East model: mixing time, cutoff and dynamical heterogeneities.

Fabio Martinelli

Dept. of Mathematics and Physics, Univ. Roma 3, Italy.

Warwick 2014
"Glassy Systems and Constrained Stochastic Dynamics"

Outline

- The East Model
  - Motivation.
  - Definition
- Mixing time and relaxation time
  - Front propagation.
  - Cutoff.
- Low temperature dynamics.
  - Coalescence and universality on finite scales.
  - Equivalence of time scales.
  - Dynamic heterogeneity.
- Scaling limit (conjectured)
- Extensions to higher dimensions.

Motivations

- The East process is a keystone for a general class of interacting particle systems featuring glassy dynamics:

Motivations

- The East process is a keystone for a general class of interacting particle systems featuring glassy dynamics:
  - Featureless stationary distribution (i.i.d.);

Motivations

- The East process is a keystone for a general class of interacting particle systems featuring glassy dynamics:
  - Featureless stationary distribution (i.i.d.);
  - Broad spectrum of time relaxation scales;

Motivations

- The East process is a keystone for a general class of interacting particle systems featuring glassy dynamics:
  - Featureless stationary distribution (i.i.d.);
  - Broad spectrum of time relaxation scales;
  - Cooperative dynamics;

Motivations

- The East process is a keystone for a general class of interacting particle systems featuring glassy dynamics:
  - Featureless stationary distribution (i.i.d.);
  - Broad spectrum of time relaxation scales;
  - Cooperative dynamics;
  - Huge relaxation times as some parameter is varied.

Motivations

- The East process is a keystone for a general class of interacting particle systems featuring glassy dynamics:
  - Featureless stationary distribution (i.i.d.);
  - Broad spectrum of time relaxation scales;
  - Cooperative dynamics;
  - Huge relaxation times as some parameter is varied.
  - Complex out-of-equilibrium dynamics.

Motivations

- The East process is a keystone for a general class of interacting particle systems featuring glassy dynamics:
  - Featureless stationary distribution (i.i.d.);
  - Broad spectrum of time relaxation scales;
  - Cooperative dynamics;
  - Huge relaxation times as some parameter is varied.
  - Complex out-of-equilibrium dynamics.
- The East process plays also a role in other unrelated MCMC e.g. the *upper triangular matrix walk* (Peres, Sly '11).

Motivations

- The East process is a keystone for a general class of interacting particle systems featuring glassy dynamics:
  - Featureless stationary distribution (i.i.d.);
  - Broad spectrum of time relaxation scales;
  - Cooperative dynamics;
  - Huge relaxation times as some parameter is varied.
  - Complex out-of-equilibrium dynamics.
- The East process plays also a role in other unrelated MCMC e.g. the *upper triangular matrix walk* (Peres, Sly '11).
- It attracted the interest of different communities: physics, probability, combinatorics.

Definition

- A "spin" $\omega_x \in \{0, 1\}$ is attached to every vertex of either $\Lambda = \{1, 2, \ldots, L\}$ or $\Lambda = \mathbb{N}$.
- Let π be the product Bernoulli(p) measure on $\{0, 1\}^\Lambda$:

$$\pi(\omega) \propto \exp(-\beta H(\omega)), \quad q = e^{-\beta}/(1 + e^{-\beta}).$$

where $H(\omega)$ = # of 0's in ω.

Definition

- A "spin" $\omega_x \in \{0,1\}$ is attached to every vertex of either $\Lambda = \{1, 2, \ldots, L\}$ or $\Lambda = \mathbb{N}$.
- Let π be the product Bernoulli(p) measure on $\{0,1\}^\Lambda$:

$$\pi(\omega) \propto \exp(-\beta H(\omega)), \quad q = e^{-\beta}/(1 + e^{-\beta}).$$

where $H(\omega) = \#$ of 0's in ω.

The East chain

1. For any vertex x with rate 1 do as follows:
   - independently toss a p-coin and sample a value in $\{0,1\}$ accordingly;
   - update ω_x to that value **iff** $\omega_{x-1} = 0$.
2. To guarantee irreducibility, the spin at $x = 1$ is always unconstrained ($\Leftrightarrow$ there is a *frozen* "0" at the origin).

