Critical correlation functions for the 4-dimensional *n*-component $|\varphi|^4$ model

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Definition of the model

- We fix a discrete torus $\Lambda = \Lambda_N = \mathbb{Z}^4 / L^N \mathbb{Z}^4$.
- For every $x \in \Lambda$, we consider n-component continuous spins $\varphi_x \in \mathbb{R}^n$.
- Given $g > 0$, $\nu \in \mathbb{R}$, let $d\varphi_x$ be the Lebesgue measure on $\mathbb{R}^n$, we define the $|\varphi|^4$ probability measure as

$$\frac{1}{Z} e^{-\sum_{x\in\Lambda}\left(\frac{1}{2}\varphi_x(-\Delta\varphi)_x + \frac{\nu}{2}|\varphi|^2 + \frac{g}{4}|\varphi|^4\right)} \prod_{x\in\Lambda} d\varphi_x.$$

- Note that for $n = 1$, this is a continuous version of the Ising model.
- We use $\langle \cdot \rangle_{g,\nu,N}$ to denote the expectation with respect the above measure. We are interested in critical correlation functions, in the infinite volume limit $\langle \cdot \rangle_{g,\nu} = \lim_{N\to\infty} \langle \cdot \rangle_{g,\nu,N}$.
- We also write $\langle F\,; G\rangle = \langle FG\rangle - \langle F\rangle\langle G\rangle$, both in finite and infinite volume, for the *correlation* or *truncated expectation* of F, G.

Critical ν_c

- We define the *susceptibility* as the limit

$$\chi(g,\nu,n) = \lim_{N\to\infty} \sum_{x\in\Lambda_N} \langle \varphi_0^1 \varphi_x^1 \rangle_{g,\nu,N}.$$

Theorem (BBS 2014)

For $g > 0$ small enough, there exists $\nu_c = \nu_c(g,n) < 0$ and a constant $C = C(g,n)$ such that as $\nu \downarrow \nu_c$,

$$\chi(g,\nu,n) \sim \frac{C}{\nu-\nu_c}\left(\log\frac{1}{\nu-\nu_c}\right)^{\frac{n+2}{n+8}}.$$

Also, $\nu_c(g,n) = -\mathrm{a}g + O(g^2)$ with $\mathrm{a} = (n+2)(-\Delta_{\mathbb{Z}^4}^{-1})_{0,0} > 0$ (the Laplacian is the lattice Laplacian on $\mathbb{Z}^4$, and its negative inverse is the massless lattice Green function).

Main result for $n = 1$

Theorem

Let $n = 1$ and $g > 0$ be sufficiently small. There exist constants $C_1, C_1' > 0$ such that as $|a-b| \to \infty$,

$$\langle \varphi_a \, ; \varphi_b \rangle_{g,\nu_c} = \frac{C_1}{|a-b|^2}\left(1 + O\left(\frac{1}{\log|a-b|}\right)\right),$$

$$\langle \varphi_a^2 \, ; \varphi_b^2 \rangle_{g,\nu_c} = \frac{C_1'}{|a-b|^4}\frac{1}{(\log|a-b|)^{\frac{2}{3}}}\left(1 + O\left(\frac{\log\log|a-b|}{\log|a-b|}\right)\right).$$

- This theorem was proven previously by K. Gawędzki and A. Kupiainen using a different renormalisation group approach.
- A closely related version was also analysed by J. Feldman, J. Magnen, V. Rivasseau, and R. Sénéor.

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$$\langle \varphi_a ; \varphi_b \rangle_{g,\nu_c} \sim \frac{C_1}{|a-b|^2} \text{ with } O\Big(\tfrac{1}{\log|a-b|}\Big) \text{ corrections,}$$

$$\langle \varphi_a^2 ; \varphi_b^2 \rangle_{g,\nu_c} \sim \frac{C_1'}{|a-b|^4} \frac{1}{(\log|a-b|)^{\frac{2}{3}}} \text{ with } O\Big(\tfrac{\log\log|a-b|}{\log|a-b|}\Big) \text{ corrections.}$$

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Main result ($n \geq 2$)

Theorem

Let $n \geq 2$ and let $g > 0$ be sufficiently small, depending on n. As $|a-b| \to \infty$, there exist constants $C_n, C_n' > 0$ such that for all i,

$$\langle (\varphi_a^i) ; (\varphi_b^i) \rangle_{g,\nu_c} \sim \frac{C_n}{|a-b|^2} \text{ with } O\left(\tfrac{1}{\log|a-b|}\right) \text{ corrections.}$$

