Ranks of elliptic curves with prescribed torsion over number fields

Filip Najman

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joint work with J. Bosman, P. Bruin, A. Dujella

Warwick, September 27, 2012.
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$$E(K) \cong T \oplus \mathbb{Z}^r,$$
where $T$ is the torsion subgroup and $r$ is the rank of $E(K)$. 
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$$E(K) \simeq T \oplus \mathbb{Z}^r,$$
where $T$ is the torsion subgroup and $r$ is the rank of $E(K)$.

We want to understand what $T$ and $r$ can be and especially how they depend on each other.
Theorem (Mazur) The torsion of an elliptic curve over $\mathbb{Q}$ is isomorphic to one of the following groups:

$$\mathbb{Z}/n\mathbb{Z}, \text{ where } n = 1, \ldots, 10 \text{ or } 12,$$

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The largest known rank of $E(\mathbb{Q})$ is 28 (Elkies).
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The rank of this curve might be odd, even, it might be 0, or 10 billion.
\[ B(T) = \sup \{ \text{rank}(E(\mathbb{Q})) : E(\mathbb{Q})_{\text{tors}} \cong T \} \]

<table>
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<tr>
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<th>( B(T) \geq )</th>
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<td>Dujella &amp; Lecacheux (09), Eroshkin (09)</td>
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<td>Dujella (05,08), Elkies (06)</td>
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Elliptic curves over number fields

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2. The torsion can tell us something about the rank
$Y_1(m, n)$ - affine curve whose $K$-rational points classify isomorphism classes of elliptic curves $E$ with a pair $(P, R)$, where $P, R \in E(K)$ generate a subgroup isomorphic to $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. 

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Hence, over a fixed number field over which some curve has $n$-torsion, for such $n$, there will be only finitely many such curves.
When we have finitely many curves with prescribed torsion, then it is not very surprising that we can say something about the rank, for example that it is bounded.
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Similarly, there is one elliptic curve with torsion $\mathbb{Z}/15\mathbb{Z}$ over $\mathbb{Q}(i, \sqrt{5})$ and one can show that

$$E(\mathbb{Q}(i, \sqrt{5}))_{\text{tors}} = \mathbb{Z}/15\mathbb{Z} \implies \text{rk}(E(\mathbb{Q}(i, \sqrt{5}))) = 1.$$
Theorem (N.)

a) The torsion of an elliptic curve over \( \mathbb{Q}(i) \) is isomorphic either to one of the groups from Mazur’s theorem or to \( \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \).

b) The torsion of an elliptic curve over \( \mathbb{Q}(\sqrt{-3}) \) is isomorphic either to one of the groups from Mazur’s theorem, or to \( \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \) or \( \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \).
Theorem (Kenku & Momose, Kamienny) Let $E$ be an elliptic curve over a quadratic field $K$. The torsion of $E(K)$ is isomorphic to one of the following groups:

$$\mathbb{Z}/n\mathbb{Z}, \text{ where } n = 1, \ldots, 16 \text{ or } 18,$$

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All of these groups appear as torsion groups for infinitely many nonisomorphic elliptic curves over quadratic fields.
The gonality of a curve $X$ is the lowest degree of a rational map from $X$ to $\mathbb{P}^1$. We call points on $X_1(m, n)$ which have degree smaller than the gonality of $X_1(m, n)$ sporadic.
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Mazur, Kenku-Momose and Kamienny - no degree 1 or 2 sporadic points.

Van Hoeij - there are degree 9 points on $X_1(29)$ and $X_1(31)$, which have gonality 11 and 12, respectively.

What is the minimal $d$ such that there is a degree $d$ sporadic point? From above it is $\leq 9$, and from Mazur, Kenku-Momose and Kamienny, it is $\geq 3$. It is exactly 3!
Sporadic points on $X_1$

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Theorem (Jeon, Kim & Schweizer)
When we run through all elliptic curves over all cubic fields the groups that appear infinitely often are exactly

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All the corresponding modular curves are of gonality $\leq 3$.

$\exists!$ pair $(E, K)$, where $E/\mathbb{Q}$ is an elliptic curve, $K$ a cubic field and $E(K)_{\text{tors}} \simeq \mathbb{Z}/21\mathbb{Z}$, so $X_1(21)$ has a sporadic point. $X_1(21)$ has gonality 4.
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The unique curve with 21-torsion is 162B1 over the field defined by $x^3 - 3x^2 + 3$ (which is $\mathbb{Q}(\zeta_9)^+$.)
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There exists a pair \((E, K)\), where \(E/\mathbb{Q}\) is an elliptic curve, \(K\) a cubic field and \(E(K)_{\text{tors}} \cong \mathbb{Z}/21\mathbb{Z}\), so \(X_1(21)\) has a sporadic point. \(X_1(21)\) has gonality 4.