Some general features

- The process evolves with *kinetic constraints*;
- The constraints try to mimic the cage effect observed in dynamics of glasses.
- The 0's are the *facilitating* sites;

Some general features

- The process evolves with *kinetic constraints*;
- The constraints try to mimic the cage effect observed in dynamics of glasses.
- The 0's are the *facilitating* sites;
- Reversible w.r.t. to π: the "constraint" at x does not involve the state of the process at x.
- π describes i.i.d random variables !

Some general features

- The process evolves with *kinetic constraints*;
- The constraints try to mimic the cage effect observed in dynamics of glasses.
- The 0's are the *facilitating* sites;
- Reversible w.r.t. to π: the "constraint" at x does not involve the state of the process at x.
- π describes i.i.d random variables !
- The process is ergodic for all $q \in (0, 1)$.
- It is not attractive/monotone: more 0's in the system allow more moves with unpredictable outcome (that's very frustrating...).
- No powerful tools like FKG inequalities, monotone coupling, censoring,... are available.

Few simple observations

- Two adjacent 'domains" of **1**'s:

$$\dots {\color{blue}0}\,\underbrace{\mathbf{1111111111111111111}\ {\color{red}\mathbf{0}}}_{L}\ \underbrace{\mathbf{111111111111111}{\color{blue}\mathbf{0}}}_{L'}\dots$$

- As long as the intermediate **0** does not flip, the second block of **1**'s evolves independently of the first one and it coincides with the East process on L' vertices.
- If the "persistence" time of 0 is large enough then the second block has time to equilibrate.
- That suggests already the possibility of a *broad spectrum* of relaxation times, hierarchical evolution.....
- Key issue: separation of time scales (more later).

Previous results

- $q = 1 - p$ is the density of the facilitating sites;

Relaxation time (inverse spectral gap) $T_{\mathrm{rel}}(L;q)$

Let $\theta_q := \log_2(1/q) = \beta/\log 2$. Then

$$\sup_L T_{\mathrm{rel}}(L;q) < +\infty \qquad \text{(Aldous-Diaconis '02)}$$

$$T_{\mathrm{rel}}(\infty;q) \sim 2^{\theta_q^2/2} \text{ as } q \downarrow 0, \qquad \text{(with Cancrini, Roberto, Toninelli '08)}.$$

Previous results

- $q = 1 - p$ is the density of the facilitating sites;

Relaxation time (inverse spectral gap) $T_{\mathrm{rel}}(L;q)$

Let $\theta_q := \log_2(1/q) = \beta/\log 2$. Then

$\sup_L T_{\mathrm{rel}}(L;q) < +\infty$ (Aldous-Diaconis '02)

$T_{\mathrm{rel}}(\infty;q) \sim 2^{\theta_q^2/2}$ as $q \downarrow 0$, (with Cancrini, Roberto, Toninelli '08) .

Exponential relaxation to π

Let $\nu \neq \pi$ be e.g. a different product measure. Then $\exists\, c, m > 0$ s.t.

$$\sup_{L,x} |\mathbb{P}_\nu(\omega_x(t) = 1) - p| \leq c \exp[-mt]$$

(with Cancrini, Schonmann, Toninelli '09)

Equilibration for small temperature ($q \searrow 0$)

- Let $L_c := 1/q$ be the natural equilibrium scale.

Four possible interesting regimes for $q \downarrow 0$

Equilibration for small temperature ($q \searrow 0$)

- Let $L_c := 1/q$ be the natural equilibrium scale.

Four possible interesting regimes for $q \downarrow 0$

1. $L \gg L_c$ (smallness of q irrelevant here);

Equilibration for small temperature ($q \searrow 0$)

- Let $L_c := 1/q$ be the natural equilibrium scale.

Four possible interesting regimes for $q \downarrow 0$

1. $L \gg L_c$ (smallness of q irrelevant here);
2. $L = O(1)$ (finite scale).

Equilibration for small temperature ($q \searrow 0$)

- Let $L_c := 1/q$ be the natural equilibrium scale.

