Abelian varieties over function fields and independence of ℓ-adic representations

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September 2012
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Plan

Definitions and Notation

ℓ - an odd prime, $\mathcal{L} = \{\ell : \quad \text{odd primes}\}$

K - a field (later on; finitely generated field)

$G_K = Gal(\bar{K}/K)$ - absolute Galois group

Assume

for every $\ell \in \mathcal{L}$ there is a representation (= cont. homomorphism) $\eta_\ell : G_K \longrightarrow Gl_n(\mathbf{Z}_l)$, and n is independent of ℓ

Denote by: $\eta : G_K \longrightarrow \prod_{\ell \in \mathcal{L}} Gl_n(\mathbf{Z}_\ell)$ the map induced by η_ℓ's.

Definition

- family $(\eta_\ell)_{\ell \in \mathcal{L}}$ is **independent** (over K) if $\eta(G_K) = \prod_{\ell \in \mathcal{L}} \eta_\ell(G_K)$
- family $(\eta_\ell)_{\ell \in \mathcal{L}}$ is **almost independent** if $\eta(H) = \prod_{\ell \in \mathcal{L}} \eta_\ell(H)$ for an open subgroup $H \subset G_K$.

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Define: $K_l = K(\eta_\ell) := \bar{K}^{\ker(\eta_\ell)}$,

then the family $(\eta_\ell)_{\ell \in \mathcal{L}}$ is independent **iff** the family of fields $(K(\eta_\ell))_{\ell \in \mathcal{L}}$ is $K-$linearly disjoint.

EXAMPLES

Let K be a finitely gen. field over $\mathbf{Q}$, i.e., finite ext. of $\mathbf{Q}(t_1\ t_2, \ldots, t_s)$.

(**1**) Let

$$\epsilon_\ell : G_K \longrightarrow \mathbf{Z}_\ell^\times$$

be the cyclotomic character. Classically known that $(\epsilon_\ell)_{\ell \in \mathcal{L}}$ is independent. Here $K(\epsilon_\ell) = K(\mu_{\ell^\infty})$ is the cyclotomic extension.

(**2**) Let A/K be an abelian variety of dim. g and let

$$\rho_{\ell,A} : G_K \longrightarrow Gl_{2g}(\mathbf{Z}_\ell) = Aut(T_\ell(A))$$

be the Tate module representation. Here $K(\rho_{\ell,A}) = K(A[l^\infty])$ is the field of ℓ-division points.

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If *trdeg* $K = 0$ and $\dim A = 1$,then $(\rho_{\ell,A})_{\ell\in\mathcal{L}}$ is almost independent.

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Same true (as for elliptic curves) for $\dim A > 1$.

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Is the family $(\rho_{\ell,A})_{\ell\in\mathcal{L}}$ almost independent, for K finitely gen. over the rationals ?

Gajda and Petersen (2011) **YES**

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$K-$ a fin. gen. field over $\mathbf{Q}$

$X/K-$ a separated scheme of finite type over K

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Relation with the Tate module - as Galois modules:

$$T_\ell(A) \otimes \mathbf{Q}_\ell = H^1_{et}(\check{A}, \mathbf{Q}_\ell(1)).$$

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Theorem A (G and Sebastian Petersen, to appear in Compositio)

If K is a finitely gen. field of char. zero and X is a separated scheme of finite type over K, then the family $(\eta_{\ell,X}^{(q)})_{\ell\in\mathcal{L}}$ is **almost independent**.

Important ingredients of the proof (more details below):

- Theorem B, below - an extension of Serre's criterion for linear independence of the family $(\eta_\ell)_{\ell\in\mathcal{L}}$. We use the classical paper by Katz and Lang on the π_1^{et}, for X smooth and proper
- for non smooth X we use alterations of de Jong - as in Katz and Laumon paper (1996).
- a specialization argument - reduction to the number field case (= Serre's theorem of 2010).

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It follows (from Theorem A):

Corollary (G and S.Petersen)
If A/K is an abelian variety over a fin. gen. field of zero char, then there exists a finite extension E/K such that the family of division fields $(E(A[l^\infty]))_{\ell\in\mathcal{L}}$ is $E-$linearly disjoint.

