MPRI – Cours 2.12.2

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Lecture I: Primality

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The slides are available on `http://www.lix.polytechnique.fr/Labo/Francois.Morain/MPRI/2012`

I. Compositeness tests.

II. Primality tests.

I. Compositeness tests

Z) Definition and classification.

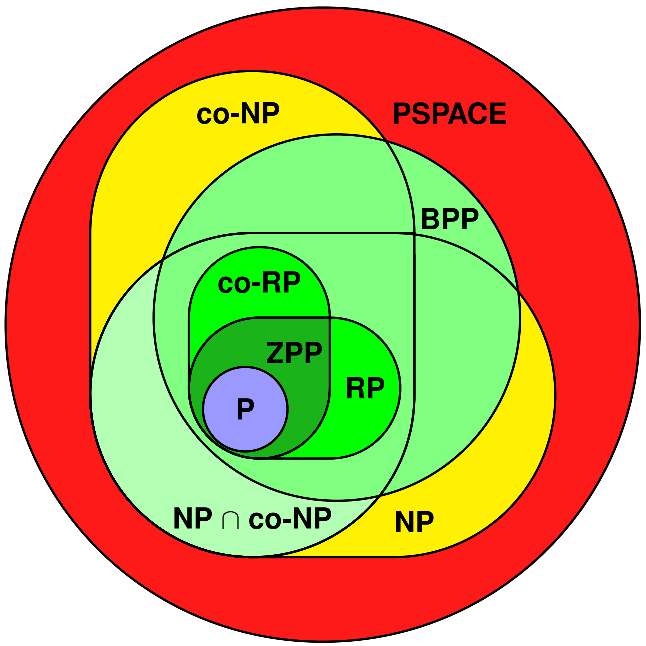
A) Fermat.

B) Euler-Solovay/Strassen.

C) Artjuhov-Miller-Rabin.

D) Other tests.

Z) Definition and classification



My view of randomized algorithms

Def. A Monte Carlo algorithm for deciding that $X \in \mathbb{A}$ returns yes or I don't know:
Proba("yes" $|X \notin \mathbb{A}) = 0$
Proba("I don't know" $|X \in \mathbb{A}) \leq 1 - \delta$, for absolute $0 < \delta < 1$.

Def. A decision problem is in **RP** if there exists a polynomial time Monte Carlo algorithm that solves it.

Rem. $\neq$ error on the answer; or a failure in the computer.

Def. A Las Vegas algorithm answers yes, no or I don't know:
Proba("yes" $|X \notin \mathbb{A}) = 0$, Proba("no" $|X \in \mathbb{A}) = 0$
Proba("I don't know") $\leq 1 - \delta$.

Def. ZPP = **RP**$\cap$ **co-RP**.

Compositeness test: deciding that N is composite.

Primality test: deciding that N is prime.

A) Fermat

Idea: if $\gcd(a, N) = 1$, then $a^{N-1} \equiv 1 \bmod N$.

But: $2^{340} \equiv 1 \bmod 341$: pseudoprime to base 2 (psp-2).

Thm. There exists an infinite number of psp-2 numbers.

Thm. (Pomerance) For $x \geq x_0$: $x^{2/7} \ll P_2(x) \ll xL(x)^{-1/2}$ with $L(x) = \exp\{\log x \log\log\log x / \log\log x\}$.

Def. $P(N) = \#\{a \in (\mathbb{Z}/N\mathbb{Z})^*, a^{N-1} \equiv 1 \bmod N\}$.

Thm. If $N = \prod_i p_i^{\alpha_i}$, $P(N) = \prod_i \gcd(p_i - 1, N - 1)$.

Proof: ■

The test

function isComposite(N)

1. Choose a at random in $\mathbb{Z}/N\mathbb{Z} - \{0\}$.
2. Compute $g = \gcd(a, N)$; **if** $g > 1$, **then** return $(\texttt{yes}, g \mid N)$.
3. **if** $a^{N-1} \not\equiv 1 \bmod N$, **then** return $(\texttt{yes}, a)$
   otherwise return I don't know.

