Large deviations for white-noise driven, nonlinear stochastic PDEs in two and three dimensions

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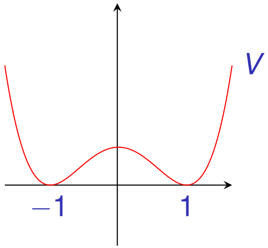
Noisy Allen-Cahn equation

$$\frac{du}{dt}(t,x) = \Delta u(t,x) - \Big(u(t,x)^3 - u(t,x)\Big) + \text{noise} \qquad \text{(AC)}$$

- $u \in \mathbb{R}$ order parameter
- $t \geq 0$ time, $x \in \mathbf{T}^d$ space
- Nonlinearity $u^3 - u = V'(u)$



Allen-Cahn II

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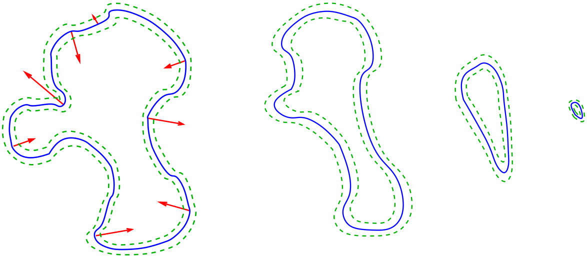
Poplular phenomenological model:

- Two phases $u \approx 1$ and $u \approx -1$. Dynamics of phases.
- Energy driven, no preservation of mass.
- Rescaled version approximates mean curvature flow.



What is the noise?

Noise $\sqrt{\varepsilon}\xi_\delta$ models termal fluctuation.

Correlation δ: ξ_δ Gaussian random field with

$$\mathbb{E}[\xi_\delta(t,x)\,\xi_\delta(s,y)] \approx \begin{cases} \delta^{-(d+2)} & \text{if } |x-y| + \sqrt{|t-s|} \ll \delta \\ 0 & \text{if } |x-y| + \sqrt{|t-s|} \gg \delta\,. \end{cases}$$

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- We are interested in “Freidlin-Wentzell type" large deviation behaviour as $\varepsilon, \delta \to 0$.

Formal derivation of large deviation behaviour

For ξ space time white noise

$$\mathbb{P}[\sqrt{\varepsilon}\xi \in du] \propto \exp\Big(-\frac{1}{2\varepsilon}\int_0^T\int_{\mathbf{T}^d} u(t,x)^2\,dt\,dx\Big)\prod_{t,x} du(t,x)\,.$$

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"Girsanov" gives for u_ε solution of (AC).

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where

$$\mathscr{I}(u) = \frac{1}{2} \int_0^T \int_{\mathbf{T}^d} \big(\partial_t u - \Delta u + u^3 - u \big)^2 \, dx \, dt \; .$$

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Large deviations

$$\lim_{\varepsilon\to 0}\varepsilon \log \mathbb{P}[u_\varepsilon \in du] = -\mathscr{I}(u)\,.$$

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- OK for $d = 1$. $u^{(\varepsilon)}$ solution of $\partial_t u = \partial_x^2 u - u^3 + u + \sqrt{\varepsilon}\xi$.

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$u^{(\varepsilon)}$ *satisfy a large deviation principle in* $\mathcal{C}([0, T] \times \mathbf{T}^d)$.

Rate function

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For every closed set $C \subseteq \mathcal{C}([0, T] \times \mathbf{T}^d)$ we have

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(C) \leq - \inf_{u \in C} \mathscr{I}(u) \ .$$

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Well-posedness problem

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Theorem (Hairer, Ryser, W. '12)

For $d = 2$ *if* $\delta, \varepsilon \to 0$

$$u_\delta^{(\varepsilon)} \to \begin{cases} 0 & \text{if } |\log(\delta)|^{-1} \ll \varepsilon \ll 1 \\ u^*_{\mathrm{det}} & \text{if } |\log(\delta)|^{-1} = \lambda^2 \varepsilon \\ u_{\mathrm{det}} & \text{if } 0 \ll \varepsilon \ll |\log(\delta)|^{-1} \end{cases} .$$

- u_{det} solution to deterministic Allen-Cahn equation.
- u^*_{det} solution to $\partial_t u = \Delta u - u^3 + u - C_\lambda u$.

Main result I

$u_\delta^{(\varepsilon)}$ solution of $\partial_t u = \Delta u + u - u^3 + \sqrt{\varepsilon}\xi_\delta$.

Theorem (Hairer, W. '14)

Space dimension $d = 2$ *or* $d = 3$.

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Renormalised solutions

$$\partial_t u = \Delta u + (C + 3\varepsilon C_\delta^{(1)} - 9\varepsilon^2 C_\delta^{(2)})u - u^3 + \sqrt{\varepsilon}\xi_\delta \ , \qquad (\widehat{AC})$$

- For $d = 2$, $C_\delta^{(2)} = 0$ and $C_\delta^{(1)} = \frac{1}{4\pi}|\log\delta|$.
- For $d = 3$, $C_\delta^{(1)} \propto \delta^{-1}$ and $C_\delta^{(2)} \propto |\log\delta|$.

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Theorem

$d = 2$ or $d = 3$. For every $\varepsilon > 0$ the $\hat{u}_\delta^{(\varepsilon)}$ converge to a limit $\hat{u}^{(\varepsilon)}$ as $\delta \to 0$. Formally

$$\partial_t \hat{u}^{(\varepsilon)} \;=\; \Delta \hat{u}^{(\varepsilon)} - \big(\hat{u}^{(\varepsilon)}\big)^{(3)} + \hat{u} \times \varepsilon\infty + \sqrt{\varepsilon}\xi \;.$$

or dynamic ϕ_2^4 model.

