Gradient interfaces with disorder

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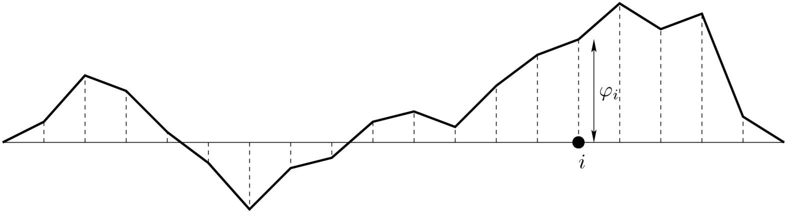
Outline

- Interface — transition region that separates different phases



- $\Lambda \subset \mathbb{Z}^d$ finite, $\quad \partial\Lambda := \{x \notin \Lambda, \; ||x-y|| = 1 \text{ for some } y \in \Lambda\}$
- Height Variables (configurations) $\phi_x \in \mathbb{R}, x \in \Lambda$
- Boundary condition ψ, such that

$$\phi_x = \psi_x, \quad \text{when} \;\; x \in \partial\Lambda.$$

- Gradients $\nabla\phi$: $\nabla\phi_b = \phi_x - \phi_y$ for $b = (x,y), ||x-y|| = 1$
- tilt $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ and tilted boundary condition $\psi^u_x = x \cdot u, x \in \partial\Lambda$.
- $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space of the disorder, $\mathbb{E}$ the expectation w.r.t $\mathbb{P}$, $\mathbb{V}$ the variance w.r.t. $\mathbb{P}$ and $\mathbb{C}$ov the covariance w.r.t $\mathbb{P}$.

Model A

- The Hamiltonian (random external field)

$$H_\Lambda^\psi[\xi](\phi) := \frac{1}{2} \sum_{\substack{x,y \in \Lambda \cup \partial\Lambda \\ |x-y|=1}} V(\phi_x - \phi_y) + \sum_{x \in \Lambda} \xi_x \phi_x,$$

- $(\xi_x)_{x \in \mathbb{Z}^d}$ are assumed to be *i.i.d.* real-valued random variables, with *finite non-zero second moments*.
- $V \in C^2(\mathbb{R})$ is an even function such that there exist $0 < C_1 < C_2$ with

$$C_1 \leq V''(s) \leq C_2 \text{ for all } s \in \mathbb{R}.$$

- The finite volume Gibbs measure on $\mathbb{R}^\Lambda$

$$\nu_\Lambda^\psi[\xi](\phi) := \frac{1}{Z_\Lambda^\psi[\xi]} \exp(-\beta H_\Lambda^\psi[\xi](\phi)) \prod_{x \in \Lambda} d\phi_x,$$

where $\phi_x = \psi_x$ for $x \in \partial\Lambda$.

- χ is the set of bonds (x, y) which satisfy the plaquette condition
- $C_b(\chi)$ is the set of continuous and bounded functions on χ
- The finite-volume $\nabla\phi$-Gibbs measure $\mu_\Lambda^\rho[\xi]$ on χ is such that it satisfies for all $F \in C_b(\chi)$

$$\int_\chi \mu_\Lambda^\rho[\xi](\mathrm{d}\eta)F(\eta) = \int_{\mathbb{R}^{\mathbb{Z}^d}} \nu_\Lambda^\psi[\xi](\mathrm{d}\phi)F(\nabla\phi),$$

where ψ is any field configuration whose gradient field is ρ. (i.e., the distribution of the $\nabla\phi$-field under the Gibbs measure ν_Λ^ψ)