Cost. $O((\log N)\mathrm{M}(\log N))$; typically $O((\log N)^3)$, asymptotically $\tilde{O}((\log N)^2)$.

Prop. Proba("I don't know") $= P(N)/(N-1)$.

Proof. Probability of yes is:

$$\left(1 - \frac{\varphi(N)}{N-1}\right) + \frac{\varphi(N)}{N-1}\left(1 - \frac{P(N)}{\varphi(N)}\right). \square$$

Rem. if N is prime, proba is 1...!

Carmichael numbers

Def. composite N s.t. $P(N) = \varphi(N)$.

Ex. 541.

Rem. $P(N)/(N-1) = \varphi(N)/(N-1)$ close to 1.

Thm.(Alford, Granville, Pomerance, 1992) There are infinitely many Carmichael numbers.

More properties of Carmichael numbers:
1. N is squarefree.
2. For all $p \mid N$, $p - 1 \mid N - 1$ (equivalently $\lambda(N) \mid N - 1$).
3. N has at least three prime factors.

B) Euler and Solovay-Strassen

Idea: (Euler) if N is prime and $\gcd(a, N) = 1$, then $a^{(N-1)/2} \equiv \left(\frac{a}{N}\right) \bmod N$.

Pb: $2^{(1105-1)/2} \equiv \left(\frac{2}{1105}\right) \bmod 1105$; this is an Euler pseudoprime to base 2 (epsp-2). There are an infinite number of them.

Prop. $E_2(x) \leq P_2(x)$.

Def. $\mathcal{E}(N) = \{a \in (\mathbb{Z}/N\mathbb{Z})^*, a^{(N-1)/2} \equiv \left(\frac{a}{N}\right) \bmod N\}$; $E(N) = \#\mathcal{E}(N)$.

Prop. $\mathcal{E}(N)$ is proper subgroup of $(\mathbb{Z}/N\mathbb{Z})^*$.

Coro. $E(N)/\varphi(N) \leq 1/2$.

The exact value of $E(N)$

Thm. (Monier) Write $N = \prod_{i=1}^k p_i^{\alpha_i}$ where p_i are distinct odd primes, $\alpha_i \geq 1$. Write $N = 1 + 2^s t$ with t odd and $p_i = 1 + 2^{s_i} t_i$ with t_i odd. Assume $s_1 \leq s_2 \leq \cdots \leq s_k$ and put $T_i = \gcd(t, t_i)$, $n_i = \gcd((N-1)/2, p_i - 1)$ and $\mathcal{N} = \prod_i n_i$. Then

$$E(N) = \delta(N)\mathcal{N}$$

where

$$\delta(N) = \begin{cases} 2 & \text{if } s = s_1 \\ 1/2 & \text{if } \exists i, \alpha_i \text{ odd and } s_i < s \\ 1 & \text{otherwise.} \end{cases}$$

Proof. exercise.

The test

function isComposite2(N)

1. Choose a at random in $\mathbb{Z}/N\mathbb{Z} - \{0\}$.
2. Compute $g = \gcd(a, N)$; **if** $g > 1$, **then** return $(\texttt{yes}, g \mid N)$.
3. **If** $a^{(N-1)/2} \not\equiv \left(\frac{a}{N}\right) \bmod N$ **then** return $(\texttt{yes}, a)$
   else return I don't know.

Prop. Proba("I don't know")$= E(N)/(N-1) \leq 1/2$.

Coro. isComposite? $\in \mathbf{RP}$ (hence **isPrime?**$\in \mathbf{co-RP}$).

Miller (1975): $a = 2, 3, \ldots$; Ankeny–Montgomery–Lenstra–Bach: if an adequate Riemann hypothesis is true, then the smallest witness is $< 2(\log N)^2$, yielding a deterministic $O((\log N)^3 \mathsf{M}(\log N))$ algorithm.