- $d = 2$ da Prato/Debussche '03, $d = 3$ Hairer '13.

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Then $\hat{u}_\delta^{(\varepsilon)}$ *satisfy a large deviation principle in* $\mathcal{C}([0,T], \mathcal{C}^\eta) \cup \{\infty\}$. *Rate function*

$$\mathscr{I}(u) = \frac{1}{2}\int_0^T \int_{\mathbf{T}^d} \left(\partial_t u - \Delta u + u^3 - Cu\right)^2 dx\, dt\ .$$

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- Renormalisation vanished on the level of large deviations. Formally $\varepsilon\infty \to 0$!

Discussion

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- Polynomial structure of non-linearity is important. For $d = 2$ arbitrary polynomial is possible.
- Low regularity space $\eta < 0$ in $d = 2$ and $\eta < -\frac{1}{2}$ in $d = 3$.

Discussion II: Related works

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- Rate function for Navier-Stokes. [Bouchet, Laurie, Zaboronski 2014].
- Study limit $\lim_{\delta\to 0}\lim_{\varepsilon\to 0}$ (i.e. $\varepsilon \ll \delta$): [Cerrai, Freidlin 2011].

Why is the one-dimensional case easy?

$$\partial_t u = \Delta u - u^3 + \xi \ .$$

Solution of the linearised equation $\Pi \overset{\bullet}{|} := K * \xi$.

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Hence LDP follows from a contraction principle.

Subcriticality: On small scales the non-linear term is lower order.

Example: scaling $x \mapsto \lambda x$, $t \mapsto \lambda^2 t$ and $u \mapsto \lambda^{\frac{d-2}{2}} u$, leaves Stochastic heat equation invariant. Under this scaling (AC) becomes

$$\partial_t \tilde{u} = \Delta \tilde{u} - \lambda^{4-d} \tilde{u}^3 + \tilde{\xi} \,.$$

A primer on regularity structures

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Formal expansion: Need to construct “by hand" a model; i.e. all $\Pi\tau$ for τ in a finite list $\mathcal{W} = \{\Xi, \mathord{\text{I}}, \mathord{\vee}, \mathord{\Psi}, \ldots\}$.

How to build that list

$$\partial_t u = \Delta u - u^3 + \xi \; . \qquad (u_3^4)$$

Integral equation (Duhamel's principle)

$$u = K * (u^3 + \xi).$$

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$$u_0 = 0.$$

$$\mathcal{W}_0 = \varnothing,$$

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$$u_2 = K * ((K * \xi)^3 + \xi).$$

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- If ξ is a smooth function, there is a canonical model Π, that can be constructed recursively.

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There is a metric on $\mathcal{M} := \{\text{models}\}$ and a solution operator $\mathcal{S}\colon \mathcal{M} \to \mathcal{C}([0,T], \mathcal{C}^\eta)$ that depends continuously on the data contained in $\mathcal{M}$.

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If Π is the canonical model constructed rom a smooth function ξ, then $\mathcal{S}(\Pi)$ maps to the classical solution.

Similar in spirit to Rough path theory.

Renormalised models

In dimension $d = 2$,

$$\hat{\Pi}_z^\delta \Psi^2 = (\hat{\Pi}_z^\delta \Psi)^2 - C_\delta^{(1)} , \quad \hat{\Pi}_z^\delta \Psi^3 = (\hat{\Pi}_z^\delta \Psi)^3 - 3C_\delta^{(1)} \hat{\Pi}_z^\delta \Psi .$$

For $d = 3$ as well

$$\hat{\Pi}_z^\delta \big(\mathcal{I}(\Psi^2)\Psi^2\big) = (\hat{\Pi}_z^\delta \mathcal{I}(\Psi^2))(\hat{\Pi}_z^\delta \Psi^2) - C_\delta^{(2)} , \quad \hat{\Pi}_z^\delta \big(\mathcal{I}(\Psi^3)\Psi^2\big) = (\hat{\Pi}_z^\delta \mathcal{I}(\Psi^3))(\hat{\Pi}_z^\delta \Psi^2) - 3C_\delta^{(2)} ,$$

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This renormalisation of the model, corresponds to the renormalisation of the equation discussed above in ($\widehat{AC}$).

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Theorem (Hairer '13)

This renormalisation of the model, corresponds to the renormalisation of the equation discussed above in ($\widehat{AC}$). These renormalised models converge in probability with respect to the metric of $\mathcal{M}$.

Large deviation for Banach-valued Wiener chaos I

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- $\mathcal{W}$ some finite set.

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Large deviation for Banach-valued Wiener chaos I

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- Natural rescaling:

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Large deviation for Banach-valued Wiener chaos II

Theorem (Hairer, W.)

$\mathbf{F}$, $\mathbf{F}^{(\varepsilon)}$, $\mathbf{F}_{\mathrm{hom}}$ *as above.*

Then $\mathbf{F}^{\varepsilon}$ *satisfy a large deviation principle on* $\mathbf{E} = \oplus_{\tau \in \mathcal{W}} E_\tau$.

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$$\mathscr{I}(\mathbf{s}) = \inf \left\{ \frac{1}{2} |h|^2_{\mathcal{H}} : h \in \mathcal{H} \text{ with } \mathbf{F}_{\mathrm{hom}}(h) = \mathbf{s} \right\} .$$

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- Similar results well known (e.g. Borell, Ledoux).

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$\Rightarrow$ Main result follows from a contraction principle.

Conclusion

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- This Wiener LDP derived by generalised contraction principle.