Four possible interesting regimes for $q \downarrow 0$

1. $L \gg L_c$ (smallness of q irrelevant here);
2. $L = O(1)$ (finite scale).
3. $L \propto L_c$ (equilibrium scale).

Equilibration for small temperature ($q \searrow 0$)

- Let $L_c := 1/q$ be the natural equilibrium scale.

Four possible interesting regimes for $q \downarrow 0$

1. $L \gg L_c$ (smallness of q irrelevant here);
2. $L = O(1)$ (finite scale).
3. $L \propto L_c$ (equilibrium scale).
4. $L \sim L_c^{\gamma}$ with $0 < \gamma < 1$ (mesoscopic scale).

Equilibration for small temperature ($q \searrow 0$)

- Let $L_c := 1/q$ be the natural equilibrium scale.

Four possible interesting regimes for $q \downarrow 0$

1. $L \gg L_c$ (smallness of q irrelevant here);
2. $L = O(1)$ (finite scale).
3. $L \propto L_c$ (equilibrium scale).
4. $L \sim L_c^\gamma$ with $0 < \gamma < 1$ (mesoscopic scale).

- Each regime has its own features.
    - (1) and (2) quite well understood.
    - (3) and (4) only partially understood.
    - Aldous and Diaconis suggested a very attractive conjecture for case (3) which is still open.

Cutoff phenomenon on length scale $L \gg L_c$

Fix $\varepsilon \in (0,1)$ and define the ε-mixing time by

$$T_{\rm mix}^{(L)}(\varepsilon) = \inf\{t : \max_{\omega} \|\mu_t^{\omega} - \pi\|_{TV} \leq \varepsilon\}.$$

Definition (Cutoff)

We say that the East process shows total variation cutoff around $\{t_L\}_{L=1}^{\infty}$ with windows $\{w_L\}_{L=1}^{\infty}$ if, for all $L \in \mathbb{N}$ and all $\epsilon \in (0,1)$,

$$T_{\rm mix}^{(L)}(\epsilon) = t_L + O_{\epsilon}(w_L).$$

Cutoff phenomenon on length scale $L \gg L_c$

Fix $\varepsilon \in (0,1)$ and define the ε-mixing time by

$$T^{(L)}_{\mathrm{mix}}(\varepsilon) = \inf\{t : \max_{\omega} \|\mu_t^{\omega} - \pi\|_{TV} \leq \varepsilon\}.$$

Definition (Cutoff)

We say that the East process shows total variation cutoff around $\{t_L\}_{L=1}^{\infty}$ with windows $\{w_L\}_{L=1}^{\infty}$ if, for all $L \in \mathbb{N}$ and all $\epsilon \in (0,1)$,

$$T^{(L)}_{\mathrm{mix}}(\epsilon) = t_L + O_{\epsilon}(w_L).$$

Theorem (with E. Lubetzky and S. Ganguly)

There exists $v > 0$ such that the East model exhibits cutoff with

$$t_L = L/v, \quad \textit{and} \quad w_L = O(\sqrt{L}).$$

Tools

- On $[1, 2, \ldots)$ start the chain from all **1**'s.
- At any later time the configuration $\omega(t)$ will have a rightmost zero (the front).
- Call $X(t)$ the position of the front.
- Behind the front all possible initial configurations have coupled.

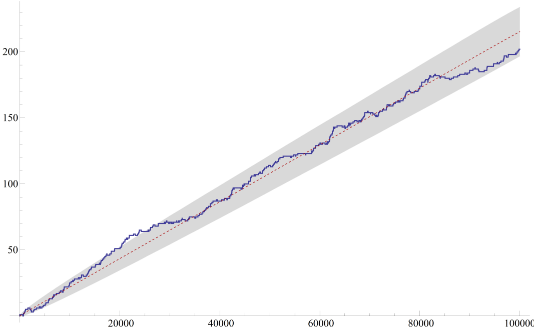


Figure: The front evolution

Results

Theorem (O. Blondel '13)

- $X(t)/t \to v > 0$ *as* $t \to \infty$ *(in probability).*
- *The law of the process seen from the front converges to a unique invariant measure* ν*. Moreover* ν *exp-close to* π *far from the front* $X(t)$*.*

Results

Theorem (O. Blondel '13)

- $X(t)/t \to v > 0$ as $t \to \infty$ (in probability).
- The law of the process seen from the front converges to a unique invariant measure ν. Moreover ν exp-close to π far from the front $X(t)$.

