at least one Nash Equilibrium.

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Let 1 = ‘rock’, 2 = ‘paper’, 3 = ‘scissors’. The game is given by (A, B):

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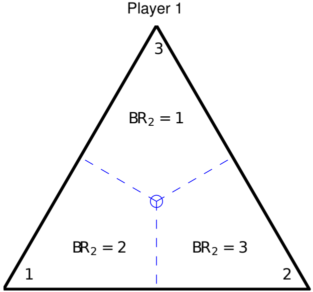
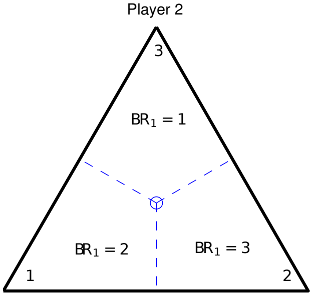
The space of mixed strategies can be visualized like this:





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Let (x_n, y_n) denote the (pure) strategies played at time $n \in \mathbb{N}$. Then **discrete-time Fictitious Play** is given by the rule

$$x_{n+1} \in \mathcal{BR}_1\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) \quad , \quad y_{n+1} \in \mathcal{BR}_2\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right).$$

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(Continuous-time) Fictitious Play Dynamics (FP)

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- Solutions for any initial conditions exist for all times $t \geq 0$ (by general theory: $\mathcal{BR}$ upper semi-continuous, has closed convex sets as values).
- Under genericity assumptions on (A, B), (FP) defines a *unique, continuous flow* for all $t \geq 0$ for a set of initial conditions which has full Lebesgue measure, is open and dense (van Strien et al., 2008).

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- (FP)-orbits converge to the set of Nash Equilibria in various specific classes of games, but generally this is not the case (example of (3×3)-game with a stable limit cycle due to Shapley, 1964).
- In a zero-sum game ($A + B = 0$), (FP)-orbits converge to the set of Nash Equilibria (Brown, 1951). The converse statement is an open conjecture (Hofbauer, 1995).

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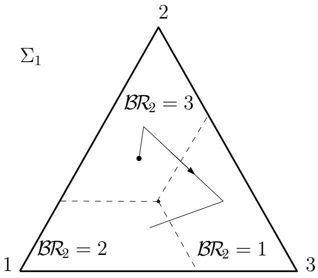
Example piece of orbit:



Non-equilibrium solution trajectories consist of straight line segments.

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Theorem (van Strien, 2011)

In this setting (FP) induces a unique continuous flow on all of $\Sigma_1 \times \Sigma_2$.

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H is a Lyapunov function for (FP) (Brown, 1951)

Let (A, B) be zero-sum with unique Nash Equilibrium (E^A, E^B). Then

1. $H(p, q) \geq 0$ and $H(p, q) = 0$ if and only if $(p, q) = (E^A, E^B)$;
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- For $A + B = 0$, the motion defined by (FP) is a product of radial motion towards the Nash Equilibrium and motion on H-level sets.
- This induced flow on $H^{-1}(1) \approx S^3$ is subject of our further interest.

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- *Ψ_t is a piecewise translation flow.*

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The Induced Flow on S^3

- ϕ_t flow of (FP) on $\Sigma_1 \times \Sigma_2$
- $\pi\colon (\Sigma_1 \times \Sigma_2) \setminus \{(E^A, E^B)\} \to H^{-1}(1)$ radial projection along rays from (E^A, E^B)

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$$A = -B = \begin{pmatrix} 1 & 0 & \beta \\ \beta & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix}, \quad \beta = \frac{\sqrt{5}-1}{2} \approx 0.618 \text{ (golden mean)}$$

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Overview

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For different zero-sum games, numerical experiments seem to suggest several possible types of induced dynamics on S^3. The following types were (numerically) observed in (O., van Strien, 2011):

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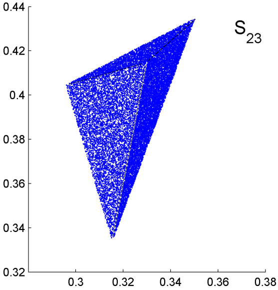
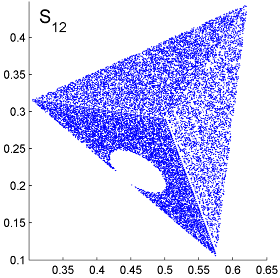
- **Completely ergodic type**