- Infinite-volume gradient Gibbs measure $\mu[\xi]$ has to satisfy the DLR equation

$$\int \mu[\xi](\mathrm{d}\rho) \int \mu_\Lambda^\rho[\xi](\mathrm{d}\eta)F(\eta) = \int \mu[\xi](\mathrm{d}\eta)F(\eta),$$

  for every finite $\Lambda \subset \mathbb{Z}^d$ and for all $F \in C_b(\chi)$.
- For $v \in \mathbb{Z}^d$, we define the shift operators τ_v:
  - For the bonds by $(\tau_v\eta)(b) := \eta(b - v)$ for b bond and $\eta \in \chi$
  - For the disorder by $(\tau_v\xi)(y) := \xi(y - v)$ for $y \in \mathbb{Z}^d$ and $\xi \in \mathbb{R}^{\mathbb{Z}^d}$.
- A measurable map $\xi \to \mu[\xi]$ is called a shift-covariant random gradient Gibbs measure if $\mu[\xi]$ is a $\nabla\phi-$ Gibbs measure for $\mathbb{P}$-almost every ξ, and if

$$\int \mu[\tau_v\xi](\mathrm{d}\eta)F(\eta) = \int \mu[\xi](\mathrm{d}\eta)F(\tau_v\eta),$$

  for all $v \in \mathbb{Z}^d$ and for all $F \in C_b(\chi)$.

Model B

- For each $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, $|x - y| = 1$, we define the measurable map $V^\omega_{(x,y)}(s) : (\omega, s) \in \Omega \times \mathbb{R} \to \mathbb{R}$.
- $V^\omega_{(x,y)}$ are random variables with *uniformly-bounded finite second moments* and jointly *stationary* distribution.
- For some given $0 < C^\omega_{1,(x,y)} < C^\omega_{2,(x,y)}, \omega \in \Omega$, with $0 < \inf_{(x,y)} \mathbb{E}\big(C^\omega_{1,(x,y)}\big) < \sup_{(x,y)} \mathbb{E}\big(C^\omega_{2,(x,y)}\big) < \infty$, $V^\omega_{(x,y)}$ obey for $\mathbb{P}$-almost every $\omega \in \Omega$ the following bounds, uniformly in the bonds (x, y)

$$C^\omega_{1,(x,y)} \le (V^\omega_{(x,y)})''(s) \le C^\omega_{2,(x,y)} \text{ for all } s \in \mathbb{R}.$$

- For each fixed $\omega \in \Omega$ and for each bond (x, y), $V^\omega_{(x,y)} \in C^2(\mathbb{R})$ is an even function.

- The Hamiltonian for each fixed $\omega \in \Omega$ (random potentials)

$$H_{\Lambda}^{\psi}[\omega](\phi) := \frac{1}{2} \sum_{x,y \in \Lambda \cup \partial\Lambda, |x-y|=1} V_{(x,y)}^{\omega}(\phi_x - \phi_y)$$

- Let $\omega \in \Omega$ be fixed. We will denote by $\mu[\tau_v \omega]$ the infinite-volume gradient Gibbs measure with given Hamiltonian $\bar{H}[\omega](\eta) := \left(H_{\Lambda}^{\rho}[\omega](\tau_v \eta)\right)_{\Lambda \subset \mathbb{Z}^d, \rho \in \chi}$. This means that we shift the field of disorded potentials on bonds from $V_{(x,y)}^{\omega}$ to $V_{(x+v,y+v)}^{\omega}$.

Questions (for general potentials V):

- Existence and (strict) convexity of infinite volume surface tension

$$\sigma(u)[\xi] = \lim_{\Lambda \uparrow \mathbb{Z}^d} \sigma_\Lambda[\xi](u), \quad \sigma_\Lambda[\xi](u) := \frac{1}{|\Lambda|} \log Z_\Lambda^{\psi^u}[\xi].$$

- Existence of shift-covariant infinite volume gradient Gibbs measure

$$\mu[\xi] := \lim_{\Lambda \uparrow \mathbb{Z}^d} \mu_\Lambda^\rho[\xi]$$

- Uniqueness of shift-covariant Gibbs measure (maybe under additional assumptions on the measure).
- Quantitative results for $\mu[\xi]$: decay of covariances with respect to $\nabla\phi$, central limit theorem (CLT) results, large deviations (LDP) results.