The unique curve with 21-torsion is 162B1 over the field defined by \(x^3 - 3x^2 + 3\) (which is \(\mathbb{Q}(\zeta_9)^{+}\)).

There might be other elliptic curves (not defined over \(\mathbb{Q}\)) with 21-torsion over cubic fields, but there can be only finitely many.
The result

Theorem (Bosman, Bruin, Dujella, N.)

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1. Any elliptic curve over any quadratic field with a point of order 13 or 18 has even rank.

2. Any elliptic curve over any quartic field with a point of order 22 has even rank.
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Complex multiplication

We want a systematic way to say something about the rank of elliptic curves with given torsion over all number fields of fixed degree $d$.

Let $E$ be an elliptic curve with complex multiplication by an order $O$ of an imaginary quadratic number field $K$. If $L$ is a number field containing $K$, then $E(L)$ is not just a $\mathbb{Z}$-module, but also an $O$-module.
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As a result, if $E(L)$ is an $O$-module of rank $n$, it is a $\mathbb{Z}$-module of rank $2n$, so the rank of $E(L)$ is necessarily even.
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Weil’s restriction of scalars $\text{Res}_{L/K} A$ is an abelian variety defined over the smaller field $K$, but with larger dimension, with the property that

$$\text{Res}_{L/K} A(K) \simeq A(L).$$
An example

Let $E : y^2 = x^3 + i$. If we want to find $E(\mathbb{Q}(i))$, we are looking for the solutions of $(c + di)^2 = (a + bi)^3 + i$, or in other words

$$c^2 + 2cdi - d^2 = a^3 + 3a^2bi - 3ab^2 - b^3i + i,$$

So we get

$$a^3 - 3ab^2 - c^2 + d^2 = 0 \quad (1)$$

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Res $\mathbb{Q}(i)/\mathbb{Q}E$ is the abelian variety defined by equations (1) and (2).
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So we get

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a^3 - 3ab^2 - c^2 + d^2 = 0, \tag{1}
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\]

\( \text{Res}_{\mathbb{Q}(i)/\mathbb{Q}} E \) is the abelian variety defined by equations (1) and (2).
In a similar way as with elliptic curves with CM, if \( \text{End}(\text{Res}_{L/K} E) \) contains an order of a quadratic field, this implies that

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We call this *false complex multiplication*. 
An elliptic curve $E$ which is isogenous to all of its $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates is called a $\mathbb{Q}$-curve.
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Let $E$ be a $\mathbb{Q}$-curve defined over a quadratic field $K$, such that $\phi : E \to E^\sigma$ is an isogeny of degree $n$, where $\langle \sigma \rangle = \text{Gal}(K/\mathbb{Q})$. 

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$\text{Res}_{K/\mathbb{Q}} E$ is isomorphic over $K$ to $E \times E^\sigma$ and if $n$ is not a square then it will follow that $E$ has false CM.
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Note that $\sigma \circ \phi : E(K) \to E(K)$ is a homomorphism of groups which is not multiplication-by-$m$, since $|\text{Ker} \sigma \circ \phi|$ is non-square.
The modular curve $X_1(18)$ is hyperelliptic of genus 2 with hyperelliptic involution $w_2$. 
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Fix a cusp $C \in X_1(18)(\mathbb{Q})$. We look at the map

$$f : \text{Sym}^2 X_1(18) \to J_1(18),$$

$$\{P, Q\} \to [P + Q - C - w_2(C)],$$

which is an isomorphism away from the fibre above 0 which consists of the pairs of points $\{P, w_2(P)\}$ which are fixed by the hyperelliptic involution $w_2$. 

Elliptic curves with points of order 18

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As $J_1(18)(\mathbb{Q}) \sim \mathbb{Z}/21\mathbb{Z}$, we can check that the inverse image of any point except 0 is a pair of cusps. Let $K$ be a quadratic field, $\text{Gal}(K/\mathbb{Q}) = \langle \sigma \rangle$. 

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Elliptic curves with points of order 18

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Fix a cusp $C \in X_1(18)(\mathbb{Q})$. We look at the map

$$f : \text{Sym}^2 X_1(18) \to J_1(18),$$

$$\{P, Q\} \to [P + Q - C - w_2(C)],$$

which is an isomorphism away from the fibre above 0 which consists of the pairs of points $\{P, w_2(P)\}$ which are fixed by the hyperelliptic involution $w_2$.