Types of Dynamics for Zero-Sum Games

- **Space decomposed into elliptic islands and ergodic regions**



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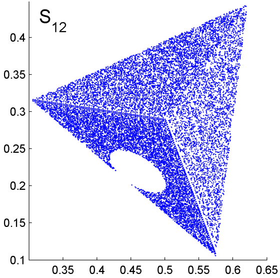
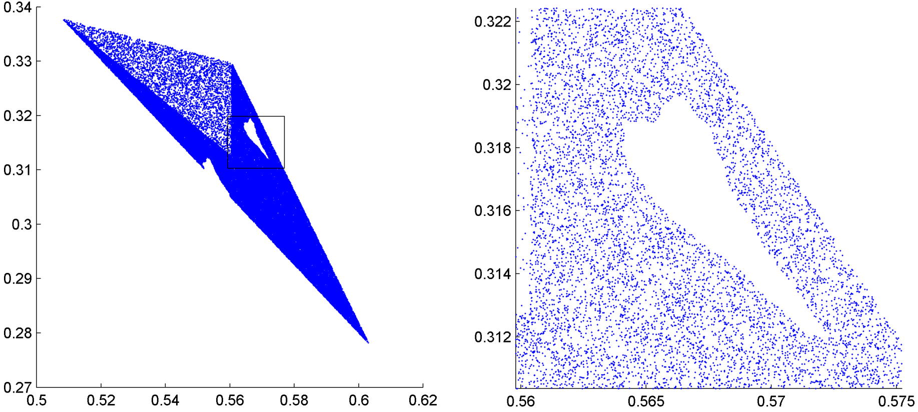


Remark (O., van Strien, 2011)

Quasi-periodic orbits in the elliptic regions exactly correspond to orbits with periodic itinerary (players switch strategies in a cyclic way).

Types of Dynamics for Zero-Sum Games

- **More complicated situation with coexistence of different quasi-periodic islands and stochastic motion**



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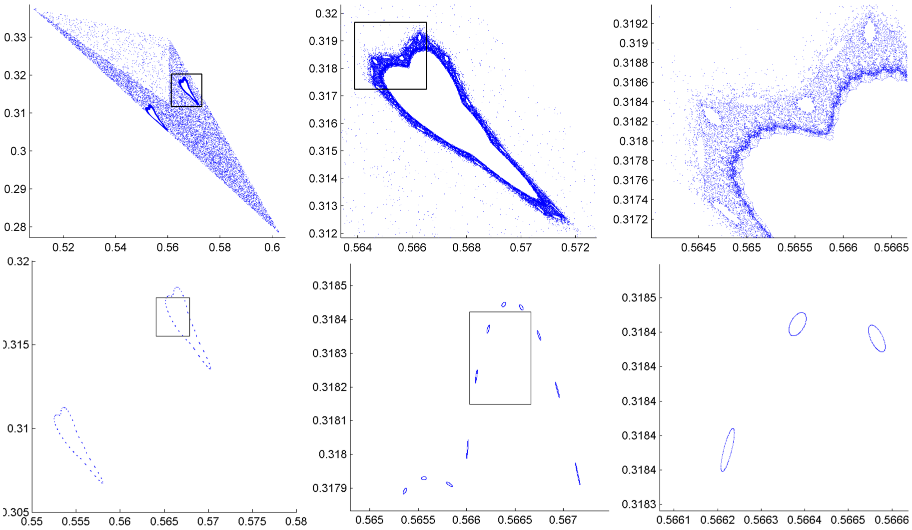

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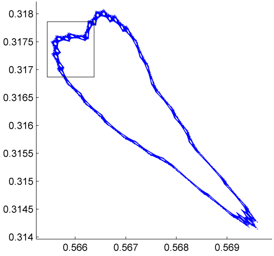

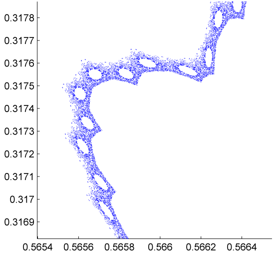

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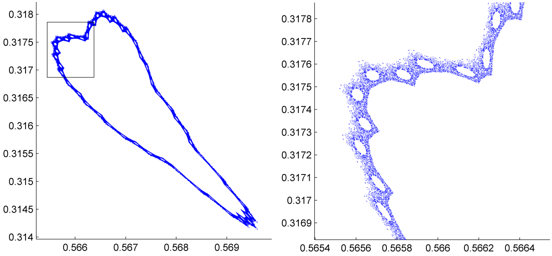

- The images indicate the occurence of ‘Arnold Diffusion’; coexistence of various families of islands of (quasi-)periodic motion (filled with invariant circles), contained in regions of stochastic (space-filling) motion.