Theorem (with E. Lubetzky and S. Ganguly)

Uniformly in all initial configurations with a front and for all t large enough, the law μ_t of the process behind the front satisfies

$$\|\mu_t - \nu\|_{TV} = O(e^{-t^\alpha}), \qquad \alpha > 0.$$

Conclusion

- It follows that the front increments

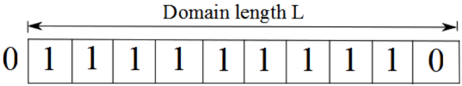
$$\xi_n := X(nt_0) - X((n-1)t_0), \quad t_0 > 0$$

behave like a stationary sequence of weakly dependent random variables $\Rightarrow$ law of large numbers + CLT.

- Thus $X(t)$ has $O(\sqrt{t})$ concentration around vt and the $O(\sqrt{L})$ cutoff window follows.

$L = O(L_c)$: *Equivalence of three basic time scales*

- *(A)* Relaxation time $T_{\mathrm{rel}}(L; q)$.
- *(B)* Mixing time $T_{\mathrm{mix}}(L; q)$ $(\epsilon = 1/4)$.
- *(C)* First passage time $T_{\mathrm{hit}}(L; q)$:= mean hitting time of $\{\omega : \omega_L = 1\}$ starting from a single 0 at $x = L$.



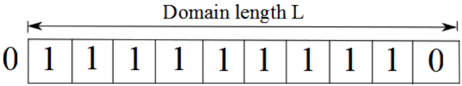
[1] $f \asymp g$ if f/g stays between two positive constants

$L = O(L_c)$: *Equivalence of three basic time scales*

(A) Relaxation time $T_{\text{rel}}(L;q)$.

(B) Mixing time $T_{\text{mix}}(L;q)$ $\qquad (\epsilon = 1/4)$.

(C) First passage time $T_{\text{hit}}(L;q)$:= mean hitting time of $\{\omega : \omega_L = 1\}$ starting from a single 0 at $x = L$.



Theorem (with Chleboun, Faggionato)

For any $L = O(L_c)$

$$T_{\text{hit}}(L;q) \asymp T_{\text{rel}}(L;q) \asymp T_{\text{mix}}(L;q), \qquad \textit{as } q \downarrow 0$$

1

[1] $f \asymp g$ if f/g stays between two positive constants

Coalescence dominated dynamics as $q \downarrow 0$ *when* $L = O(L_c)$

- On length scales $L = O(L_c)$, as $q \downarrow 0$ dynamics dominated by removing excess of **0**'s (Evans-Sollich).

Coalescence dominated dynamics as $q \downarrow 0$ *when* $L = O(L_c)$

- On length scales $L = O(L_c)$, as $q \downarrow 0$ dynamics dominated by removing excess of **0**'s (Evans-Sollich).
- Energy barrier: # of extra **0**'s that are required.

Coalescence dominated dynamics as $q \downarrow 0$ *when* $L = O(L_c)$

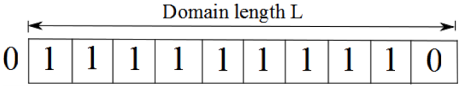
- On length scales $L = O(L_c)$, as $q \downarrow 0$ dynamics dominated by removing excess of **0**'s (Evans-Sollich).
- Energy barrier: # of extra **0**'s that are required.
- Subtle interplay between energy and entropy (number of ways to create the extra **0**'s).

Coalescence dominated dynamics as $q \downarrow 0$ *when* $L = O(L_c)$

- On length scales $L = O(L_c)$, as $q \downarrow 0$ dynamics dominated by removing excess of **0**'s (Evans-Sollich).
- Energy barrier: # of extra **0**'s that are required.
- Subtle interplay between energy and entropy (number of ways to create the extra **0**'s).
- If $L = O(1)$ entropy is negligible compared to energy.