Remark (work in progress in positive char) a similar theorem holds over char. $p > 0$:

Theorem (G.Böckle, G.W., S.Petersen, 2012)
For K a fin. gen. field of char. $p > 0$, then the family $(\eta^{(q)}_{\ell,X})_{\ell\in\mathcal{L}}$ is almost independent over the field $\bar{\mathbf{F}}_p K$.

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For E, any field extension of $\mathbf{Q}$ we define **the constant field** of E:

$$\kappa_E := \{x \in E : \quad x \text{ is algebraic over } \mathbf{Q}\}$$

For an algebraic extension E/K we have:

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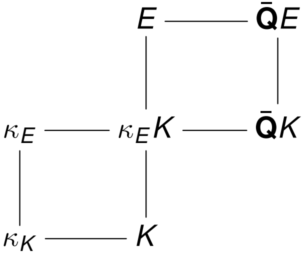

Denote

K - the function field of an affine, normal $\mathbf{Q}$-variety S
$S^{(E)}$ - the normalization of S in E

Call

- E/K **constant** if $\kappa_E K = E$
- E/K **geometric** if $\kappa_E = \kappa_K$
- E/K **unramified along** S if the map $S^{(E')} \longrightarrow S$ is etale for every finite subextension E'/K
- $K_{S,nr}$ - the maximal unramified along S ext. of K
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Note that $\pi_1^{et}(S) = Gal(K_{S,nr}/K)$

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Facts (EGA V)

For an abelian variety A/K

- (replacing S by an affine open, if necessary) A extends to an abelian scheme $\mathcal{A} \longrightarrow S$ (i.e., $\mathcal{A}_\eta = A$)
- $\mathcal{A}[n] \longrightarrow S$ is a finite etale group scheme (since residue chars of S are 0)

  Hence each

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- for $P \in S$ a closed point

$A_P := \mathcal{A} \times_S k(P)$ - the corr. fibre at P over the number field $k(P)$

P_{nr} - a point of S_{nr} over P. Easy to check: $k(P_{nr}) = \bar{\mathbf{Q}}$. We have a **specialization diagram:**

$$\begin{array}{ccc}
G(K_{S,\mathrm{nr}}/K) & \xrightarrow{\rho_{\ell,A}} & Aut(V_\ell(A)) \\
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Serre's result (1986) implies that $(\rho_{\ell,A_P})_{\ell\in\mathcal{L}}$ is almost independent, **hence** $(\rho_{\ell,A}|D(P_{\mathrm{nr}}))_{\ell\in\mathcal{L}}$ is almost independent.

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Serre's result (1986) implies that $(\rho_{\ell,A_P})_{\ell \in \mathcal{L}}$ is almost independent, **hence** $(\rho_{\ell,A}|D(P_{\mathrm{nr}}))_{\ell \in \mathcal{L}}$ is almost independent.

- for $P \in S$ a closed point
  $A_P := \mathcal{A} \times_S k(P)$ - the corr. fibre at P over the number field $k(P)$
  P_{nr} - a point of S_{nr} over P. Easy to check: $k(P_{nr}) = \bar{\mathbf{Q}}$. We have a **specialization diagram:**

$$
\begin{array}{ccc}
G(K_{S,\mathrm{nr}}/K) & \xrightarrow{\rho_{\ell,A}} & Aut(V_\ell(A)) \\
\uparrow (\hookrightarrow) & & \\
D(P_{nr}) := D_{K_{S,\mathrm{nr}}/K}(P_{\mathrm{nr}}) & & \downarrow \equiv \\
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Remarks

- in terms of fields this means:

$$K_{\ell_1} K_{\ell_2} \dots K_{\ell_s} \cap K_{\ell_{s+1}} \quad \text{is geometric over } K$$

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Let S be a normal $\mathbf{Q}$-variety and let $K = \mathbf{Q}(S)$.

Consider a family of continuous representations

$$\eta_\ell : \pi_1^{et}(S) \longrightarrow Gl_n(\mathbf{Z}_\ell),$$

where $\ell \in \mathcal{L}$

$(*)$ Assume
there is a point $\hat{P} \in S_{nr}$ such that $(\eta_\ell | D)_{\ell \in \mathcal{L}}$ is almost independent for an open subgroup $D \subset D(\hat{P})$.

