On multiple peaks of Gaussian Processes   Jw R. Eldan & A. Zhai

Gaussian process $\{X_i \ 1 \le i \le n\}$, $\mathrm{var}\, X_i \le \sigma^2$, $\mathbb{E} X_i = 0$ $\forall\, 1 \le i \le n$

Write $R(i,j) = \mathbb{E} X_i X_j$, $M = \sup_{1 \le i \le n} X_i$, $m = \mathbb{E} M$, $\sigma_M^2 = \mathrm{var}\, M$

Borell-Sudakov-Tsirelson (70's)

$$\mathbb{P}(|M - m| \ge z) \le 2 e^{-z^2/2\sigma^2}$$, in particular $\sigma_M^2 \le \sigma^2$

Some Examples (a) REM $\sigma^2 = 1$, $\sigma_M^2 \simeq \frac{1}{\log n}$

(b) DGFF in 2D $\sigma^2 \simeq \log n$, $\sigma_M^2 \simeq 1$

(c) 2D last passage percolation $\sigma^2 = 2n$, $\sigma_M^2 \simeq n^{2/3}$ → prediction

(d) SK spin glass model

$H_n \in \{-1,1\}^n$ $X_\tau = \frac{1}{\sqrt{n}} \sum_{1 \le i,j \le n} \tau_i \tau_j g_{ij}$ $\tau \in H_n$ where $g_{ij} \overset{iid}{\sim} N(0,1)$

$\sigma^2 = n$ $\sigma_M^2 \simeq n^\nu$ $\nu < 1$

Some Examples (a) REM $\sigma_m^2 \sim \log n$
(b) DGFF in 2D

Question (a) when would "$\sigma_m^2 << \sigma^2$" occur?
(b) Any structure properties when "$\sigma_m^2 << \sigma^2$"

Chatterjee 2008, 2009

(i) superconcentration $\sigma_m^2 << \sigma^2$ (ii) <u>Chaos</u> $X \overset{d}{=} \sqrt{1-\varepsilon}\, X' + \sqrt{\varepsilon}\, X''$

$\exists\, \varepsilon << 1$, st $R(\underline{I}_m, \underline{I}_m') \ll \sigma^2$ $\quad I_m$ - maximizer of X $\quad I_m'$

(iii) Multiple peaks $\exists\, \delta, \varepsilon << 1$, $\ell >> 1$ st w.h.p $\exists A$ with
(a) $|A| \geq \ell$ (b) $X_i \geq (1-\delta) m \ \forall i \in A$. (c) $|R(i,j)| \leq \varepsilon \sigma^2 \ \forall i \neq j \in A$

Chatterjee Results

① $\sigma_m^2 << \sigma^2$ for many models including LPP.

② $\sigma_\beta^2 = \mathrm{var}\left(\frac{1}{\beta} \log \sum_{i=1}^{N} e^{\beta X_i}\right) << \sigma^2$ for SK

③ superconcentration $\Longleftrightarrow$ Chaos $\Longrightarrow$ multiple peak $\rightarrow$ non-negative correlated

D.E.Z 2013 $\forall \varepsilon, \delta$, $\exists C = C(\varepsilon, \delta)$ s.t. with prob tending to 1, $\exists A$

s.t. (a) $|A| \geq e^{c\sigma/\sigma_m^2}$, (b) $X_i \geq (1-\delta)m$ for all $i \in A$ (c) $|R(i,j)| \leq \varepsilon \sigma^2$ $\forall i,j \in A$

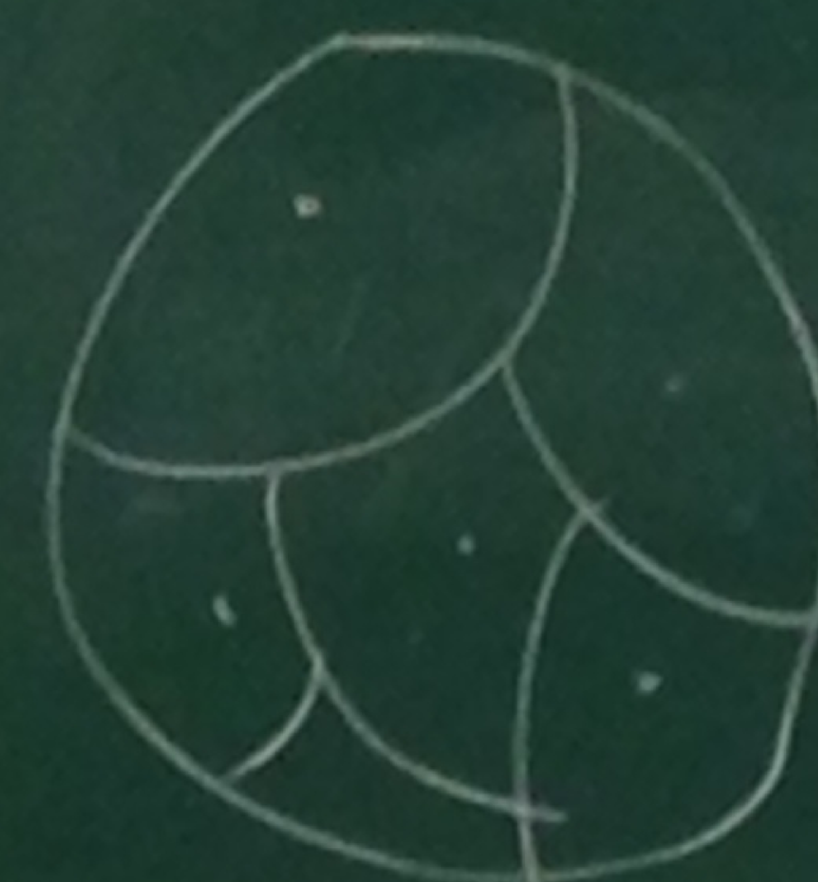
Step 1 if $\sigma_m^2 \ll \sigma^2$, does there exist A s.t. $|A| \geq e^{c\sigma^2/\sigma_m^2} = K$

and $|R(i,j)| \leq \varepsilon \sigma^2$

Pf Assume $\sigma = 1$ Suppose # of centers is $\leq K$

$[n] = \bigcup_{i=1}^{K} B_i$ where $\exists i^*$ s.t.

$\forall j \in B_i$, $|R(j, i^*)| \geq \varepsilon$



Represent $X_i = \langle V_i, y \rangle$

For $y \in \mathbb{R}^n$, let i_y be argmax $\langle V_i, y \rangle$ and let i_y^* be s.t. $i_y \in B_{i_y^*}$

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Def map f $y \longrightarrow y + \eta V_{i_y^*}$ where $\eta = \frac{1}{10\sqrt{\log k}}$

<u>Step</u> take $\alpha = 1 - \delta$, and $A_\alpha = \{ i : X_i \geq \alpha m \}$

Consider an independent copy X' but restricted to A_α

wish to argue $\operatorname{var} \sup_{i \in A_\alpha} X' \simeq \sigma_m^2$ quench on A_α

$$\tilde{X} = \sqrt{1-t}\, X + \sqrt{t}\, X \qquad \text{for some } t$$

need to show $\tilde{I}_m \in A_\alpha$ for some suitable t