Overview

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Trying to construct simple (non-trivial) examples of maps with these properties shows that the properties are quite restrictive.

A Model Map

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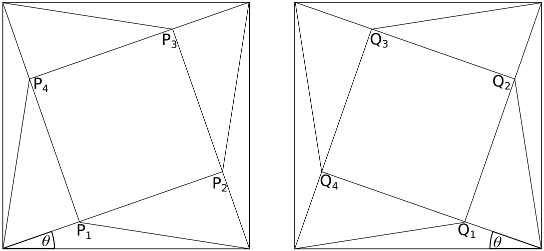

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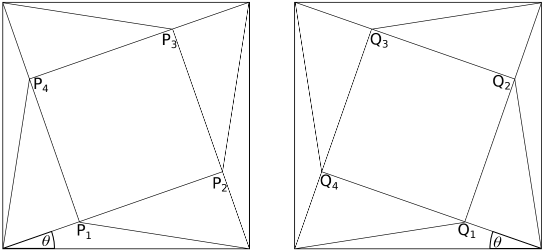

The map F_θ for $\theta \in (0, \frac{\pi}{4}]$ is defined by declaring:

1. $F_\theta = id$ on the boundary of the square
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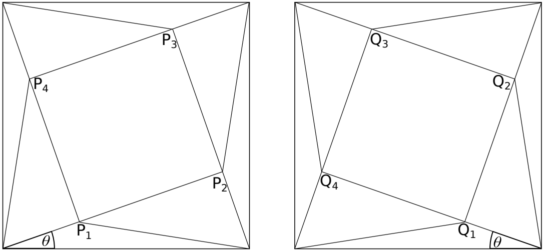

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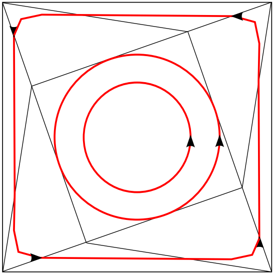
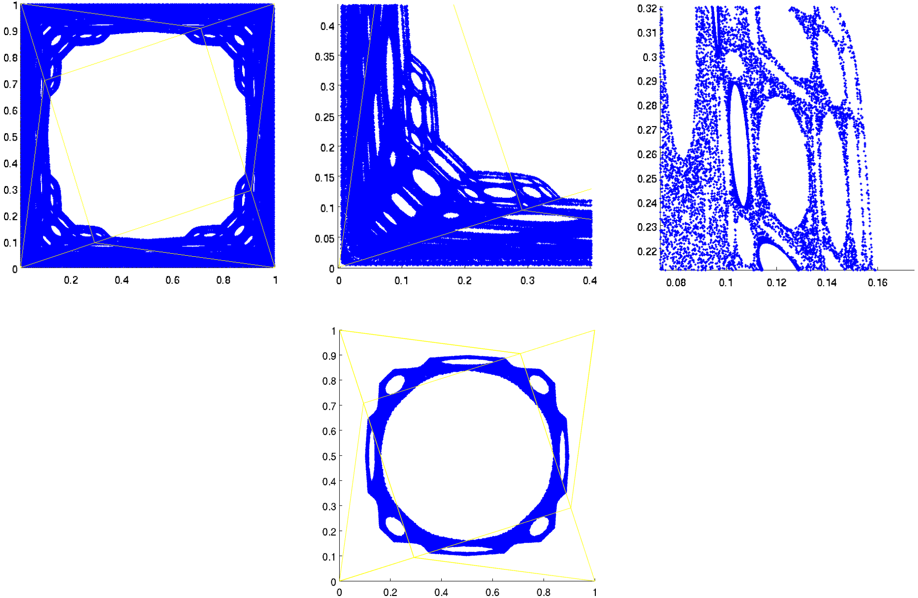
- The map acts similar to a twist map. Intuitively, all points approximately rotate counterclockwise with rotation number decreasing to zero as points approach the boundary.

