Half integral weight modular forms

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Explicit Methods for Modular Forms
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Actually η turns out to be weight $1/2$ but with a character of order 24.

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It is not hard to see that if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, then

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So $\theta(z)^2 \in M_1(\Gamma_0(4), \chi_{-1})$.

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We denote by $M_{k/2}(4N, \psi)$ the space of such forms and $S_{k/2}(4N, \psi)$ the subspace of cuspidal ones.

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4. If terms of q-expansion, let $\omega = \frac{k-1}{2}$, then T_{p^2} acts like

$$a_{p^2n} + \psi(n)\left(\frac{-1}{p}\right)^{\omega}\left(\frac{n}{p}\right)p^{\omega-1}a_n + \psi(p^2)p^{k-1}a_{n/p^2}.$$

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Hence there exists a basis of eigenforms for the Hecke operators prime to $4N$.

Shimura's Theorem

Theorem (Shimura)

For each square-free positive integer n, there exists a $\mathbb{T}_0$-linear map

$$\mathrm{Shim}_n : S_{k/2}(4N, \psi) \to M_{k-1}(2N, \psi^2).[1]$$

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What information encode the non-square Fourier coefficients?

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$$a_{n_1}^2 L(F, \psi_0^{-1}\chi_{n_2}, \omega)\psi\left(\frac{n_2}{n_1}\right) n_2^{k/2-1} = a_{n_2}^2 L(F, \psi_0^{-1}\chi_{n_1}, \omega) n_1^{k/2-1}$$

where $\psi_0(n) = \psi(n)\left(\frac{-1}{n}\right)^\omega$, χ_n is the quadratic character corresponding to the field $\mathbb{Q}[\sqrt{n}]$

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$$a_n^2 = \kappa L(F, \psi_0^{-1}\chi_n, \tfrac{k-1}{2})\psi(n)n^{k/2-1}$$

where κ is a global constant.

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If $n \in \mathbb{N}$ is odd, (assuming BSD) it is a congruent number iff

$$\#\{(x,y,z) \in \mathbb{Z}^3 \; : \; n = 2x^2 + y^2 + 32z^2\} = \frac{1}{2}\#\{(x,y,z) \in \mathbb{Z}^3 \; : \; n = 2x^2 + y^2 + 8z^2\}$$

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For even n, iff

$$\#\{(x,y,z) \in \mathbb{Z}^3 \ : \ n/2 = 4x^2 + y^2 + 32z^2\} = \frac{1}{2}\#\{(x,y,z) \in \mathbb{Z}^3 \ : \ n/2 = 4x^2 + y^2 + 8z^2\}.$$

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For simplicity we will consider the case of weight $k = 2$ (where modular forms correspond with elliptic curves).

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$$\langle [\mathfrak{a}_i], [\mathfrak{a}_j] \rangle = \begin{cases} 0 & \text{if } i \neq j, \\ \frac{1}{2} \# R_r(\mathfrak{a}_i)^\times & \text{if } i = j. \end{cases}$$

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Given $m \in \mathbb{N}$ and $\mathfrak{a} \in \mathcal{J}(R)$, let

$$t_m(\mathfrak{a}) = \{\mathfrak{b} \in \mathcal{J}(R) \ : \ \mathfrak{b} \subset \mathfrak{a}, [\mathfrak{a} : \mathfrak{b}] = m^2\}.$$

Hecke operators

For $m \in \mathbb{N}$, the Hecke operators $T_m : M(R) \to M(R)$ is

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Let $e_0 = \sum_{i=1}^{n} \frac{1}{\langle \mathfrak{a}_i, \mathfrak{a}_i \rangle}[\mathfrak{a}_i]$. It is an eigenvector for the Hecke operators. Denote by $S(R)$ its orthogonal complement.

