Half integral weight modular forms

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Explicit Methods for Modular Forms
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Motivation

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1. The Dedekind eta function $\eta(z) = e^{\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})$. It is known that $\eta(z)^24 = \Delta(z)$, a weight 12 cusp form, so $\eta(z)$ should be of weight $1/2$. Actually, $\eta(z)$ turns out to be of weight $1/2$ but with a character of order 24.
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\[ \eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}). \]
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It is not hard to see that if \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) \), then

\[ \left( \frac{\theta(\gamma z)}{\theta(z)} \right)^2 = \left( \frac{-1}{d} \right) (cz + d). \]
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So \( \theta(z)^2 \in \mathcal{M}_1(\Gamma_0(4), \chi_{-1}) \).
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**Definition**

A modular form of weight $k/2$, level $4N$ and character $\psi$ is an holomorphic function $f : \mathcal{H} \rightarrow \mathbb{C}$ such that

$$f(\gamma z) = J(\gamma, z) \psi(d) f(z) \quad \forall \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(4N).$$
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We denote by \( M_{k/2}(4N, \psi) \) the space of such forms and \( S_{k/2}(4N, \psi) \) the subspace of cuspidal ones.
Hecke operators

Via a double coset action, one can define Hecke operators $\{T_n\}_{n \geq 1}$ acting on $S_{k/2}(4N, \psi)$. They satisfy the properties:
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4. If terms of q-expansion, let \( \omega = \frac{k-1}{2} \), then \( T_p^2 \) acts like

\[
a_{p^2 n} + \psi(n) \left( \frac{-1}{p} \right)^\omega \left( \frac{n}{p} \right) p^{\omega-1} a_n + \psi(p^2) p^{k-1} a_{n/p^2}.
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Hence there exists a basis of eigenforms for the Hecke operators prime to $4N$. 
Shimura’s Theorem

Theorem (Shimura)

For each square-free positive integer \( n \), there exists a \( \mathbb{T}_0 \)-linear map

\[
\text{Shim}_n : S_{k/2}(4N, \psi) \rightarrow M_{k-1}(2N, \psi^2). \]

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Furthermore, if \( f \in S_{k/2}(4N, \psi) \) is an eigenform for all the Hecke operators with eigenvalues \( \lambda_n \), then \( \text{Shim}_n(f) \) is (up to a constant) given by

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\prod_p (1 - \lambda_p p^{-s} + \psi(p^2) p^{k-2-2s})^{-1}.
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What information encode the non-square Fourier coefficients?

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**Theorem (Waldspurger)**

Let $n_1, n_2$ be square free positive integers such that $n_1/n_2 \in (\mathbb{Q}_p^\times)^2$ for all $p | 4N$. 

If we fixed $n_1$, for all $n_2$ as above $a_{2n_1} = \kappa L(F, \psi - 10 \chi_{n_2}, k - 1/2)$ where $\psi_0(n) = \psi(n)(-1/n)$ and $\chi_{n_2}$ is the quadratic character corresponding to the field $\mathbb{Q}[\sqrt{n}]$. 

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a_{n_1}^2 L(F, \psi_0^{-1} \chi_{n_2}, \omega) \psi\left(\frac{n_2}{n_1}\right) n_2^{k/2-1} = a_{n_2}^2 L(F, \psi_0^{-1} \chi_{n_1}, \omega) n_1^{k/2-1}
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where \( \psi_0(n) = \psi(n) \left(\frac{-1}{n}\right)^\omega \), \( \chi_n \) is the quadratic character corresponding to the field \( \mathbb{Q}[\sqrt{n}] \).
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If we fixed $n_1$, for all $n$ as above

$$a_n^2 = \kappa L(F, \psi_0^{-1} \chi_n, \frac{k-1}{2}) \psi(n) n^{k/2-1}$$

where $\kappa$ is a global constant.
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**Theorem (Tunnell)**

*If $n \in \mathbb{N}$ is odd, (assuming BSD) it is a congruent number iff*

$$\# \{ (x, y, z) \in \mathbb{Z}^3 : n = 2x^2 + y^2 + 32z^2 \} = \frac{1}{2} \# \{ (x, y, z) \in \mathbb{Z}^3 : n = 2x^2 + y^2 + 8z^2 \}$$

For even $n$, iff

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What we would like to do:

1. Given $F \in S_k^2(N,1)$, construct preimages under Shim.
2. Give an explicit constant in Waldspurger Theorem.
3. Generalize this to Hilbert modular forms.