C) Artjuhov-Miller-Rabin

N being odd, write $N - 1 = 2^s t$ with $s \geq 1$ and odd t.

$$a^{N-1} - 1 = (a^t - 1)(a^t + 1)(a^{2t} + 1) \cdots (a^{2^{s-1}t} + 1)$$

$$(AMR_a) : a^t \equiv 1 \bmod N \text{ or } \exists j, 0 \leq j < s, a^{2^j t} \equiv -1 \bmod N.$$

Pb: $N = 2047 = 23 \times 89$ is s.t. $N - 1 = 2 \times 1023$ and $2^{(N-1)/2} \equiv 1 \bmod N$: strong-pseudoprime to base 2 (spsp-2).

Thm. spsp-a $\Rightarrow$ epsp-a.

Def. $F(N) = \#\{a \in (\mathbb{Z}/N\mathbb{Z})^*, (AMR_a) \text{ is satisfied}\}$.

Thm. (Monier)

$$F(N) = \left[1 + \frac{2^{ks_1} - 1}{2^k - 1}\right] \prod_{i=1}^k T_i.$$

Coro. $F(N)/(N-1) \leq 1/4$.

The test

function isComposite3(N)

1. Choose a at random in $\mathbb{Z}/N\mathbb{Z} - \{0\}$.
2. Compute $g = \gcd(a, N)$; **if** $g > 1$, **then** return $(\texttt{yes}, g \mid N)$.
3. **If** (AMR_a) **then** return $(\texttt{yes}, a)$
   else return I don't know.

Prop. Proba("I don't know")$= F(N)/(N-1)$.

Numerical tables

x	$P_2(x)$	$E_2(x)$	$F_2(x)$	$C(x)$	$\pi(x)$
10^4	22	12	5	7	1229
10^5	78	36	16	16	9592
10^6	245	114	46	43	78498
10^7	750	375	162	105	664579
10^8	2057	1071	488	255	5761455
10^9	5597	2939	1282	646	50847534
10^{10}	14884	7706	3291	1547	455052511
25×10^9	21853	11347	4842	2163	1091987405
10^{11}	38975	20417	8607	3605	4118054813
10^{12}	101629	53332	22407	8241	37607912018
10^{13}	264239	139597	58897	19279	346065536839
10^{14}				44706	3204941750802
10^{15}				105212	29844570422669
10^{16}				246683	279238341033925

Building primes?

function randomProbablePrime(b)

repeat
  choose odd N at random in $[2^{b-1}, 2^b[$
until N passes k tests.

$p_{b,k} = \text{Proba}(X = N \text{ is composite} | Y_k = N \text{ passes } k \text{ tests}) = ?$

Rem. What we know is
$\text{Proba}(Y_k = N \text{ passes } k \text{ tests} | X = N \text{ is composite}) \leq (1/4)^k$.

Thm. (Burthe, 1996) $\forall b \geq 2, \forall k \geq 1, p_{b,k} \leq 4^{-k}$.

Other tests

Goal: reduce the non-answer probability while keeping the computations fast.

- Algebraic extensions: Lucas (degree 2), Adams & Shanks, Gurak.
- Elliptic curves: Gordon.
- Combinations of the preceding: no examples known of spsp-a and Lucas pseudoprime, for instance.
- Frobenius pseudoprimes à la Grantham: $\leq 1/7710$. Cf. also Zhang.

II. Primality tests

A) Fermat

B) En route for **P**.

C) Agrawal, Kayal, Saxena.

A) Fermat

Thm. N is prime if and only if $(\mathbb{Z}/N\mathbb{Z})^*$ is cyclic of ordre $N-1$:

$$\left.\begin{array}{l} a^{N-1} \equiv 1 \bmod N \\ \forall\, p \mid N-1,\ a^{\frac{N-1}{p}} \not\equiv 1 \bmod N \end{array}\right\} \Rightarrow N \text{ is prime}$$

Certificate: $(N, \{p \mid N-1\}, a)$ $\Rightarrow$**isPrime?**$\in$**NP**.

