A short introduction to the brief introduction to motives

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Objects that give rise to 'arithmetic' L-functions ($\Leftrightarrow$ algebraic coefficients)

- elliptic curves over any number field
- hyperelliptic curves over any number field
- abelian surfaces
- algebraic varieties over any number field
- number fields
- Artin representations
- modular forms:
  - holomorphic
  - Siegel
  - Bianchi
  - Hilbert
  - paramodular
- motives

NOT Maass forms

Axioms of an ‘arithmetic’ L-function

From representation theory,

(Farmer, Pitale, Ryan, and Schmidt)

explicit, concrete expressions are given for:

Dirichlet series

- motivic weight, w $\quad$ (0 OR $2\cdot\max\{\nu_1, \nu_2, \ldots, \nu_k\}$)
  $L_{\text{analytic}}(s) = L_{\text{arithmetic}}(s + \frac{w}{2}) \Rightarrow a_n\, n^{\frac{w}{2}}$ is an algebraic integer

Functional equation

- level, N
- sign, ε
- spectral parameters, μ_j, ν_k

  (Langlands parameters, Hodge parameters, Γ-shifts)

Euler product

- degree, d
- central character, χ

Also, $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$

(normalised Γ functions)

Definition of 'arithmetic' L-functions (analytic normalisation)

i.e., L-functions with algebraic coefficients

- Dirichlet series: $$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad a_n \ll n^{\varepsilon}$$

- Functional equation:

$$\Lambda(s) := N^{\frac{s}{2}} \prod_{j=1}^{J} \Gamma_{\mathbb{R}}(s+\mu_j) \prod_{k=1}^{K} \Gamma_{\mathbb{C}}(s+\nu_k)\, L(s) \;=\; \varepsilon\, \bar{\Lambda}(1-s)$$

where $\mu_j \in \{0, 1\}$ and $\nu_k \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \cdots\}$

- Euler product:

$$L(s) = \prod_p f_p(p^{-s})^{-1} \quad \text{where} \quad f_p(z) = 1 - a_p z + \cdots + (-1)^d \chi(p) z^d,$$

$$d = J + 2K, \quad \text{and} \quad \chi(-1) = (-1)^{(\sum \mu_j + \sum(2\nu_k+1))}.$$

Special Case: degree 4, trivial character

$\chi(-1) = 1 \Rightarrow$ not every combination of $\Gamma_\mathbb{R}$ and $\Gamma_\mathbb{C}$ is possible

$w = 0$

$\Gamma_\mathbb{R}(s)^4$
$\Gamma_\mathbb{R}(s)^2\ \Gamma_\mathbb{R}(s+1)^2$
$\Gamma_\mathbb{R}(s+1)^4$

$$\chi(-1) = (-1)^{(\sum \mu_j + \sum (2\nu_k+1))}$$

w : 0 OR $2 \cdot \max\{\nu_1, \nu_2, ..., \nu_k\}$

$w = 1$

$\Gamma_\mathbb{R}(s)^2\ \Gamma_\mathbb{C}(s+\frac{1}{2})$
$\Gamma_\mathbb{R}(s+1)^2\ \Gamma_\mathbb{C}(s+\frac{1}{2})$
$\Gamma_\mathbb{C}(s+\frac{1}{2})^2$

$w = 2$

$\Gamma_\mathbb{R}(s)\ \Gamma_\mathbb{R}(s+1)\ \Gamma_\mathbb{C}(s+1)$
$\Gamma_\mathbb{C}(s+1)^2$

$w = 3$

$\Gamma_\mathbb{R}(s)^2\ \Gamma_\mathbb{C}(s+3/2)$
$\Gamma_\mathbb{R}(s+1)^2\ \Gamma_\mathbb{C}(s+3/2)$
$\Gamma_\mathbb{C}(s+1/2)\ \Gamma_\mathbb{C}(s+3/2)$
$\Gamma_\mathbb{C}(s+3/2)^2$

Specialising further to the case of rational integer coefficients:

Good News: We can search effectively for these.

Bad News: Some uninteresting examples arise.