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Let $B$ be a quaternion algebra over $\mathbb{Q}$ ramified at $\infty$. 
Quaternionic modular forms

Let $B$ be a quaternion algebra over $\mathbb{Q}$ ramified at $\infty$. Let $R \subset B$ be an Eichler order of level $N$.

Let $\mathcal{J}(R)$ be the set of left $R$-ideals and let $\{[a_1], \ldots, [a_n]\}$ be ideal classes representatives.
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$$\langle [a_i], [a_j] \rangle = \begin{cases} 0 & \text{if } i \neq j, \\ \frac{1}{2} \#R_r(a_i) & \text{if } i = j. \end{cases}$$
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Given $m \in \mathbb{N}$ and $a \in \mathcal{J}(R)$, let

$$t_m(a) = \{b \in \mathcal{J}(R) : b \subset a, [a : b] = m^2\}.$$
For $m \in \mathbb{N}$, the Hecke operators $T_m : M(R) \rightarrow M(R)$ is

$$T_m([a]) = \sum_{b \in t_m(a)} \frac{[b]}{\langle b, b \rangle}.$$
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**Proposition**

*The Hecke operators satisfy:*

- are self adjoint (all of them).
- commute with each other.

Let $e_0 = \sum_{i=1}^{n} \langle a_i, a_i \rangle [a_i]$. It is an eigenvector for the Hecke operators. Denote by $S(R)$ its orthogonal complement.
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Theorem (Eichler)

There is a natural map of $\mathbb{T}_0$-modules $S(R) \times S(R) \to S_2(N)$. 
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- In general, considering other orders, any weight 2 form which has a non-principal series prime is in the image (J-L).
Basis problem

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$$S(R) \to S_2(N) \quad \text{Shim} \quad S_3^{+}(4N)$$

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\]
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If $a \in \mathcal{J}(R)$, and $d \in \mathbb{N}$, let

$$a_d(a) = \# \{ [x] \in R_r(a)/\mathbb{Z} : \Delta(x) = -d \}.$$
Ternary forms

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For $d \in \mathbb{N}_0$, let $e_d \in M(R)$ be given by

$$e_d = \sum_{i=1}^{n} \frac{a_d(a_i)}{\langle a_i, a_i \rangle} [a_i].$$
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$$S(R) \xrightarrow{\Theta} S_{3/2}(4N) \xrightarrow{\text{Shim}} S_2(N)$$
Here are some questions:

- Given $F \in S_2(N)$, how to choose $R$ such that $\Theta(v_F)$ is non-zero?
- Do we have an explicit formula relating the coefficients to central values of twisted $L$-series?
- It should be the case that for all fundamental discriminants $d$ in some residue classes, the following formula should hold:
  
  $$L(F, 1) L(F, d, 1) = \star <F, F> \sqrt{|d| a_{F, O}(d)} 2 \langle v_F, v_F \rangle.$$ 

  Done by Gross if $N = p$.

  Done by Böcherer and Schulze-Pillot if $N$ is odd and squarefree.

  Done by P. and Tornaría if $N = p^2$. 