Invariant curves

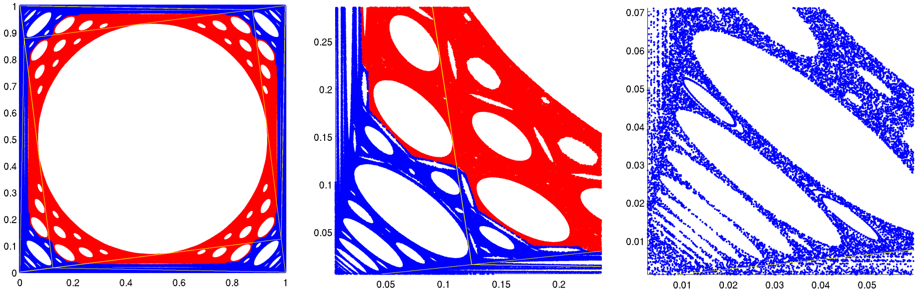


- For many parameter values of θ, families of invariant circles consisting of straight line segments and accumulating on the boundary can be constructed explicitly. On these invariant circles, F_θ has rational rotation number.

Stochastic Regions



Open Questions



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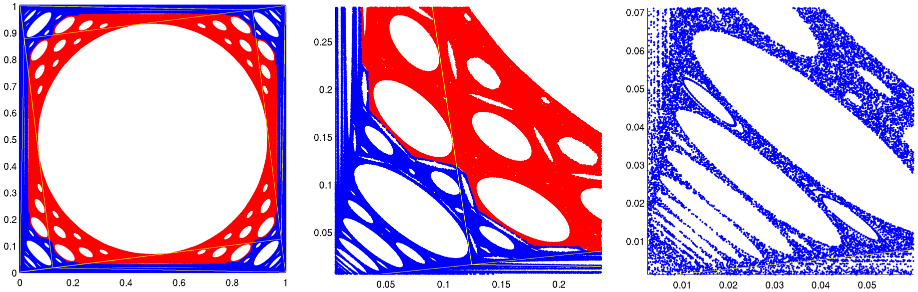

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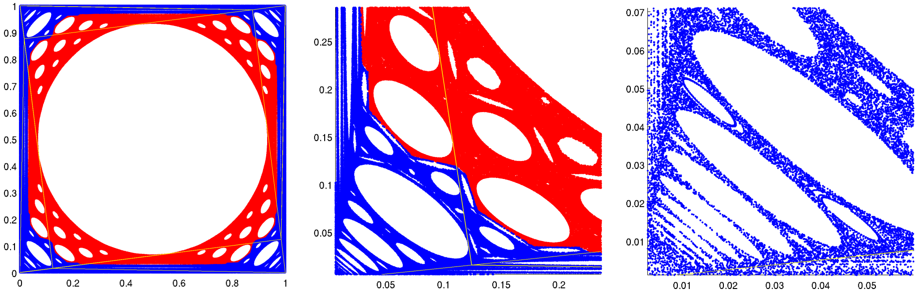

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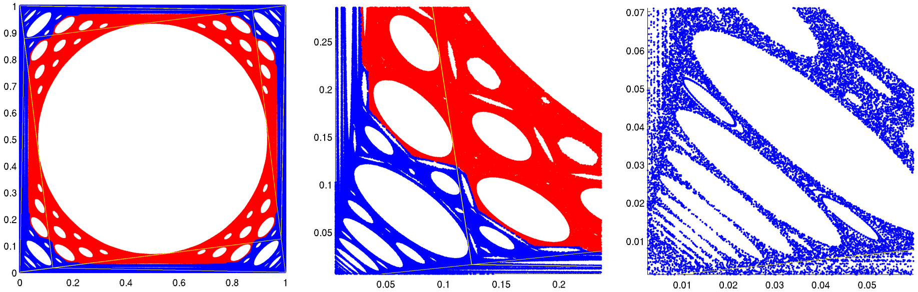

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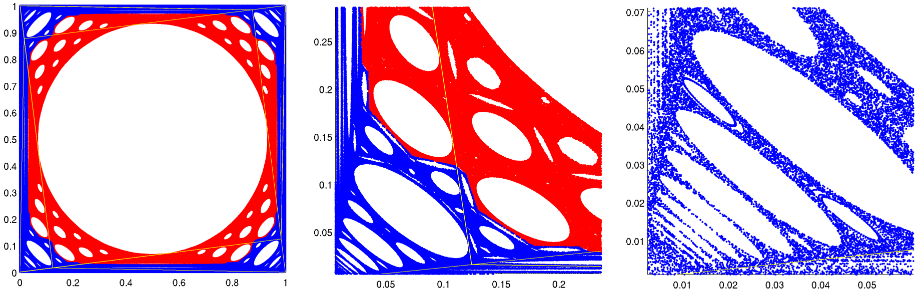

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- What are the possible itineraries for quasi-periodic orbits?