(products of L-functions)

If $L(s) = L_1(s) \cdot L_2(s)$,
then

$$N = N_1 \cdot N_2$$
$$\varepsilon = \varepsilon_1 \cdot \varepsilon_2$$
$$\chi = \chi_1 \cdot \chi_2$$
$$d = d_1 + d_2$$
$$w = \max\{w_1, w_2\}$$

$$a_p = a_{p,1} + a_{p,2}$$

Degree 4, weight 0, rational integer coefficients

Case 1: weight $= 0$ of form $N^{s/2}\,\Gamma_{\mathbb{R}}(s+1)^4$.

Computational Theorem: For $N \leq 80$ and trivial character, no such L-functions exist.

There is an L-function with $N = 81$: $L(s,\chi_3)^4$.

$L(s,\chi_3)$:	$3^{s/2}\,\Gamma_{\mathbb{R}}(s+1)$	character $= \chi_3$
$L(s,\chi_3)^4$:	$81^{s/2}\,\Gamma_{\mathbb{R}}(s+1)^4$	character $= (\chi_3)^4 =$ trivial

Computational Theorem: For degree 4, motivic weight 0, and trivial character, the only L-functions with $N < 200$ come from products of Dirichlet L-functions.

Case 2: weight $= 1$ of form $N^{s/2}\,\Gamma_{\mathbb{R}}(s+1)^2\,\Gamma_{\mathbb{C}}(s+1/2)$

Computational Theorem: There are no L-functions with rational integer coefficients with $N < 200$.

So, why didn't we find something at $N = 99$?
e.g., $L(s,\chi_3)^2\ L(s,E_{11})$

$L(s,\chi_3)$:	$3^{s/2}\,\Gamma_{\mathbb{R}}(s+1)$	$w = 0$, character $= \chi_3$
$L(s,E_{11})$:	$11^{s/2}\,\Gamma_{\mathbb{C}}(s+1/2)$	$w = 1$, character $=$ trivial

$L(s,\chi_3)^2\cdot(s,E_{11})$: $99^{s/2}\,\Gamma_{\mathbb{R}}(s+1)^2\Gamma_{\mathbb{C}}(s+1/2)$
$w = 1$, character $=$ trivial

p^{th} coefficient: $2\ \chi_3(p) + a_p$ (analytic)
$(2\ \chi_3(p) + a_p)\sqrt{p}$ (arithmetic)

$\cdots$ some questions

Some Questions

1. Are there any L-functions with functional equation
$$\Lambda(s) = N^{\frac{s}{2}} \Gamma_{\mathbb{R}}(s+1)^2 \; \Gamma_{\mathbb{C}}(s+1/2) \; L(s) = \Lambda(1-s)$$
with rational integer coefficients (in the arithmetic normalisation)? If so, do they come from a motive?

2. From what objects do the dozen possible degree 4, weight ≤ 3 cases arise? Could they all come from motives?

3. For those that do come from a motive, are there additional restrictions, say, on the Euler factors?

4. If we find such an L-function and suspect that it comes from a motive, how can we find the motive?

5. Why are the first few L-functions non-primitive?

From motives: L-functions of degree 4, trivial character

motivic weight	Γ factors	Hodge vector
0	$\Gamma_{\mathbb{R}}(s)^4$	4
	$\Gamma_{\mathbb{R}}(s)^2\ \Gamma_{\mathbb{R}}(s+1)^2$	4
	$\Gamma_{\mathbb{R}}(s+1)^4$	4
1	$\Gamma_{\mathbb{R}}(s)^2\ \Gamma_{\mathbb{C}}(s+\frac{1}{2})$	
	$\Gamma_{\mathbb{R}}(s+1)^2\ \Gamma_{\mathbb{C}}(s+\frac{1}{2})$	
	$\Gamma_{\mathbb{C}}(s+\frac{1}{2})^2$	2 2
2	$\Gamma_{\mathbb{R}}(s)\ \Gamma_{\mathbb{R}}(s+1)\ \Gamma_{\mathbb{C}}(s+1)$	1 2 1
	$\Gamma_{\mathbb{C}}(s+1)^2$	2 0 2
3	$\Gamma_{\mathbb{R}}(s)^2\ \Gamma_{\mathbb{C}}(s+3/2)$	
	$\Gamma_{\mathbb{R}}(s+1)^2\ \Gamma_{\mathbb{C}}(s+3/2)$	
	$\Gamma_{\mathbb{C}}(s+1/2)\ \Gamma_{\mathbb{C}}(s+3/2)$	1 1 1 1
	$\Gamma_{\mathbb{C}}(s+3/2)^2$	2 0 0 2