Known results for gradients without disorder

$$0 < C_1 \leq V'' \leq C_2 :$$

- Existence and strict convexity of the surface tension for $d \geq 1$.
- Gibbs measures ν do not exist for $d = 1, 2$.
- $\nabla\phi$-Gibbs measures μ exist for $d \geq 1$.
- (Funaki-Spohn) For every $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$ there exists a **unique shift-invariant ergodic** $\nabla\phi$- Gibbs measure μ with $E_\mu[\phi_{e_k} - \phi_0] = u_k$, for all $k = 1, \ldots, d$.
- Decay of covariances results, CLT results, LDP results
- Important properties for proofs: shift-invariance, ergodicity and extremality of the infinite volume Gibbs measures

Results for gradients with disorder

- For model A, van Enter-Külske (2007): For $d = 2$, there exists no shift-covariant gradient Gibbs measure $\mu[\xi]$ with $\mathbb{E}\left|\int \mu[\xi](d\eta)V'(\eta(b))\right| < \infty$ for all bonds b.
- For model A, Cotar-Külske (2010): For $d = 3, 4$, there exists no shift-covariant Gibbs measure.
- Cotar-Külske (2014): **(Model A)** Let $d \geq 3$, $\xi(0)$ with symmetric distribution and $u \in \mathbb{R}^d$. Assume $0 < C_1 \leq V'' \leq C_2$. Then there exists exactly one shift-covariant random gradient Gibbs measure $\xi \to \mu[\xi]$ with $\mathbb{E}\left(\int \mu[\xi]\right)$ ergodic and such that

$$\mathbb{E}\left(\int \mu[\xi](\mathrm{d}\eta)\eta_b\right) = \langle u, y_b - x_b\rangle \text{ for all } b = (x_b, y_b).$$

Moreover $\mu[\xi]$ satisfies the integrability condition $\mathbb{E}\int \mu[\xi](\mathrm{d}\eta)(\eta_b)^2 < \infty$ for all bonds b.

- **(Model B)** Let $d \geq 1$ and $u \in \mathbb{R}^d$. Assume $0 < C_1 \leq (V^{\omega}_{(i,j)})'' \leq C_2$ for all ω. Then there exists exactly one shift-covariant random gradient Gibbs measure $\omega \to \mu[\omega]$ with $\mathbb{E}\left(\int \mu[\omega]\right)$ ergodic and such that

$$\mathbb{E}\left(\int \mu[\omega](\mathrm{d}\eta)\eta_b\right) = \langle u, y_b - x_b \rangle \text{ for all } b = (x_b, y_b).$$

Moreover $\mu[\omega]$ satisfies

$$\mathbb{E}\int \mu[\omega](\mathrm{d}\eta)(\eta_b)^2 < \infty \text{ for all } b.$$

For our 2nd main result, we need

- Poincaré inequality assumption on the distribution γ of the disorder $\xi(0)$, (respectively of $V^{\omega}_{(0,e_1)}$): There exists $\lambda > 0$ such that for all smooth enough real-valued functions f on Ω, we have for the probability measure γ

$$\lambda \text{var}_{\gamma}(f) \leq \int |\nabla f|^2 \, d\gamma, \tag{1}$$

  where $|\nabla f|$ is the Euclidean norm of the gradient of f smooth enough.
- Milman (2010, 2012)-enough for the above to have semi-convexity assumption
- Let

$$\partial_b F(\eta) := \frac{\partial F(\eta)}{\partial \eta_b}, \; ||\partial_b F||_\infty := \sup_{\eta \in \chi} |\partial_b F(\eta)| \text{ and }]|b|[= \max\{|x_b|, 1\}.$$