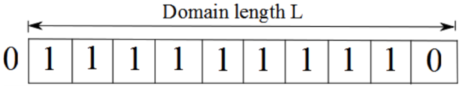
Energy barrier

- $L \in [2^{n-1} + 1,\ 2^n]$



Energy barrier

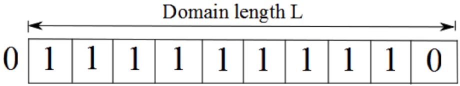
- $L \in [2^{n-1} + 1,\ 2^n]$



- Has to create at least **n** simultaneous zeros (Chung-Diaconis-Graham and Evans-Sollich).

Energy barrier

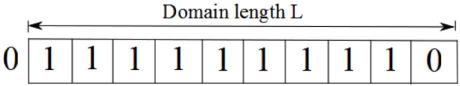
- $L \in [2^{n-1}+1,\ 2^n]$



- Has to create at least **n** simultaneous zeros (Chung-Diaconis-Graham and Evans-Sollich).
- Energy cost $\Delta H = \mathbf{n}$.
- Activation time: $\exp(\beta \Delta H) \sim (1/q)^n = 2^{n\theta_q}$.

Energy barrier

- $L \in [2^{n-1} + 1,\ 2^n]$



- Has to create at least **n** simultaneous zeros (Chung-Diaconis-Graham and Evans-Sollich).
- Energy cost ΔH = **n**.
- Activation time: $\exp(\beta \Delta H) \sim (1/q)^n = 2^{n\theta_q}$.
- Actual killing of last zero is (relatively) instantaneous. Metastable dynamics.

Entropy

- Let $V(n)$ be the number of configurations with **n** zeros reachable from the empty configuration using at most **n** zeros.

$$c_1^n\, n!\, 2^{\binom{n}{2}} \le V(n) \le c_2^n\, n!\, 2^{\binom{n}{2}},$$

  (Chung, Diaconis, Graham '01)

- Entropy could reduce the activation time;
- Very subtle question: need to determine how many of the $V(n)$ configurations lie at the bottleneck.
- Answer: roughly a fraction proportional to $(1/n!)^2$.

Relaxation Time and Energy-Entropy Balance

- Fix $L = 2^n \leq L_c =: 2^{\theta_q}$ $\qquad (\theta_q = \log_2 1/q)$.

Theorem (with Chleboun and Faggionato)

$$T_{\mathrm{rel}}(L; q) = 2^{n\theta_q - \binom{n}{2} + n \log n + O(\theta_q)}$$

- Energy/Entropy contribution:

$$2^{n\theta_q} \equiv \exp[\,\beta\Delta H]; \quad 2^{-\binom{n}{2} + n \log n} \sim \exp[-\log(V_n/(n!)^2)\,]$$

- When $n = \theta_q$ that gives

$$T_{\mathrm{rel}}(L_c; q) = 2^{\theta_q^2/2 + \theta_q \log \theta_q + O(\theta_q)}$$

which is also the correct scaling $\forall\ L \geq L_c$.

Main tools

Upper bound

- Very precise recursive inequality for $T_{\rm rel}(L\,;q)$ on scales $L_j \approx 2^j$.
- Auxiliary block chain key tool to establish the recursion.

Main tools

Upper bound

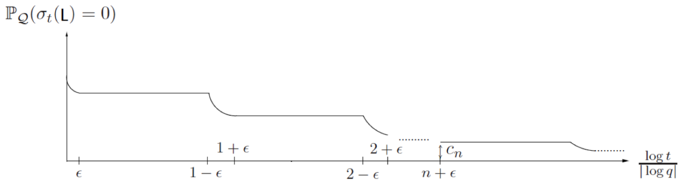
- Very precise recursive inequality for $T_{\mathrm{rel}}(L\,;q)$ on scales $L_j \approx 2^j$.
- Auxiliary block chain key tool to establish the recursion.