For the correlation of squares, we require that $i \neq j$. Then

$$\langle (\varphi_a^i)^2 ; (\varphi_b^i)^2 \rangle_{g,\nu_c} \sim \frac{1}{n} \frac{C_n'}{|a-b|^4} \left[\frac{n-1}{(\log|a-b|)^{\frac{4}{n+8}}} + \frac{1}{(\log|a-b|)^{2\frac{n+2}{n+8}}} \right],$$

$$\langle (\varphi_a^i)^2 ; (\varphi_b^j)^2 \rangle_{g,\nu_c} \sim \frac{1}{n} \frac{C_n'}{|a-b|^4} \left[-\frac{1}{(\log|a-b|)^{\frac{4}{n+8}}} + \frac{1}{(\log|a-b|)^{2\frac{n+2}{n+8}}} \right],$$

both with $O\left(\frac{\log\log|a-b|}{\log|a-b|}\right)$ corrections.

Main result ($n \geq 2$)

Theorem

Let $n \geq 2$ and let $g > 0$ be sufficiently small, depending on n. As $|a-b| \to \infty$, there exist constants $C_n, C'_n > 0$ such that for all i,

$$\langle (\varphi_a^i) ; (\varphi_b^i) \rangle_{g,\nu_c} \sim \frac{C_n}{|a-b|^2} \text{ with } O\left(\frac{1}{\log|a-b|}\right) \text{ corrections.}$$

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$$\langle (\varphi_a^i)^2 ; (\varphi_b^j)^2 \rangle_{g,\nu_c} \sim \frac{1}{n} \frac{C'_n}{|a-b|^4} \left[\underset{-}{\downarrow} \frac{1}{(\log|a-b|)^{\frac{4}{n+8}}} + \frac{1}{(\log|a-b|)^{2\frac{n+2}{n+8}}} \right],$$

both with $O\left(\frac{\log\log|a-b|}{\log|a-b|}\right)$ corrections.

Remarks about negative correlations

(for the case $n \geq 2$)

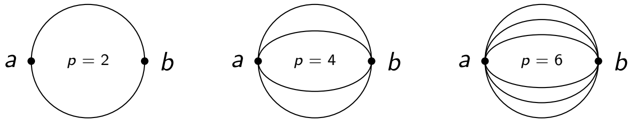
- Note that $(\varphi_a^i)^2$ is more highly correlated with $(\varphi_b^i)^2$ than is $|\varphi_a|^2$ with $|\varphi_b|^2$, due to cancellations with the negative correlations of $(\varphi_a^i)^2$ with $(\varphi_b^j)^2$ for $i \neq j$. More precisely,

$$\langle |\varphi_a|^2 \,; |\varphi_b|^2 \rangle_{g,\nu_c} \sim n \frac{C_n'}{|a-b|^4} \frac{1}{(\log |a-b|)^{2\frac{n+2}{n+8}}}.$$

- In fact, we prove that $\langle |\varphi_a|^2 \rangle < \infty$, the field has a typical size.
- Making one component large must come at the cost of making another one small. Thus locally, negative correlations between different components are to be expected.
- Our results show that this effect persists over long distances at the critical point.

Continuous-time WSAW model as $n = 0$ case

- Our result also covers the computation of the critical generating function for the "watermelon" network of p mutually- and self-avoiding walks, as the $n = 0$ case of the $|\varphi|^4$ model.



- The parameter p corresponds to the power p in $\langle (\varphi_a^1)^p \,; (\varphi_b^1)^p \rangle_{g,\nu_c}$. In the $|\varphi|^4$ model, we only allow $p = 1, 2$, but for the WSAW model, we prove a formula valid for all $p \geq 1$, namely

$$\mathrm{W}_{a,b}^{(p)}(g, \nu_c(0)) \sim \frac{C}{|a-b|^{2p}} \frac{1}{(\log|a-b|)^{\frac{1}{2}\binom{p}{2}}}.$$

- This extends the work of R. Bauerschmidt, D. Brydges, and G. Slade on the critical two-point function of the 4-dimensional weakly self-avoiding walk.