- Cotar-Külske (2014): Let $u \in \mathbb{R}^d$.
  - (a) **(Model A)** Let $d \geq 3$. Assume that $(\xi(x))_{x \in \mathbb{Z}^d}$ are i.i.d with mean 0 and the distribution of $\xi(0)$ satisfies (1). Then if $\xi \to \mu[\xi]$ is the shift-covariant gradient Gibbs measure from uniqueness result, $\xi \to \mu[\xi]$ satisfies the following decay of covariances for all $F, G \in C_b^1(\chi)$

$$|\mathbb{C}\mathrm{ov}\,(\mu[\xi](F(\eta)), \mu[\xi](G(\eta)))\,| \leq c \sum_{b,b'} \frac{||\partial_b F||_\infty ||\partial_{b'} G||_\infty}{]|b-b'|[^{d-2}},$$

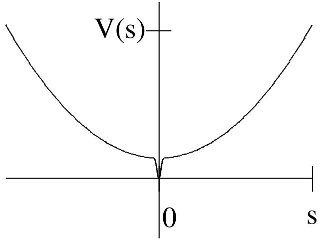
    for some $c > 0$ which depends only on d, C_1, C_2 and on the number of terms b, b' in F and G.
  - (b) **(Model B)** Let $d \geq 1$. Assume that $V^\omega_{(x,y)}$ are i.i.d., and they also satisfy (1) for $\mathbb{P}$-almost every ω and uniformly in the bonds (x, y). Then if $\omega \to \mu[\omega]$ is the shift-covariant gradient Gibbs measure from uniqueness result, $\omega \to \mu[\omega]$ satisfies the following decay of covariances for all $F, G \in C_b^1(\chi)$

$$|\mathbb{C}\mathrm{ov}\,(\mu[\omega](F(\eta)), \mu[\omega](G(\eta)))\,| \leq c \sum_{b,b'} \frac{||\partial_b F||_\infty ||\partial_{b'} G||_\infty}{]|b-b'|[^{d}}.$$

Conjecture for disordered non-convex potentials

- For the potential

$$e^{-V(s)} = pe^{-k_1\frac{s^2}{2}} + (1-p)e^{-k_2\frac{s^2}{2}}, \; \beta = 1, k_1 << k_2, \; p = \left(\frac{k_1}{k_2}\right)^{1/4}$$



- Biskup-Kotecký (2007): $d = 2$. Existence of two ergodic $\nabla\phi$-Gibbs measures with same expected tilt $E_\mu[\phi_{e_k} - \phi_0] = 0$, but with different variances.

- Cotar-Deuschel (2012): $d \geq 2$.
  Let

$$V = V_0 + g,\ C_1 \leq V_0'' \leq C_2,\ g'' < 0.$$

If

$$C_0 \leq g'' < 0 \ \text{ and } \quad \sqrt{\beta}||g''||_{L^1(\mathbb{R})} \ \text{ small}(C_1, C_2).$$

uniqueness for shift-invariant $\nabla\phi$-Gibbs measures μ such that $E_\mu\left[\phi_{e_k} - \phi_0\right] = u_k$ for $k = 1, 2, \ldots, d$. Our results includes the Biskup-Kotecký model, but for different range of choices of p, k_1 and k_2.

- Consider the corresponding disordered model

$$e^{-V_b(\eta_b)} := e^{-\omega_b(\eta_b)^2}(pe^{-k_1(\eta_b)^2} + (1-p)e^{-k_2(\eta_b)^2}).$$

Naive Aizenman-Wehr argument hints at: uniqueness for low enough $d \leq d_c$ and uniqueness/non-uniqueness phase transition for high enough $d > d_c \geq 2$.

For $0 < C_1 \leq V'' \leq C_2$:

- Brascamp-Lieb Inequality: for all $x \in \Lambda$ and for all $i \in \Lambda$

$$\text{var}_{\nu_\Lambda^\psi}(\phi_i) \leq \text{var}_{\tilde{\nu}_\Lambda^\psi}(\phi_i),$$

  $\tilde{\nu}_\Lambda^\psi$ is the Gaussian Free Field with potential $\tilde{V}(s) = C_1 s^2$.