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- Theorem B **implies** Theorem A by the base change theorems in cohomology due to Katz and Laumon (1996) and by Serre and Illusie result (2010).
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Plan

Problem

Compute images of $\rho_{\bar{\ell},A}$ and $\rho_{\ell,A}$ in terms of linear algebraic groups

Serre (1972)

If A/F is a non-CM elliptic curve (where F is a # field), then $Im\,\rho_{\bar{\ell},A} = GL_2(\mathbb{F}_\ell)$ for almost all ℓ.

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If A/F is a principally polarized Abelian variety, $g = 2, 6$, or is an odd integer, and $End\,A = \mathbb{Z}$, then $(Im\,\rho_{\bar{\ell},A})' = Sp_{2g}(\mathbb{F}_\ell)$ for almost all ℓ.

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$$Im\,\rho_{\bar{\ell},A} = \{M \in GL_2(\mathcal{O}_E/\ell) : \quad \det M \in \mathbb{F}_\ell^\times\}$$

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Less classical result:

C.Hall (2009)
Let A/F be a principally polarized Abelian variety and $End\ A = \mathbb{Z}$. Assume that there is a prime of $\mathcal{O}_F$ at which A has semistable reduction of toric dimension one. Then $(Im\,\rho_{\bar{\ell},A})' = Sp_{2g}(\mathbb{F}_\ell)$ for almost all ℓ.

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Assume:

K - a finitely generated field (over its prime field), e.g.,

$K = F(S)$ the function field of a smooth, geom. connected variety S over a global field k

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$\mathcal{A} \to S$ is an abelian scheme over S, and $\mathcal{A}_\eta = A$.

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Compute images of Galois representations attached to A at ℓ in terms of linear algebraic groups (A over a finitely generated field K).

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Compare images of Galois at the generic $\mathcal{A}_\eta$ and special fibres $\mathcal{A}_s$ at closed points $s \in S$.

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Abelian varieties of Hall type

Definition

A/K is of **Hall type** if:

(1) $End\, A = \mathbf{Z}$ and

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Recall:

Condition (2) means that there is an exact sequence of group schemes:

$$1 \longrightarrow T \longrightarrow \mathcal{N}_v^o \longrightarrow B \longrightarrow 0$$

$T = \mathbb{G}_m$ - a torus over $\kappa(v) = \mathcal{O}_v/m_v$

B - an abelian variety over $\kappa(v)$

$\mathcal{N} \longrightarrow Spec\, \mathcal{O}_v$ - the Neron model of A over $\mathcal{O}_v$= integers of K_v,

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Let

$f \in \mathbf{Z}[x]$ - a monic, square-free polynomial, $deg\, f = n \geq 5$

C_f - smooth, projective curve; affine part $y^2 = f(x)$

$A = Jac(C_f)$

Properties

(a) Zarhin proved in 2007: If $Gal(Spl(f)/\mathbf{Q}) = S_n$, then $End\, A = \mathbf{Z}$.

(b) A has semistable reduction of toric dim. one at a prime p : **if** $\bar{f} = f_1 f_2$ mod p, $(f_1, f_2) = 1$ and $f_1 = (x - \alpha)^2$ with $\alpha \in \mathbf{F}_p$, and f_2 is square-free of degree $n-2$.

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Theorem C (Arias-de-Reyna, G., to appear in JPAA)

Every abelian variety A/K of Hall type defined over a finitely generated field (of arbitrary characteristic) has **big monodromy**, i.e., the image $Im\,\rho_{\bar{\ell},A}$ contains $Sp_{2g}(\mathbb{F}_\ell)$, for $\ell >> 0$.

- Hall proved Theorem C for number fields in 2009.

Proof of Theorem C uses:

- $A[l]$ is a simple $\mathbb{F}_l[Im\,\bar{\rho}_l]$-module
- $Im\,\bar{\rho}_l$ contains a transvection, i.e., an unipotent endomorphism of $\mathbb{F}_l^{2g}$ with dim $Eig(u,1) = 2g-1$
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A/K an abelian variety over a field K

$G_K^e := G_K \times G_K \times \cdots \times G_K, \quad$ for $\quad e \geq 1$

$K(\sigma) := \bar{K}^{<\sigma>}$, for $\sigma \in G_K^e$ the subgroup $< \sigma > \subset G_K$ is generated by coordinates of σ.