Approximation by the free field

- We restrict our attention to the case $n = 1$, for simplicity.
- The $|\varphi|^4$ measure on $\Lambda = \Lambda_N$ has the density

$$\frac{1}{Z} e^{-U_{g,\nu,1}(\varphi)} d\varphi, \quad U_{g,\nu,z}(\varphi) = \sum_{x \in \Lambda} \left[\frac{1}{2} z \varphi_x (-\Delta \varphi)_x + \frac{1}{2} \nu \varphi_x^2 + \frac{1}{4} g \varphi_x^4 \right].$$

- For a given $z_0 > -1$, we split $U_{g,\nu,1}(\varphi)$ into a Gaussian part and a perturbation

$$U_{g,\nu,1}(\varphi) = U_{0,m^2,1}((1+z_0)^{-1/2}\varphi) + U_{g_0,\nu_0,z_0}((1+z_0)^{-1/2}\varphi).$$

- So that for $C = (-\Delta_{\Lambda_N} + m^2)^{-1}$,

$$\langle F(\varphi) \rangle_{g,\nu,N} = \frac{\mathbb{E}_C \left[F\left((1+z_0)^{1/2}\varphi\right) e^{-U_{g_0,\nu_0,z_0}(\varphi)} \right]}{\mathbb{E}_C \left[e^{-U_{g_0,\nu_0,z_0}(\varphi)} \right]}.$$

Observable fields

- A standard approach to compute the correlation function is to introduce *observable fields* and then differentiate.
- This leads us to define for $\sigma_a, \sigma_b \in \mathbb{R}$,

$$V_0 = U_0 - \sum_{x \in \Lambda} \varphi_x^p \left(\sigma_a \mathbb{1}_{x=a} + \sigma_b \mathbb{1}_{x=b} \right).$$

- Then we have for $p = 1, 2$

$$\left\langle \varphi_a^p \, ; \, \varphi_b^p \right\rangle_{g,\nu,N} = (1 + z_0)^p \left. \frac{\partial}{\partial \sigma_a} \frac{\partial}{\partial \sigma_b} \right|_0 \log \mathbb{E}_C e^{-V_0}.$$

Progressive integration

- We begin by decomposing the covariance

$$\left(\Delta_{\mathbb{Z}^4} + m^2\right)^{-1} = \sum_{j=1}^{\infty} C_j.$$

- Since each C_j is independent of the torus Λ by the finite range property, $(C_j)_{xy} = 0$ if $|x-y| > \frac{1}{2}L^j$, for each N, we have

$$\left(-\Delta_{\Lambda_N} + m^2\right)^{-1} = \sum_{j=1}^{N-1} C_j + C_{N,N}.$$

- We now set $(\mathbb{E}_C \theta F)(\varphi) = \mathbb{E}_C^{\zeta} F(\varphi + \zeta)$ to obtain the formula

$$\mathbb{E}_C F = \mathbb{E}_{C_{N,N}} \circ \left(\mathbb{E}_{C_{N-1}} \theta \circ \cdots \circ \mathbb{E}_{C_1} \theta F\right),$$

with $Z_0 = e^{-V_0(\Lambda)}$ and $Z_{j+1} = \mathbb{E}_{C_{j+1}} \theta Z_j$. Then $Z_N = \mathbb{E}_C Z_0$.

Cumulant expansion

- If we can define V_{j+1} in such a way so that $\mathbb{E}_{C_{j+1}}\theta e^{-V_j} \approx e^{-V_{j+1}}$, we can iterate and rewrite the evolution $Z_j \mapsto Z_{j+1}$ in terms of the much simpler $V_j \mapsto V_{j+1}$. We use the cumulant expansion

$$\mathbb{E}_C\theta e^{-V} = \exp\left(-\mathbb{E}_C\theta V + \frac{1}{2}\mathbb{E}_C\theta(V;V) + O(V^3)\right).$$

- $\mathbb{E}_C\theta(A;B) = \mathbb{E}_C\theta\,(AB) - (\mathbb{E}_C\theta A)\,(\mathbb{E}_C\theta B)$.
- For any polynomial P,

$$\mathbb{E}_C\theta P = e^{\mathcal{L}_C}P = \left(\sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{L}_C^n\right)P, \quad \mathcal{L}_C = \frac{1}{2}\sum_{u,v\in\Lambda} C_{uv}\frac{\partial}{\partial\varphi_u}\frac{\partial}{\partial\varphi_v}.$$

- Also $\mathbb{E}_C P = e^{\mathcal{L}_C}\big|_0 P$, where $\mathcal{L}_C\big|_0$ is the $\mathcal{L}_C$ operator with derivatives taken at $\varphi = 0$.