- Random Walk Representation Deuschel-Giacomin-Ioffe (2000): Representation of Covariance Matrix in terms of the Green function of a particular random walk.
  - GFF: If $x, y \in \Lambda$

$$\text{cov}_{\nu_\Lambda^0}(\phi_x, \phi_y) = G_\Lambda(x, y),$$

    where $G_\Lambda(x, y)$ is the Green's function, that is, the expected number of visits to y of a simple random walk started from x killed when it exits Λ.
  - General $0 < C_1 \leq V'' \leq C_2$:
    $0 \leq \text{cov}_{\nu_\Lambda^\psi}(\phi_x, \phi_y) \leq \frac{C}{]|x-y|[^{d-2}}$, $|\text{cov}_{\mu_\Lambda^\rho}(\nabla_i \phi_x, \nabla_j \phi_y)| \leq \frac{C}{]|x-y|[^{d-2+\delta}}$

- The dynamic: SDE satisfied by $(\phi_x)_{x\in\mathbb{Z}^d}$

$$d\phi_x(t) = -\frac{\partial H}{\partial \phi_x}(\phi(t))dt + \sqrt{2}dW_x(t), \;\; x \in \mathbb{Z}^d,$$

  where $W_t := \{W_x(t), x \in \mathbb{Z}^d\}$ is a family of independent 1-dim Brownian Motions.

- Komlos (1967): If $(\zeta_n)_{n\in\mathbb{N}}$ is a sequence of real-valued random variables with $\liminf_{n\to\infty} \mathbb{E}(|\zeta_n|) < \infty$, then there exists a subsequence $\{\theta_n\}_{n\in\mathbb{N}}$ of the sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ and an integrable random variable θ such that for any arbitrary subsequence $\{\tilde{\theta}_n\}_{n\in\mathbb{N}}$ of the sequence $\{\theta_n\}$, we have almost surely that

$$\lim_{n\to\infty} \frac{\tilde{\theta}_1 + \tilde{\theta}_2 + \ldots + \tilde{\theta}_n}{n} = \theta.$$

- Gloria-Otto (2012)/ Ledoux (2001): Fix $n \in \mathbb{N}$ and let $a = (a_i)_{i=1}^n$ be independent random variables with uniformly-bounded finite second moments on $(\Omega, \mathcal{F}, \mathbb{P})$. Let X, Y be Borel measurable functions of $a \in \mathbb{R}^n$ (i.e. measurable w.r.t. the smallest σ-algebra on $\mathbb{R}^N$ for which all coordinate functions $\mathbb{R}^n \ni a \to a_i \in \mathbb{R}$ are Borel measurable). Then $|\text{cov}\,(X, Y)| \leq$ $\max_{1 \leq i \leq n} \text{var}\,(a_i) \sum_{i=1}^n \left(\int \sup_{a_i} \left| \frac{\partial X}{\partial a_i} \right|^2 \text{d}\mathbb{P} \right)^{1/2} \left(\int \sup_{a_i} \left| \frac{\partial Y}{\partial a_i} \right|^2 \text{d}\mathbb{P} \right)^{1/2}$ where $\sup_{a_i} \left| \frac{\partial Z}{\partial a_i} \right|$ denotes the supremum of

$$\frac{\partial Z}{\partial a_i}(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n)$$

of Z with respect to the variable a_i, for $Z = X, Y$.
- The independence assumption can be relaxed by using, for example, Marton (2013) and Caputo, Menz, Tetali (2014)

We will first prove:
Fix $u \in \mathbb{R}^d$. Let for all $\alpha \in \{1, 2, \ldots, d\}$

$$E_\alpha := \{\eta \mid \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \eta(b_{x,\alpha}) = u_\alpha\},$$

along the sequence with $b_{x,\alpha} := (x + e_\alpha, x) \in \chi$.
Then there exists a unique shift-covariant random gradient Gibbs measure $\xi \to \mu[\xi]$ which satisfies for $\mathbb{P}$-almost every ξ

$$\mu[\xi](E_\alpha) = 1, \; \alpha \in \{1, 2, \ldots, d\}.$$

Moreover, $\mu[\xi]$ satisfies the integrability condition

$$\mathbb{E} \int \mu[\xi](d\eta)(\eta(b))^2 < \infty \text{ for all bonds } b \in \chi.$$

Existence: We consider first case $u = 0$.

