Proving full scale invariance for a massless ϕ^4 theory

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Gradient Random Fields - University of Warwick

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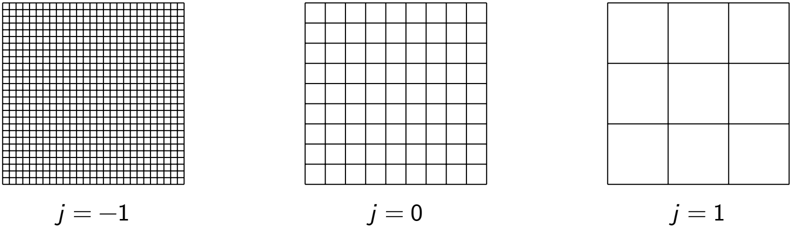

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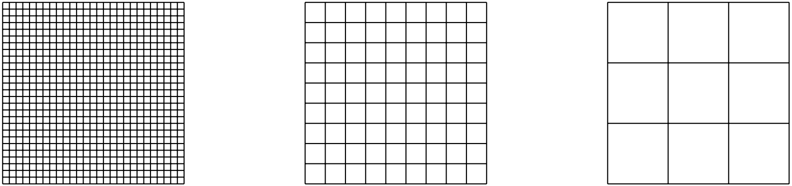

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- Define a sequence of independent Gaussian random fields $\{\xi_j(x)\}_{j \in \mathbb{Z}}$
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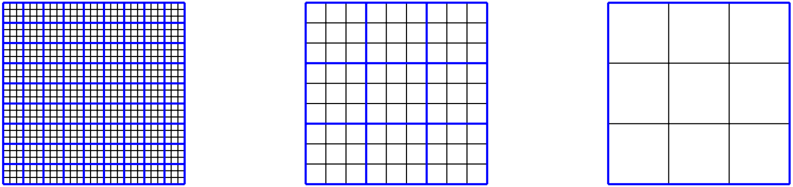

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- Definition of $|\cdot|$ and ϕ is natural if $\mathbb{Q}_p^d$ is used as the space-time - in which case ϕ is given by the Gaussian measure on $S'(\mathbb{Q}_p^d)$ determined by the covariance

$$(-\Delta)^{-\frac{d-2\kappa}{2}}$$

The hierarchical ϕ^4 model

- Following Brydges, Mitter, Scoppola CMP 2003 we fix $d = 3$, $\kappa = \frac{3-\epsilon}{4}$ so our covariance is given by

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$$d\nu = \frac{1}{\mathcal{Z}} \exp\left[-\int_{\mathbb{Q}_p^3} d^3x \ g \ \phi^4(x) + \mu \ \phi^2(x) \right] d\mu_{C_{-\infty}}(\phi)$$

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- $d\mu_{C_{-\infty}}$ is the law of the Gaussian field with covariance $C_{-\infty}$
  $g > 0$, $\mu < 0$

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- Removing UV cut-off (scaling limit) means taking $r \to -\infty$, removing IR cut-off (thermodynamic limit) means taking $s \to \infty$.

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$$
\begin{aligned}
\mathcal{Z}_{\underline{r},\underline{s}} &:= \int \exp\left[- \int_{\Lambda_{\underline{s}}} d^3x \; L^{-\epsilon r} g \; \phi_{\underline{r}}^4(x) + L^{-\frac{3+\epsilon}{2} r} \mu \; \phi_{\underline{r}}^2(x) \right] d\mu_{C_{\underline{r}}}(\phi_{\underline{r}}) \\
&= \int \exp\left[- \int_{\Lambda_{\ell(s-r)}} d^3x \; g \; \phi_0^4(x) + \mu \; \phi_0^2(x) \right] d\mu_{C_0}(\phi_0) \\
&= \int \prod_{\substack{\Delta \in \mathcal{B}_0 \\ \Delta \subset \Lambda_{\ell(s-r)}}} F_0(\phi_{0,\Delta}) d\mu_{C_0}(\phi_0)
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Where the new functional is given by:

$$F_{j+1}(\phi_{0,\Delta}) = e^{-\delta b_{j+1}} \int \prod_{\substack{\Delta'\in\mathcal{B}_0 \\ \Delta'\subset L\Delta}} F_j(L^{-\kappa}\phi_{0,\Delta} + \zeta_{\Delta'}) \, \mathrm{d}\mu_\Gamma(\zeta)$$