Approximating the flow

- A natural candidate for V_{j+1} thus comes from the cumulant expansion

$$V_{j+1} \approx \mathbb{E}_{C_{j+1}} \theta V_j - \frac{1}{2} \mathbb{E}_{C_{j+1}} \theta (V_j ; V_j)$$

- We maintain *locality* of V, that is $V = \sum_{x \in \Lambda} V_x$, using a projection operator Loc, that we will not discuss here.
- We are able to preserve the form of V_j, that is for all j and $p = 1, 2$,

$$V_{j;x} = \frac{g_j}{4} \varphi_x^4 + \frac{\nu_j}{2} \varphi_x^2 + \frac{z_j}{2} \varphi_x (-\Delta \varphi)_x - u_j \\ - \left(\lambda_j \varphi_x^p + t_j \right) \left(\sigma_a \mathbb{1}_{x=a} + \sigma_b \mathbb{1}_{x=b} \right) - q_j \sigma_a \sigma_b ,$$

where $\lambda_0 = 1$, $u_0 = t_0 = q_0 = 0$ and g_0, ν_0, z_0 are obtained from g, ν.
- Constant terms: since $Z_N = \mathbb{E}_{C_{N,N}} Z_{N-1}$, the last integration will set all $\varphi = 0$ in Z_N. Thus, $Z_N \approx e^{u_N + t_N(\sigma_a + \sigma_b) + q_N \sigma_a \sigma_b}$ and

$$\langle \varphi_a^p ; \varphi_b^p \rangle_{g,\nu,N} = (1 + z_0)^p \left. \frac{\partial^2}{\partial \sigma_a \partial \sigma_b} \right|_0 \log Z_N \approx \text{const} \cdot q_N .$$

Flow of coupling constants

- We set $w_j = \sum_{i=1}^{j} C_i$,

$$\beta_j = (n+8) \sum_{x \in \Lambda} \Big((w_{j+1})_{0x}^2 - (w_j)_{0x}^2 \Big) \text{ and } \gamma = \begin{cases} 0 & (p=1) \\ \frac{n+2}{n+8} & (p=2). \end{cases}$$

- We also define the *coalescence scale* j_{ab} to be the smallest j such that $(C_j)_{ab} \neq 0$. By the finite range property, $(C_j)_{ab} = 0$ if $|a-b| > \frac{1}{2} L^j$, so $j_{ab} = \lfloor \log_L (2|a-b|) \rfloor$.
- Then the coefficients in V_{j+1} are given by

$$\begin{aligned} g_{j+1} &= g_j - \beta_j g_j^2 + \dots \\ \lambda_{j+1} &= \begin{cases} \lambda_j (1 - \gamma \beta_j g_j) + \dots & (j < j_{ab}) \\ \lambda_{j_{ab}} + \dots & (j \geq j_{ab}) \end{cases} \\ q_{j+1} &= q_j + p! \lambda_j^2 \left[(w_{j+1})_{ab}^p - (w_j)_{ab}^p \right] + \dots \end{aligned}$$

- Providing the control of the non-perturbative part of V, that we indicated by the $(\dots)$, is a major challenge, but is not part of our discussion here.

Sketch of proof for q

- Let us assume that $V_{j+1} \approx \mathbb{E}_{C_{j+1}}\theta V_j - \frac{1}{2}\mathbb{E}_{C_{j+1}}\theta(V_j; V_j)$ and

$$V_{j;x} = \frac{g_j}{4}\varphi_x^4 + \frac{\nu_j}{2}\varphi_x^2 + \frac{z_j}{2}\varphi_x(-\Delta\varphi)_x - u_j$$
$$- \left(\lambda_j\varphi_x^p + t_j\right)\left(\sigma_a \mathbb{1}_{x=a} + \sigma_b \mathbb{1}_{x=b}\right) - q_j\sigma_a\sigma_b.$$

- We want to reach out and grab the coefficient of the constant monomial containing $\sigma_a\sigma_b$ in V_{j+1}, call it $\pi_{ab}V_{j+1}$.
- $\pi_{ab}\mathbb{E}_{C_{j+1}}\theta\left(V_j\right) = -q_j$ and $\pi_{ab}\frac{1}{2}\mathbb{E}_{C_{j+1}}\theta(V_j; V_j) = \frac{2}{2}\mathbb{E}_{C_{j+1}}\theta(\lambda_j\varphi_a^p; \lambda_j\varphi_b^p)$.

$$\pi_{ab}\frac{1}{2}\mathbb{E}_{C_{j+1}}\theta(V_j; V_j) = \pi_{ab}\lambda_j^2\sum_{n=1}^{p}\frac{1}{n!}\mathcal{L}_{C_{j+1}}^n(\varphi_a^p\varphi_b^p) = \frac{(p!)^2}{p!}\lambda_j^2(C_{j+1})_{ab}^p.$$