- Step 1: Define

$$\bar{\mu}^0_\Lambda[\xi] := \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\mu^{\rho_0}_{\Lambda+x}[\xi],$$

where $\Lambda + x := \{z + x : z \in \Lambda\}$. Then there exists a deterministic subsequence $(m_i)_{i\in\mathbb{N}}$ such that for $\mathbb{P}$-almost every ξ

$$\hat{\mu}^0_k[\xi] := \frac{1}{k}\sum_{i=1}^{k}\bar{\mu}^0_{\Lambda_{m_i}}[\xi]$$

converges as $k \to \infty$ weakly to $\mu[\xi]$, which is a shift-covariant random gradient Gibbs measure.

- Step 2: It suffices to show

$$\liminf_{n\to\infty}\liminf_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k}\frac{1}{|\Lambda_{m_i}|}\sum_{w\in\Lambda_{m_i}}\mathbb{E}\mu^{\rho_0}_{\Lambda_{m_i}+w}[\xi]\left(\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\eta(b_{x,\alpha})\right)^2=0.$$

- Step 3: We need to estimate the following 3 terms

$$\mathbb{E}\left(\operatorname{var}_{\mu^{\rho_0}_{m_i+w}[\xi]}\left(\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\eta(b_{x,\alpha})\right)\right)$$
$$+\mathbb{V}\left(\mu^{\rho_0}_{m_i+w}[\xi]\left(\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\eta(b_{x,\alpha})\right)\right)$$
$$+\left(\mathbb{E}\mu^{\rho_0}_{m_i+w}[\xi]\left(\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n}\eta(b_{x,\alpha})\right)\right)^2.$$

General $u \in \mathbb{R}^d$ case.

Define

$$\nu^{\psi}_{\text{shift},\Lambda}[\xi](\mathrm{d}\phi) := \frac{1}{Z^{\psi}_{\text{shift},\Lambda}[\xi]} e^{-\frac{1}{2}\sum_{\substack{x\in\Lambda, y\in\Lambda\cup\partial\Lambda \\ |x-y|=1}} V(\phi(x)-\phi(y)-\langle u, x-y\rangle)}$$

$$e^{\sum_{x\in\Lambda}\xi(x)\phi(x)}\,\mathrm{d}\phi_\Lambda \delta_\psi(\mathrm{d}\phi_{\mathbb{Z}^d\setminus\Lambda}).$$

Proceed now as in Steps 1-3.

Uniqueness:

- Suppose that there exist two shift-covariant measures $\xi \to \mu[\xi], \xi \to \bar{\mu}[\xi], \mu[\xi], \bar{\mu}[\xi] \in \mathcal{P}(\chi)$, stationary for the dynamics which satisfy for $\mathbb{P}$-almost every ξ

$$\mu[\xi](E_\alpha) = 1, \ \bar{\mu}[\xi](E_\alpha) = 1, \ \ \alpha \in \{1, 2, \ldots, d\},$$

  and which satisfy the integrability condition

$$\mathbb{E} \int \mu[\xi](\mathrm{d}\eta)(\eta(b))^2 < \infty, \mathbb{E} \int \bar{\mu}[\xi](\mathrm{d}\eta)(\eta(b))^2 < \infty, \ \text{for all } b.$$

- For each fixed $\xi \in \Omega$, we construct two independent χ_r-valued random variables $\eta = \{\eta(b)\}_{b \in (\mathbb{Z}^d)^*}$ and $\bar{\eta} = \{\bar{\eta}(b)\}_{b \in (\mathbb{Z}^d)^*}$ on a common probability space $(\Upsilon, \mathcal{L}, \mathbb{Q}[\xi])$ in such a manner that η and $\bar{\eta}$ are distributed by $\mu[\xi]$ and $\bar{\mu}[\xi]$ under $\mathbb{Q}[\xi]$, respectively.