Lower bound

- Potential analysis tools.
- Algorithmic construction of an approximate solution of the Dirichlet problem associated to the hitting time $T_{\mathrm{hit}}(L\,;q)$.
- Bottleneck.

The case $L = O(1)$

- Fix $L = 2^n$ with $n \gg 1$ independent of q !
- $T_{\rm rel}(L;q) \asymp 1/q^n$ (only energy counts).
- Non-equilibrium dynamics:
  - Distribute the initial 0's according to a renewal process Q.
  - The function $t \mapsto \mathbb{P}_Q(\omega_L(t) = 0)$ exhibits plateau behavior.



(with Faggionato, Roberto, Toninelli '10)

Main Result

- Fix $\epsilon > 0$ and let $t_k^\pm = (1/q)^{k(1\pm\epsilon)}$.
- Recall that $L = 2^n$ with n independent of q.

Theorem (Universality)

Fix $k \le n$. Then

$$\lim_{q\to 0} \sup_{t_k^- \le t \le t_k^+} \left|\mathbb{P}_Q(\omega_L(t) = 0) - \left(\frac{1}{2^k + 1}\right)^{\mu(1+\epsilon_k)}\right| = 0$$

with $\lim_{k\to\infty} \epsilon_k = 0$ and $\mu = 1$ if Q has finite mean and $\mu = \alpha$ if $Q \sim \alpha$-stable law.

- As observed by Evans-Sollich exactly the same scaling behavior occurs in several other coalescence models in stat-physics (Derrida)!

Time Scales Separation for $L = O(L_c)$.

- If $L = O(1)$ then $T_{\rm rel}(2L; q) \asymp (1/q) T_{\rm rel}(L; q)$.
- The above phenomenon is called time scale separation.

Theorem (with Chleboun and Faggionato)

Given $0 \le \gamma < 1$ *there exists* $\lambda > 1$ *and* $\alpha > 0$ *such that, for all* $L = O(L_c^\gamma)$,

$$\frac{T_{\rm rel}(\lambda L; q)}{T_{\rm rel}(L; q)} \ge (1/q)^\alpha \quad \text{as } q \downarrow 0.$$

$\lambda = 2$ *if* $\gamma < 1/2$.

- Consider initial **0**'s with at least $c \times L_c^\gamma$ **1**'s on its left.
  - If $c \gg 1$ then **0** will survive until time $T_{\rm rel}(L_c^\gamma; q)$.
  - It will disappear before time $T_{\rm rel}(L_c^\gamma; q)$ if $c \ll 1$.
- Dynamic heterogeneities.

Scaling limit as $q \downarrow 0$ *for the stationary East process*

On the basis of numerical simulations it was assumed in the physical literature that *continuous time scale separation* occurs at the equilibrium scale L_c.

Definition (Continuous time scale separation)

Given $\gamma \in (0,1]$ we say that *continuous time scale separation* occurs at length scale L_c^γ if for all $d' > d$ there exists $\alpha > 0$ such that

$$\frac{T_{\rm rel}(d' L_c^\gamma; q)}{T_{\rm rel}(d L_c^\gamma; q)} \asymp (1/q)^\alpha$$

Theorem (with Chleboun and Faggionato)

Fix $\gamma = 1$. *For any* $d' > d$ *there exists* $\kappa(d', d)$ *such that*

$$\frac{T_{\rm rel}(d' L_c; q)}{T_{\rm rel}(d L_c; q)} \le \kappa \quad \forall q.$$

The Aldous-Diaconis conjecture

- Rescale space and time: $x' = qx$ and $t' = t/T_{\mathrm{rel}}(L_c; q)$.
- Under this rescaling $L_c \to 1$ and $T_{\mathrm{rel}}(L_c; q) \to 1$.

The Aldous-Diaconis conjecture

- Rescale space and time: $x' = qx$ and $t' = t/T_{\mathrm{rel}}(L_c; q)$.
- Under this rescaling $L_c \to 1$ and $T_{\mathrm{rel}}(L_c; q) \to 1$.

Conjecture

As $q \downarrow 0$ the rescaled stationary East process in $[0, +\infty)$ converges to the following limiting point process X_t on $[0, +\infty)$:

(i) At any time t, X_t is a Poisson(1) process (particles $\Leftrightarrow$ **0**'s).