- Therefore, we get $-q_{j+1} = -q_j - p!\lambda_j^2(C_{j+1})_{ab}^p$. By expanding the definition of w_j, we have $(C_{j+1})_{ab}^p = \left((w_{j+1})_{ab} - (w_j)_{ab}\right)^p$, while the formula that I claimed to be true was

$$q_{j+1} = q_j + p!\lambda_j^2\left[(w_{j+1})_{ab}^p - (w_j)_{ab}^p\right].$$

Analyzing the g flow

- Let us begin by analyzing the recursion

$$g_{j+1} = g_j - \beta_j g_j^2.$$

- For $m^2 = 0$, $\beta_j \to \beta_\infty = \bar{\beta} \log L$, where $\bar{\beta} = \frac{n+8}{16\pi^2}$.
- If we assume that $\beta_j = \beta_\infty$, we can solve the recursion and obtain a formula for g_j

$$g_j \sim \frac{g_0}{1 + g_0 \beta_\infty j} \sim \frac{1}{\beta_\infty j} \text{ as } j \to \infty.$$

- In fact, we prove that, as $|a-b| \to \infty$,

$$g_{j_{ab}} = \frac{1}{\bar{\beta} \log |a-b|} \left(1 + O\left(\frac{\log \log |a-b|}{\log |a-b|} \right) \right) \sim \frac{1}{\log |a-b|}.$$

- Recall that $j_{ab} = \lfloor \log_L(2|a-b|) \rfloor$, so that $\beta_\infty j_{ab} \sim \bar{\beta} \log |a-b|$.

Analyzing the λ flow

- The recursion defining λ depends on g_j and the coalescence scale j_{ab}.

$$\lambda_{j+1} = \begin{cases} \lambda_j(1-\gamma\beta_j g_j) & (j < j_{ab}) \\ \lambda_{j_{ab}} & (j \geq j_{ab}), \end{cases} \quad \text{where } \gamma = \begin{cases} 0 & (p=1) \\ \frac{n+2}{n+8} & (p=2). \end{cases}$$

- Since $\frac{g_{j+1}}{g_j} \sim 1 - \beta_j g_j$, we can write $\left(\frac{g_{j+1}}{g_j}\right)^\gamma \sim 1 - \gamma\beta_j g_j$.
- Inserting this into the recursion, produces a simpler expression for the leading terms in the flow of λ for both choices $p = 1, 2$

$$\lambda_{j+1} = \lambda_j(1-\gamma\beta_j g_j) \sim \lambda_0 \prod_{k=0}^{j}(1-\gamma\beta_k g_k) \sim \left(\frac{g_{j+1}}{g_0}\right)^\gamma.$$

- In particular, the limit $\lim_{m^2\downarrow 0} \lambda_{j_{ab}}^2$ exists and obeys, as $|a-b| \to \infty$,

$$\lim_{m^2\downarrow 0} \lambda_{j_{ab}}^2 \sim \left(\frac{1}{\log|a-b|}\right)^{2\gamma}.$$

Analyzing the q flow

- As $(C_j)_{ab} = 0$ for all $j < j_{ab}$, we can sum the equation for q_j and obtain a telescoping sum

$$q_N \approx \sum_{j=j_{ab}}^{N} p!\lambda_j^2 \left[(w_{j+1})_{ab}^p - (w_j)_{ab}^p \right] = p!\lambda_{j_{ab}}^2 (w_N)_{ab}^p.$$

- By definition, as $N \to \infty$, $q_N \to q_\infty(m^2) = p!\lambda_{j_{ab}}^2 (-\Delta_{\mathbb{Z}^4} + m^2)_{ab}^{-p}$.
- The limit of $(-\Delta_{\mathbb{Z}^4} + m^2)_{ab}^{-p}$ as $m^2 \downarrow 0$ is well-known

$$\lim_{m^2 \downarrow 0} (-\Delta_{\mathbb{Z}^4} + m^2)_{ab}^{-1} = (-\Delta_{\mathbb{Z}^4})_{ab}^{-1} \sim \frac{1}{|a-b|^2}.$$

- Finally, for $\nu_c = \nu_c(g, 1)$ and as $|a - b| \to \infty$,

$$\langle \phi_a \, ; \, \phi_b \rangle_{g,\nu_c} \sim \frac{1}{|a-b|^2},$$

$$\left\langle \phi_a^2 \, ; \, \phi_b^2 \right\rangle_{g,\nu_c} \sim \frac{1}{|a-b|^4 (\log|a-b|)^{\frac{2}{3}}},$$

since $\gamma = \frac{n+2}{n+8} = \frac{1}{3}$.

Thank you.