Ariel Pacetti

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Let $F$ be a totally real number field, and $a = \{\tau : F \hookrightarrow \mathbb{R}\}$. 
Hilbert modular forms

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$$j(\alpha, z) = \prod_{\tau \in \mathfrak{a}} j(\tau(\alpha), z_\tau).$$
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Let $\mathcal{O}_F$ denotes the ring of integers of $F$. If $r, n$ are ideals, let

$$\Gamma(r, n) = \left\{ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2^+(F) : \det(\alpha) \in \mathcal{O}_F^\times \text{ and } a, d \in \mathcal{O}_F, b \in r^{-1}, c \in rn \right\}.$$
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Main properties

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- There are Hecke operators indexed by integral ideals (satisfying the same properties).
- The action can be given in terms of $q$-expansion.
- There is a theory of new subspaces.
Let

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For \( n \) an integral ideal in \( \mathcal{O}_F \), let
\[ \tilde{\Gamma}[2^{-1}\delta, n] = \Gamma[2^{-1}\delta, n] \cap \text{SL}_2(F). \]
**Definition**

If $\psi$ is a Hecke character of conductor $n$, a Hilbert modular form of parallel weight $3/2$, level $4n$ and character $\psi$, is a holomorphic function $f$ on $\mathfrak{H}$ satisfying:

$$f(\gamma z) = \psi(d)J(\gamma, z)f(z) \quad \forall \gamma \in \tilde{\Gamma}[2^{-1}\delta, 4n].$$
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- There is a formula relating the Hecke operators with the Fourier expansion at different ideals.
Shimura map for HMF

Theorem (Shimura)

For each \( \xi \in F^+ \), there exists a \( \mathbb{T}_0 \) linear map

\[
\text{Shim}_\xi : M_{3/2}(4n, \psi) \to M_2(2n, \psi^2).
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How do we compute preimages? $\leadsto$ use quaternionic forms.
We have to make some small adjustments to the classical picture.
Quaternionic HMF

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- Take $B/F$ a quaternion algebra ramified at least all the infinite places, and $R$ an Eichler order in it.
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- Let $M(R)$ be the $\mathbb{C}$-v.s. spanned by class ideal representatives with the inner product

$$\langle [a_i], [a_j] \rangle = \begin{cases} 
0 & \text{if } i \neq j, \\
[R_r(a_i) \times : \mathcal{O}_F^\times] & \text{if } i = j.
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- Define the Hecke operators in the same way as before.
- They commute, and the adjoint of $T_p$ is $p^{-1} T_p$. 

Theorem (J-L, Hida)

There is a natural map of $\mathbb{T}_0$-modules $S(R) \times S(R) \rightarrow S_2(N)$.

The same remarks as in the classical case apply.
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Let $\Theta : M(R) \rightarrow M_{3/2}(4\mathfrak{n}(R), \chi_R)$ be given by

$$\Theta(v)(z) = \sum_{\xi \in \mathcal{O}_F^+} \langle v, e_\xi \rangle q^\xi.$$

Theorem (Sirolli)

The map $\Theta$ is $\mathbb{T}_0$-invariant. Furthermore, the image lies in the Kohnen space. $\Theta(v)$ is cuspidal iff $v$ is cuspidal.
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General picture

\[ S(R) \xrightarrow{\Theta} S_{3/2}^+(4n) \xrightarrow{\text{Shim}_\xi} S_2(n) \]
Let $F = \mathbb{Q}(\sqrt{5})$, $\omega = \frac{1+\sqrt{5}}{2}$, and consider the elliptic curve

$$E : y^2 + xy + \omega y = x^3 - (1 + \omega)x^2.$$ 

This curve has conductor $n = (5 + 2\omega)$ (an ideal of norm 31).

- Let $B/F$ be the quaternion algebra ramified at the two infinite primes, and $R$ an Eichler order of level $n$.
- The space $M_2(R)$ has dimension 2 (done by Lassina). The element $v = [R] - [a]$ is a Hecke eigenvector.
- If we compute $\theta(v)$, we get a form whose $q$-expansion is “similar” to Tunnell result.
- There are 5 non-trivial zero coefficients with trace up to 100, and the twists of the original curve by this discriminants all have rank 2.