Thm. (**Pocklington**, 1914) Let s s.t. $s \mid N-1$

$$\left.\begin{array}{l} a^{N-1} \equiv 1 \bmod N \\ \forall\, q \text{ prime } \mid s, \gcd(a^{\frac{N-1}{q}} - 1, N) = 1 \end{array}\right\} \Rightarrow \forall\, p \mid N,\ p \equiv 1 \bmod s$$

Coro. $s > \sqrt{N} \Rightarrow N$ is prime.

Rem. factorisation is not polynomial time in the classical world (see later), but polynomial quantic; search for a is not either (except if Riemann is true or randomized approach).

Example of use

Hyp. We know how to find all prime factors < 20.

$$\begin{array}{rcrl} N_0 &=& 100003, & N_0 - 1 = 2 \times 3 \times 7 \times N_1, \\ N_1 &=& 2381, & N_1 - 1 = 2^2 \times 5 \times 7 \times 17 \end{array}$$

p	2	5	7	17
$3^{(N_1-1)/p} \bmod N_1$	2380	1347	1944	949

$\Rightarrow N_1$ is prime

$$s = N_1 > \sqrt{N_0}$$

$$2^{N_0-1} \equiv 1 \bmod N_0, \gcd(2^{(N_0-1)/N_1} - 1, N_0) = 1$$

$\Rightarrow N_0$ is prime

Rem. We have got a (recursive) primality proof of depth $O(\log N)$.

The $N+1$ test

For a_0 and a_1 integers, let:

$$A_N = A_N(a_0, a_1) = \mathbb{Z}/N\mathbb{Z}[T]/(T^2 + a_1 T + a_0)$$

and $\Delta = a_1^2 - 4a_0$.
Elements of A_N are $u + v\alpha$ with u, v dans $\mathbb{Z}/N\mathbb{Z}$, computations made using $\alpha^2 = -a_1\alpha - a_0$.
Thm. Let p be a prime $\nmid \Delta$.

- if $(\Delta/p) = -1$, then $A_p \sim \mathbb{F}_{p^2}$;
- if $(\Delta/p) = +1$, then $A_p \sim \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Proof : If $(\Delta/p) = -1$, $T^2 + a_1 T + a_0$ is irreducible, hence we recover the classical construction of $\mathbb{F}_{p^2}$.
If $(\Delta/p) = +1$, $T^2 + a_1 T + a_0 = (T-u)(T-v)$ with $u \not\equiv v \bmod p$.
Therefore

$$A_p \sim (\mathbb{Z}/p\mathbb{Z})[T]/(T-u)) \times (\mathbb{Z}/p\mathbb{Z})[T]/(T-v)) \sim \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}.\square$$

Thm. Let N be an odd integer. Assume that we found a_0, a_1 s.t. $\Delta = a_1^2 - 4a_0$ satisfies $(\Delta/N) = -1$. Write $N+1 = \prod_i q_i^{\beta_i}$. Suppose we have found $\theta \in A_N = A_N(a_0, a_1)$ s.t.

$$\theta^{N+1} = 1 \text{ in } A_N,$$

and for all i :

$$\theta^{(N+1)/q_i} = u_i + v_i\alpha \text{ with } (u_i - 1, v_i, N) = 1.$$

Then N is prime.
Proof : assume N is composite and let $p \mid N$ with $p \leq \sqrt{N}$.
Reduce A_N mod p towards A_p :

$$\tau = \theta \bmod p = (u \bmod p) + (v \bmod p)\alpha.$$

We get

$$\tau^{N+1} = 1 \text{ in } A_p,$$

and

$$\tau^{(N+1)/q_i} \neq 1 \text{ in } A_p$$

which proves τ has ordre $N+1$ in $(A_p)^*$.
Hence $N+1 \leq \#A_p = p^2$, contradiction. $\square$

Choosing θ**:** using $\overline{\alpha} = -a_1 - \alpha$ (conjugate), enough to choose

$$\theta = \frac{\alpha + m}{\overline{\alpha} + m} = \frac{(m^2 - a_0) + (2m - a_1)\alpha}{m(m - a_1) + a_0}$$

for varying m.