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Theorem (Eichler)

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$$\begin{array}{ccc} S(R) & \longrightarrow & S_2(N) \\ & & \nearrow_{\text{Shim}} \\ & S^+_{3/2}(4N) & \end{array}$$

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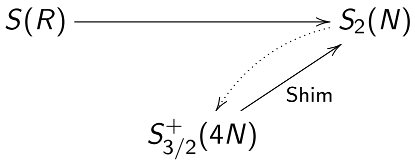

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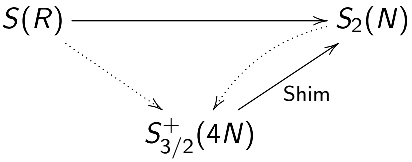

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For $d \in \mathbb{N}_0$, let $e_d \in M(R)$ be given by

$$e_d = \sum_{i=1}^{n} \frac{a_d(\mathfrak{a}_i)}{\langle \mathfrak{a}_i, \mathfrak{a}_i \rangle}[\mathfrak{a}_i].$$

Theta map

Let $\Theta : M(R) \to M_{3/2}(4N(R), \chi_R)$ be given by

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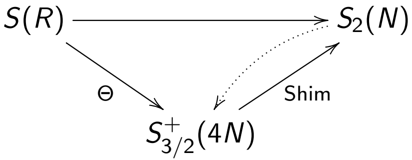

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It should be the case that for all fundamental discriminants d in some residue classes, the following formula should hold

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- There is a theory of new subspaces.

Half integral weight HMF

Let

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For $\mathfrak{n}$ an integral ideal in $\mathcal{O}_F$, let

$$\tilde{\Gamma}[2^{-1}\delta, \mathfrak{n}] = \Gamma[2^{-1}\delta, \mathfrak{n}] \cap \mathrm{SL}_2(F).$$

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Definition

If ψ is a Hecke character of conductor $\mathfrak{n}$, a Hilbert modular form of parallel weight $3/2$, level $4\mathfrak{n}$ and character ψ, is a holomorphic function f on $\mathfrak{H}^a$ satisfying:

$$f(\gamma \mathbf{z}) = \psi(d) J(\gamma, \mathbf{z}) f(\mathbf{z}) \qquad \forall\, \gamma \in \tilde{\Gamma}[2^{-1}\delta, 4\mathfrak{n}].$$

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- There is a theory of Hecke operators as in the classical case.
- There is a formula relating the Hecke operators with the Fourier expansion at different ideals.

Shimura map for HMF

Theorem (Shimura)

For each $\xi \in F^+$, there exists a $\mathbb{T}_0$ linear map

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How do we compute preimages? $\rightsquigarrow$ use quaternionic forms.

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- They commute, and the adjoint of $T_\mathfrak{p}$ is $\mathfrak{p}^{-1} T_\mathfrak{p}$.

Preimages

Theorem (J-L,Hida)

There is a natural map of $\mathbb{T}_0$-modules $S(R) \times S(R) \to S_2(N)$.

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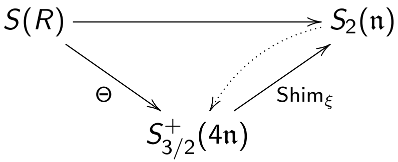
- $\Theta(\mathbf{v})$ is cuspidal iff $\mathbf{v}$ is cuspidal.

General picture



Example

Let $F = \mathbb{Q}(\sqrt{5})$, $\omega = \frac{1+\sqrt{5}}{2}$, and consider the elliptic curve

$$E : y^2 + xy + \omega y = x^3 - (1+\omega)x^2.$$

This curve has conductor $\mathfrak{n} = (5 + 2\omega)$ (an ideal of norm 31).

- Let B/F be the quaternion algebra ramified at the two infinite primes, and R an Eichler order of level $\mathfrak{n}$.
- The space $M_2(R)$ has dimension 2 (done by Lassina). The element $v = [R] - [\mathfrak{a}]$ is a Hecke eigenvector.
- If we compute $\theta(v)$, we get a form whose q-expansion is "similar" to Tunnell result.
- There are 5 non-trivial zero coefficients with trace up to 100, and the twists of the original curve by this discriminants all have rank 2.