The RG Flow

$$F_j(\phi_\Delta) = \exp\left[-g_j\ \phi_\Delta^4 - \mu_j\ \phi_\Delta^2\right] + R_j(\phi_\Delta) \to (g_j, \mu_j, R_j) \in (0,\infty) \times \mathbb{R} \times \mathcal{X}$$

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- Can show existence of non-trivial hyperbolic fixed point of RG denoted (g_*, μ_*, R_*) with $g_* > 0$, along with its local stable manifold
- In particular there is an analytic function $\mu_{\mathrm{crit}}(\cdot)$ defined on a small non-empty neighborhood $U \subset (0,\infty)$ such that

$$\lim_{n\to\infty} RG^n\left[(g, \mu_{\mathrm{crit}}(g), 0)\right] \ = \ (g_*, \mu_*, R_*)$$

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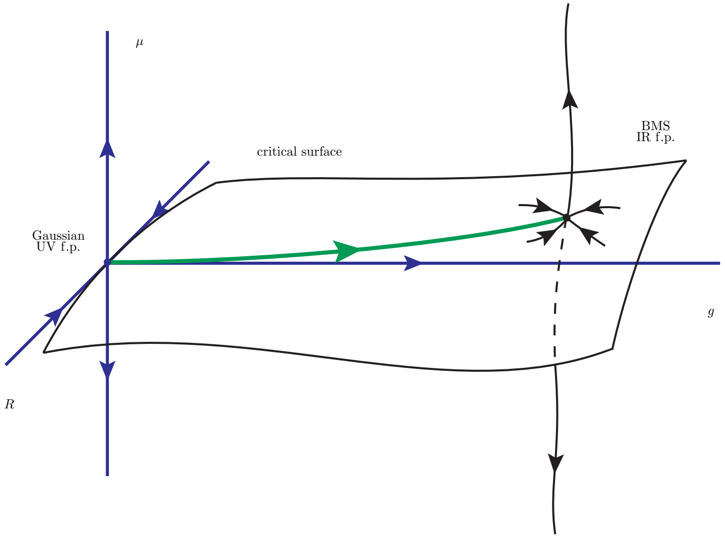
Sketch of RG Phase Portrait



The Flow of Observables

To construct concrete measures corresponding to the ϕ field and $\mathcal{N}[\phi^2]$ field we control the following quantity for suitable test functions f and j:

$$\frac{\mathcal{Z}_{\underline{r},\underline{s}}(f,j)}{\mathcal{Z}_{\underline{r},\underline{s}}(0,0)} = \mathbb{E}_{\underline{r},\underline{s}}\left[e^{\phi(f)+\mathcal{N}[\phi^2](j)}\right]$$

where

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$$\times \exp\left[\int_{\Lambda_{\underline{s}}} d^3x\ \phi_{\underline{r}}(x)f(x) + L^{-\eta r}\left(\phi_{\underline{r}}^2(x) - L^{-(3-2\kappa)r}\gamma_{\ell,0}\right)j(x)\right] d\mu_{C_{\underline{r}}}(\phi_{\underline{r}})$$

- Observables require us to work in a larger dynamical system with an RG transformation that acts on a space of spatially varying potentials
- Constructing the composite field $\mathcal{N}[\phi^2]$ requires a correction due to eigenvalue of RG_ℓ at the non-trivial fixed point along the unstable manifold, a partial linearization theorem in the direction of the unstable manifold is used to show that with this correction one has constructed a non-zero, non-infinite composite field.

Main Result

Theorem (Abdesselam, C., Guadagni)

For any p prime, for ℓ sufficiently large, and for ϵ sufficiently small there exist a non-empty neighborhood $U \subset (0, \infty)$, analytic map $\mu_{\ell,\mathrm{crit}}(\cdot)$ on U such that

- *The measures $\nu_{\ell r, \ell s}$ converge to a limiting measure ν^{ℓ} in the sense of moments as $r \to -\infty, s \to \infty$*
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- *Additionally the fields constructed have the following partial scale invariance (which hold in the two field's joint law) - there exists $\eta_{\ell} > 0$ such that:*

$$\Big(\phi(x), \mathcal{N}[\phi^2](y)\Big) \stackrel{d}{=} \Big(L^{-\kappa}\phi(L^{-1}x), L^{-2\kappa-\eta_{\ell}}\mathcal{N}[\phi^2](L^{-1}y)\Big)$$