We will first show

- For all $u \in \mathbb{R}^d$, we have

$$\lim_{T\to\infty} \int \frac{1}{T} \int_0^T \sum_b e^{-2r|x_b|} \mathbb{E}_{\mathbb{Q}[\xi]} \left[(\eta_t(b) - \bar{\eta}_t(b))^2 \right] \mathrm{d}t \mathbb{P}(\mathrm{d}\xi) = 0.$$

- There exists a deterministic sequence $(m_r)_{r\in\mathbb{N}}$ in $\mathbb{N}$ such that for $\mathbb{P}$-almost every ξ

$$\lim_{k\to\infty} \frac{1}{k} \left(\sum_{i=1}^{k} \frac{1}{m_i} \int_0^{m_i} \sum_b e^{-2r|x_b|} \mathbb{E}_{\mathbb{Q}[\xi]} \left[(\eta_t(b) - \bar{\eta}_t(b))^2 \right] \mathrm{d}t \right) = 0.$$

- This implies for $\mathbb{P}$-almost all ξ

$$\lim_{k\to\infty} \int |\eta - \bar{\eta}|_r^2 \hat{\mathbb{P}}_k[\xi](d\eta d\bar{\eta}) = 0,$$

  where $\hat{\mathbb{P}}_k[\xi]$ is a shift-covariant probability measure on $\chi \times \chi$ defined by

$$\hat{\mathbb{P}}_k[\xi](d\eta d\bar{\eta}) := \frac{1}{k}\left(\sum_{i=1}^{k} \frac{1}{m_i} \int_0^{m_i} \mathbb{Q}[\xi](\{\eta_t(b), \bar{\eta}_t(b)\}_b \in d\eta d\bar{\eta}) \ \mathrm{d}t \right).$$

  The first marginal of $\hat{\mathbb{P}}_k[\xi]$ is $\mu[\xi]$ and the second one is $\bar{\mu}[\xi]$.
- This implies that the Wasserstein distance between μ and $\bar{\mu}$ is zero and hence $\mu[\xi] = \bar{\mu}[\xi]$ for $\mathbb{P}$-almost all ξ.

Ergodicity of the averaged measure:

- Let $\mathcal{F}_{inv}(\chi)$ the σ-algebra of shift-invariant events on χ. Let

$$\mu_{av} = \left(\int \mathbb{P}(d\xi)\mu[\xi] \right) (\,\mathrm{d}\eta).$$

  We need to show that for all $A \in \mathcal{F}_{inv}(\chi)$, we have $\mu_{av}(A) = 0$ or $\mu_{av}(A) = 1$. We will show that this holds by contradiction.

- Suppose that there exists $A \in \mathcal{F}_{inv}(\chi)$ such that $0 < \mu_{av}(A) < 1$. Then, for $\mathbb{P}$-almost all ξ we have $0 < \mu[\xi](A) < 1$. We define now for all ξ the *distinct* measures on χ

$$\mu_A[\xi](B) := \frac{\mu[\xi](B \cap A)}{\mu[\xi](A)} \quad \text{and} \quad \mu_{A^c}[\xi](B) := \frac{\mu[\xi](B \cap A^c)}{\mu[\xi](A^c)}, \quad \forall B \in \mathcal{T},$$

  where we denoted by $\mathcal{T} := \sigma(\{\eta_b : b \in \chi\})$ the smallest σ-algebra on χ generated by all the edges in χ.

THANK YOU!