(ii) For each $\ell > 0$ and some rate $r(\ell)$ each particle deletes all particles to its right up to a distance ℓ and replaces them by a new Poisson (rate 1) process of particles.

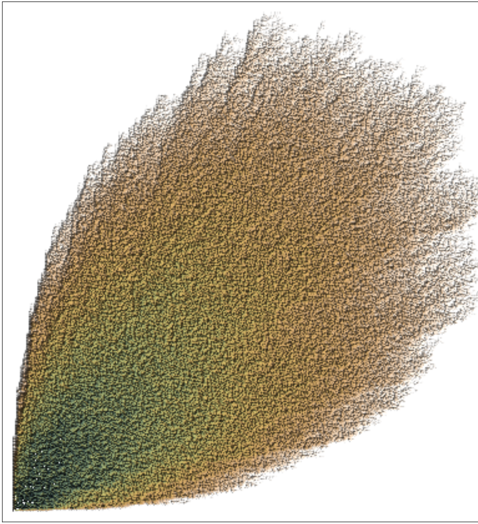
The above conjecture is very close to the "super-spins" description of Evans-Sollich.

East Model in Higher Dimensions

- D. Chandler and J. Garrahan suggested that a *realistic* model of glassy dynamics involves *d*-dimensional analog of the East process.
- E.g. on $\mathbb{Z}^2$ consider the constraint requiring at least one **0** between the South and West neighbor of a vertex.
- We call the corresponding process the East-like process (or *South-or-West process*).

Limiting Shape

- Dynamics in the positive quadrant.
- Boundary conditions: only the origin is unconstrained.
- Initial condition: all **1**'s.
- Black dots: vertices that have flipped at least once within time t.
- Dynamics seems to be much faster along the diagonal.



Main results for q small (with Chleboun, Faggionato)

Theorem (Relaxation time on infinite volume)

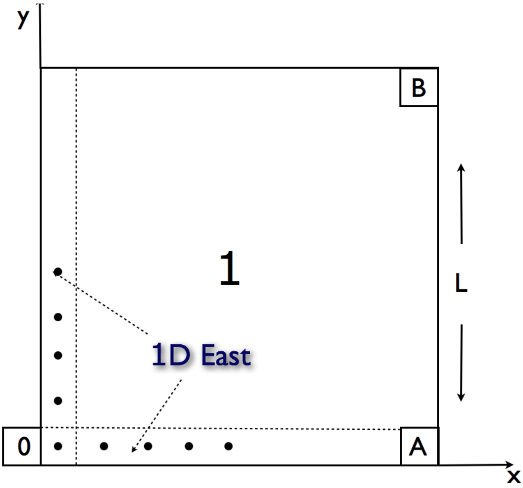
$$T_{\rm rel}(\mathbb{Z}^d; q) = 2^{\frac{\theta_q^2}{2d}(1+o(1))}, \quad \text{as } q \downarrow 0.$$

In particular

$$T_{\rm rel}(\mathbb{Z}^d; q) = T_{\rm rel}(\mathbb{Z}; q)^{\frac{1}{d}(1+o(1))}.$$

The result confirms massive simulations by D.J. Ashton, L.O. Hedges and J.P. Garrahan and indicates that dimensional effects play an important role, contrary to what originally assumed.

Angle dependence of the first passage time



Let $T_{\mathrm{hit}}(A;q)$ and $T_{\mathrm{hit}}(B;q)$ be the first passage times for A and for B.

Theorem

Let $L = 2^n$.

- *If $n = \alpha\theta_q$, $\alpha \in (0,1]$, then,*

$$\frac{T_{\mathrm{hit}}(B;q)}{T_{\mathrm{hit}}(A;q)} = O(2^{-\alpha^2\theta_q^2/2}), \qquad \text{as } q \downarrow 0.$$

- *If n indep. of q then $T_{\mathrm{hit}}(A;q) \ll T_{\mathrm{hit}}(B;q)$.*
- *Crossover induced by entropy.*

***