As $J_1(18)(\mathbb{Q}) \simeq \mathbb{Z}/21\mathbb{Z}$, we can check that the inverse image of any point except 0 is a pair of cusps. Let $K$ be a quadratic field, $\text{Gal}(K/\mathbb{Q}) = \langle \sigma \rangle$.

Now take a non-cusp point $Q$ in $X_1(18)(K)$. Then $f(\{Q, Q^\sigma\}) \in J_1(18)(\mathbb{Q})$, thus it has to be 0, so $Q^\sigma = w_2(Q)$. 
If in the moduli interpretation of $Q \in X_1(18)$ represents $E$, then $Q^\sigma$ represents $E^\sigma$ and $w_2(Q)$ represents a curve that is 2-isogenous to $E$. Hence, all elliptic curves with a point of order 18 are isogenous to their Galois conjugate by an isogeny of degree 2. It follows that all elliptic curves with a point of order 18 over quadratic fields have false CM. It can be shown that $\text{End}(\text{Res}_{K/Q}E) \cong \mathbb{Z}[\sqrt{-2}]$ for all such curves $E$. So, all elliptic curves with torsion $\mathbb{Z}/18\mathbb{Z}$ over quadratic fields have even rank.
Elliptic curves with points of order 18

If in the moduli interpretation of $Q \in X_1(18)$ represents $E$, then $Q^\sigma$ represents $E^\sigma$ and $\nu_2(Q)$ represents a curve that is 2-isogenous to $E$.

Hence, all elliptic curves with a point of order 18 are isogenous to their Galois conjugate by an isogeny of degree 2.
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It follows that all elliptic curves with a point of order 18 over quadratic fields have false CM. It can be shown that

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So, all elliptic curves with torsion $\mathbb{Z}/18\mathbb{Z}$ over quadratic fields have even rank.
The modular curve $X_1(18)$ has a model

$$X_1(18) : y^2 = f(x) = x^6 + 2x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1 \ (3)$$
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As we have seen, the only quadratic points on $X_1(18)$ are the ones fixed by the hyperelliptic involution. These are the points in the model (3) with rational $x$-s.
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But for every $x \in \mathbb{R}$, $f(x)$ is positive. As all the quadratic points are of the form $(x, \sqrt{f(x)})$, $x \in \mathbb{Q}$, this implies that all of them are defined over real quadratic fields.
The modular curve $X_1(13)$ is hyperelliptic of genus 2 with hyperelliptic involution $\langle 5 \rangle$. It can be shown that for every point $P \in Y_1(13)(K)$, $\langle 5 \rangle P = P_\sigma$. This implies that an elliptic curve $E$ with a point of order 13 over a quadratic field $K$ is isomorphic to its Galois conjugate $E_\sigma$, and $\text{End}(\text{Res}_{K/Q}E) \cong \mathbb{Z}[\sqrt{-1}]$. It follows that all elliptic curves over quadratic fields with a point of order 13 have even rank.
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It follows that all elliptic curves over quadratic fields with a point of order 13 have even rank.
In exactly the same way as for elliptic curves with $\mathbb{Z}/18\mathbb{Z}$ torsion, it can be shown that elliptic curves with 13-torsion cannot exist over imaginary quadratic fields.
The modular curve $X_1(22)$ is a genus 6 curve, which is \textit{bielliptic}, meaning that it has a degree 2 map to an elliptic curve (11A3 in this case).
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The moduli interpretation of $Y_1(22)/\iota$ implies that every elliptic curve with a point of order 22 over a quartic field $L$ is a $K$-curve (meaning that it is isogenous to all of its $\text{Gal}(\overline{K}/K)$-conjugates), where $K$ is a quadratic subfield of $L$. 
The modular curve $X_1(22)$ is a genus 6 curve, which is *bielliptic*, meaning that it has a degree 2 map to an elliptic curve ($11A3$ in this case).

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The moduli interpretation of $Y_1(22)/\iota$ implies that every elliptic curve with a point of order 22 over a quartic field $L$ is a $K$-curve (meaning that it is isogenous to all of its $\text{Gal}(\overline{K}/K)$-conjugates), where $K$ is a quadratic subfield of $L$.

Furthermore, $\text{End}(\text{Res}_{L/K} E) \cong \mathbb{Z}[\sqrt{-2}]$, so it follows that every elliptic curve with a point of order 22 over a quartic field has even rank.
Thank you for your attention!