Earlier Work:
(Bleher, Sinai 73), (Collet, Eckmann '77), (Gawedzki, Kupianien 83 & 84), (Bleher, Major 87): Hierarchical model
(Brydges, Mitter, Scoppola 03): Euclidean model, non-trivial fixed point
(Abdesselam 06): Euclidean model, construction of a trajectory between Gaussian and non-trivial fixed points

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- Goal of what follows is to prove the following - if one applies the above construction for some sufficiently large ℓ and also $\ell + 1$

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- *Additionally the fields constructed have the following partial scale invariance (which hold in the two field's joint law) - there exists $\eta_{\ell} > 0$ such that:*

$$\left(\phi(x), \mathcal{N}[\phi^2](y)\right) \stackrel{d}{=} \left(L^{-\kappa}\phi(L^{-1}x), L^{-2\kappa-\eta_{\ell}}\mathcal{N}_{\ell}[\phi^2](L^{-1}y)\right)$$

- Goal of what follows is to prove the following - if one applies the above construction for some sufficiently large ℓ and also $\ell+1$ then the functions $\mu_{\ell,\mathrm{crit}}$ and $\mu_{\ell+1,\mathrm{crit}}$ must coincide on the intersection of their domains.

Main Result

Theorem (Abdesselam, C., Guadagni)

For any p prime, for ℓ sufficiently large, and for ϵ sufficiently small there exists a non-empty neighborhood $U \subset (0, \infty)$, analytic map $\mu_{\ell,\mathrm{crit}}(\cdot)$ on U such that

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- By passing to a common subsequence of scales it would follow that $\nu^{\ell} = \nu^{\ell+1}$ which means this measure is fully scale invariant. Easy to check that this also forces $\eta_{\ell} = \eta_{\ell+1}$ and that the laws of $N_{\ell}[\phi^2]$ and $N_{\ell+1}[\phi^2]$ must coincide.

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- Proceed by contradiction:
  We write $\mu_1(\cdot) = \mu_{\text{crit},\ell}(\cdot)$, $\mu_2(\cdot) = \mu_{\text{crit},\ell+1}(\cdot)$ and suppose that $\mu_1(\cdot) > \mu_2(\cdot)$ on some open interval.

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- *Formally* define measures on lattice field configurations $\phi = \{\phi_x\}_{x\in\mathbb{L}}$ via

$$d\nu[g,\mu,\beta,h](\phi) \ = \ \frac{1}{Z}\exp\left[\beta\sum_{x,y\in\mathbb{L}} J_{x,y}\phi_x\phi_y + \sum_{x\in\mathbb{L}} h\phi_x\right]\left(\prod_{x\in\mathbb{L}} d\rho(\phi_x)\right)$$

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$$\inf_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle_{\nu[g, \mu_i(g)]} = 0 \text{ absence of long range order (LRO)}$$

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- Scaling the field allows us translate this into an intermediate phase in the β parameter for *fixed* g, μ.

- Should be a contradiction as transition of ϕ^4 Ising ferromagnets should be *sharp*. In particular if one fixes g, μ (suppressed) and $h = 0$ and defines

$$\beta_{LRO} = \inf \left\{ \beta | \inf_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle_{\mu[\beta,0]} > 0 \right\} \quad \beta_{\chi} = \sup \left\{ \beta | \sum_{x \in \mathbb{L}} \langle \phi_0 \phi_x \rangle_{\mu[\beta,0]} < \infty \right\}$$

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- Natural way to show this - classifying Gibbs measures/pure states for our Ising models. Defining Gibbs measures in this context takes some care - unbounded spin system with interactions of slow decay.

- In particular (i) interaction is not well defined for arbitrary boundary conditions, (ii) need some compactness to prove convergence of measures
- Key tools $\longrightarrow$ Ruelle's superstablity estimates and tempered Gibbs measures [Ruelle 1970], [Lebowitz, Presutti 1976]

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- These bounds give: compactness of finite volume measures, existence of pressure independent of boundary conditions
- Can also construct analogs of $+$ and $-$ boundary conditions along with corresponding extremal measures $\nu[\beta, h, +]$, $\nu[\beta, h, -]$

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- For such β one can show that for all Gibbs measures ν one has:

$$\lim_{\Lambda \to \mathbb{L}} \int_{U_\infty} \frac{dp_\Lambda}{d\beta}(\beta, S) d\nu(S) = \sum_{x \in \mathbb{L} \setminus 0} J(x) \langle \phi_0 \phi_x \rangle_\nu = \frac{dp}{d\beta}(\beta, 0)$$

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- Thanks for listening to my talk and thanks to the organizers for a great conference!