Geyer-Jarden conjecture (1978)

(a) For almost all $\sigma \in G_K$, the group $A(K(\sigma))_{Tors}$ is infinite
- moreover $A(K(\sigma))[l] \neq 0$ for infinitely many l
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The **GJC** holds **true** for all abelian varieties with **big monodromy**.

In particular Theorem D (+ Theorem C + extension of Serre's theorem to fin. gen. fields) imply:

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The **GJC** is **true**:

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Important proof ingredients of Theorem D:

- the classical lemma of Borel and Cantelli of measure theory
- counting of certain symplectic matricies (e.g., with eigenvalue 1) in the Galois image - a delicate matter, if char > 0.

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Proof sketch of part (b) of **GJC**, for A as above:

Claim

Let $e \geq 2$ and A/K is an abelian variety over fin. gen. field with big monodromy. **Then** for almost all $\sigma \in G_K^e$ there exists **only** finitely many primes l s.t.

$$A(K_{sep}(\sigma)[l]) \neq 0.$$

Lemma of Borel-Cantelli

Let $\{X_l\}_{l \in \mathcal{L}}$ be a sequence of measurable sets in a measure space (X, μ) s.t. $\mu(X) = 1$.

(b) If $\sum_{l \in \mathcal{L}} \mu(X_l) < \infty$, **then** almost every $x \in X$ (outside of a set of measure zero) belongs to at most finitely many of $X_l's$.

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Let $\{X_l\}_{l\in\mathcal{L}}$ be a sequence of measurable sets in a measure space (X, μ) s.t. $\mu(X) = 1$.

(b) If $\sum_{l\in\mathcal{L}} \mu(X_l) < \infty$, **then** almost every $x \in X$ (outside of a set of measure zero) belongs to at most finitely many of $X_l's$.

(a) If $\mu(\cap_{l\in I} X_l) = \prod_{l\in I} \mu(X_l)$ for any finite $I \subset \mathcal{L}$ and $\sum_{l\in\mathcal{L}} \mu(X_l) = \infty$,

then almost every $x \in X$ belongs to infinitely many of the $X_l's$.

To prove Claim we take in (b) of Borel-Cantelli Lemma:

$$\mathcal{L} = \{primes\}$$

$$X_l := \{\sigma \in G_K^e : \quad A(K(\sigma))[l] \neq 0\},$$

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$$\mu(G^e_{K(P)}) = [K(P) : K]^{-e}$$

Since $A[l]$ is a simple $\mathbf{F}_l(G_K)$-module, **we get** easily:

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But:

$$G_l \supset Sp_{2g}(\mathbf{F}_l) \quad \text{for all } l > l_0 \quad \text{(big monodromy)}$$

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$$s_l := |Sp_{2g}(\mathbf{F}_l)| = l^{g^2} \prod_{i=1}^{g} (l^{2i} - 1)$$

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THANK YOU

an easy proof

Proposition (Serre)

If A/K is a principally polarized Abelian variety over K of char. zero, $g = 2, 6$, or is an odd integer, and $End\, A = \mathbb{Z}$, then A

$$(Im\, \bar{\rho}_l)' = Sp_{2g}(\mathbb{F}_l)$$

for almost all l.

- Proposition extends Serre's theorem of 1986 to finitely generated fields of zero characteristic
- A similar result for elliptic curves is true over $\mathbf{F}_p(t)$ (proven by Igusa in the 50th).
- **Question** Does Igusa's theorem hold true in higher dimensions (say, for A over $\mathbf{F}_p(t)$) ?

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Proof of **Proposition** (by induction on $trdeg(K/\mathbf{Q})$):

$trdeg(K/\mathbf{Q}) = 0$ - the open image theorem of Serre.

Assume that $trdeg(K/\mathbf{Q}) = d > 0$

$F/\mathbf{Q}$ - fin. gen. field, $trdeg(F/\mathbf{Q}) = d - 1$

C/F - smooth curve over F with $F(C) = K$

A - extends to an abelian group scheme $\mathcal{A} \longrightarrow C$.

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