Ex. Consider $N = 101$; $N + 1 = 2 \times 3 \times 17$. Take $a_1 = -2$, $a_0 = -1$, $\Delta = 8$ and $\left(\frac{8}{101}\right) = \left(\frac{2}{101}\right) = -1$. Take $\theta = 1 + 2\alpha$ (using $m = 1$)

$$\theta^{102} = 1, \theta^{102/2} = 100, \gcd(100 - 1, N) = 1,$$

$$\theta^{102/3} = 47T + 3, \gcd(3 - 1, 47, N) = 1,$$

$$\theta^{102/17} = 23T + 85, \gcd(85 - 1, 23, N) = 1.$$

Remarks

- Pocklington-like theorems exist.
- Deduce from this the degree 2 pseudoprimes.
- All this can be reformulated in terms of Lucas sequences.
- **Lucas-Lehmer:** $M_m = 2^m - 1$ is prime iff for $L_0 = 4$, $L_{n+1} = L_n^2 - 2 \bmod M_m$, one has $L_{m-2} = 0$ [using $\sqrt{3}$].

  $\Rightarrow$ largest known primes, e.g., $M_{43112609}$ with $12, 978, 189$ decimal digits.

  Lower bound (?) for primality proving algorithms: $O((\log N)\mathsf{M}(M_p))$ (super fast arithmetic!).

B) En route for **P**

- Gauss and Jacobi sums: L. Adleman, C. Pomerance, S. Rumely (1980, 1983); H. Cohen, H. W. Lenstra, Jr (1981 – 1984) ; H. Cohen, A. K. Lenstra (1982, 1987). W. Bosma & M.-P. van der Hulst (1990) ; P. Mihăilescu (1998). deterministic $O((\log N)^{c_1 \log\log\log N})$.
- almost **RP**: Goldwasser and Kilian using elliptic curves (1986); practical algorithm by Atkin (1986; later FM).
- **RP**: Adleman and Huang using hyperelliptic curves (1986ff).

C) Agrawal, Kayal, Saxena (AKS)

First idea: (Agrawal, Biswas – 1999)

Prop. N is prime iff $P(X) = (X + 1)^N - X^N - 1 \equiv 0 \bmod N$.

In pratice: choose $Q(X) \in \mathbb{Z}/N\mathbb{Z}[X]$ at random of degree $O(\log N)$. If

$$(X + 1)^N \not\equiv X^N + 1 \bmod (Q(X), N)$$

then N is composite.

The probability of failure is bounded by $1 - 1/(4 \log N)$.

Conjecture: If N is composite, there exists $1 \leq r \leq \log N$ s.t. $P(X)$ is not divisible by $X^r - 1$ modulo N.

Agrawal, Kayal, Saxena

Thm. Let N, s be integers, r a prime number and $q = P(r-1)$. If:

(0)
$$\binom{q-1+s}{s} > N^{2\lfloor\sqrt{r}\rfloor};$$

(i) $N \neq M^k$, $k > 1$;

(ii) N has no prime factor $\leq s$;

(iii) $N^{(r-1)/q} \bmod r \notin \{0, 1\}$;

(iv) $\forall a, 1 \leq a \leq s, (X-a)^N \equiv X^N - a \bmod (X^r - 1, N)$;

then N is prime.

For a proof, see FM's Bourbaki article.

Analysis

Cost: s computations of X^N modulo $(X^r - 1, N)$; one computation costs $O(\log N)$ products of degree r polynomials, hence:

$$O(s(\log N)\mathsf{M}_P(r)\mathsf{M}(\log N)).$$

Prop. If $s = \lfloor 2\lfloor\sqrt{r}\rfloor \log N / \log 2\rfloor + 1$ and $q \geq 2s$, then

$$\binom{q-1+s}{s} > N^{2\lfloor\sqrt{r}\rfloor}.$$

Proof:
$$\binom{q-1+s}{s} > (q/s)^s \geq 2^s > N^{2\lfloor\sqrt{r}\rfloor}.$$

Coro. $O((\log N)^2 r^{1/2}\mathsf{M}_P(r)\mathsf{M}(\log N))$.

Analytical number theory: we can find $r = (\log N)^{2/(2\delta-1)}$ for $\delta \in]0.5, 0.676]$.

At last...

Using $r = (\log N)^{2/(2\delta-1)}$.
Coro. There exists a deterministic primality proving algorithm whose running time is

$$O\left((\log N)^{(8\delta+1)/(2\delta-1)}\right)$$

using $\mathsf{M}_P(r) = r^2$, $\mathsf{M}(\log N) = (\log N)^2$; and

$$\tilde{O}\left((\log N)^{6\delta/(2\delta-1)}\right)$$

with $\mathsf{M}_P(r) = \tilde{O}(r)$, $\mathsf{M}(\log N) = \tilde{O}(\log N)$.
Proof:
$$L^2 r^{1/2}\mathsf{M}_P(r)\mathsf{M}(\log N) = L^{2+2/(2\delta-1)+4/(2\delta-1)+2}.\Box$$

Ex. (AKS original) $\delta = 2/3$: 19, 12; $\delta = 1$ (Sophie Germain): 9, 6.

Rem. Jacobi $O((\log N)^{c \log\log\log N})$, ECPP $\tilde{O}((\log N)^4)$.

Rem. Non effective.

What next?

- cf. D. Bernstein homepage for more on the history of improvements to the basic test.
- Including: H. W. Lenstra, Jr. ($\tilde{O}_{eff}((\log N)^{12})$ or $\tilde{O}((\log N)^8)$), S. David.
- Cleaner version of AKS: $\tilde{O}_{eff}((\log N)^{10.5})$ or $\tilde{O}((\log N)^{7.5})$.
- H. W. Lenstra, C. Pomerance : $\tilde{O}_{eff}((\log N)^6)$.
- P. Berrizbeitia / Q. Cheng :
  Let r prime s.t. $r^\alpha \mid\mid N-1$, $r \geq \log^2 N$; $1 < a < N$ s.t.
  $a^{r^\alpha} \equiv 1 \bmod N$, $\gcd(a^{r^{\alpha-1}} - 1, N) = 1$,
  $(X+1)^N = X^N + 1 \bmod (X^r - a, N)$, then N is prime. Heuristic complexity would be $\tilde{O}((\log N)^4)$ for these numbers.
- D. Bernstein, P. Mihăilescu: use $e \mid N^d - 1$; inject cyclotomic ideas, $\tilde{O}((\log N)^4)$.
- D. Bernstein has an AKS example for $2^{1024} + 643$ (13 hours on 800 MHz PC, 200 Mb memory).

V. Conclusions for primality

Which algorithm?

- easy to understand / implement, fast: compositeness tests;
- fast, proven: Jacobi;
- fast, heuristic: ECPP;
- certificate: ECPP;
- deterministic polynomial: AKS.

Some records:
14/07/03: FM, 7000dd with mpifastECPP.
19/08/03: Franke/Kleinjung/Wirth, 10000dd.
06/06: FM, 20,562 dd with mpifastECPP.
15/10/10: FM, 25,050 dd with mpifastECPP (2000 CPU days).
11/12/12: Franke/Kleinjung/Decker/Grosswendt, 30,008 using CIDE.

What's left to be done?

Open questions:

- Is $\tilde{O}((\log N)^4)$ the best running time for all numbers?
  Compare: $\tilde{O}((\log N)^2)$ for Fermat or Mersenne numbers.

  Claim (Lukes, Patterson, Williams): $\tilde{O}((\log N)^3)$ under GRH? (pseudosquares or pseudocubes).

- Combination of tests?