Simple random walk on the two-dimensional uniform spanning tree and its scaling limits

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UNIFORM SPANNING TREE IN TWO DIMENSIONS

Let $\Lambda_n := [-n, n]^2 \cap \mathbb{Z}^2$.

A subgraph of the lattice is a **spanning tree** of Λ_n if it connects all vertices, no cycles.

Let $\mathcal{U}^{(n)}$ be a spanning tree of Λ_n selected uniformly at random from all possibilities.

The UST on $\mathbb{Z}^2$, $\mathcal{U}$, is then the local limit of $\mathcal{U}^{(n)}$.
NB. Wired/free boundary conditions unimportant.

Almost-surely, $\mathcal{U}$ is a spanning tree of $\mathbb{Z}^2$.

[Aldous, Benjamini, Broder, Häggström, Kirchoff, Lyons, Pemantle, Peres, Schramm. . .]

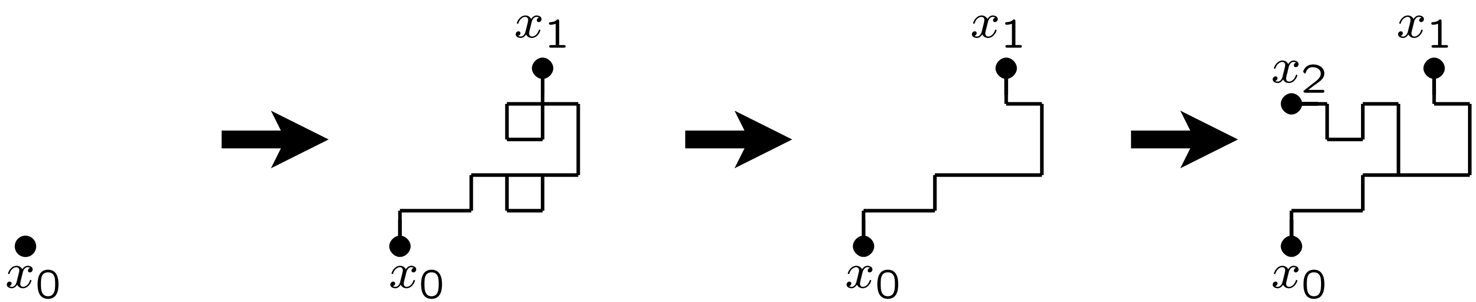
WILSON'S ALGORITHM ON $\mathbb{Z}^2$

Let $x_0 = 0, x_1, x_2, \ldots$ be an enumeration of $\mathbb{Z}^2$.

Let $\mathcal{U}(0)$ be the graph tree consisting of the single vertex x_0.

Given $\mathcal{U}(k-1)$ for some $k \geq 1$, define $\mathcal{U}(k)$ to be the union of $\mathcal{U}(k-1)$ and the loop-erased random walk (LERW) path run from x_k to $\mathcal{U}(k-1)$.

The UST $\mathcal{U}$ is then the local limit of $\mathcal{U}(k)$.

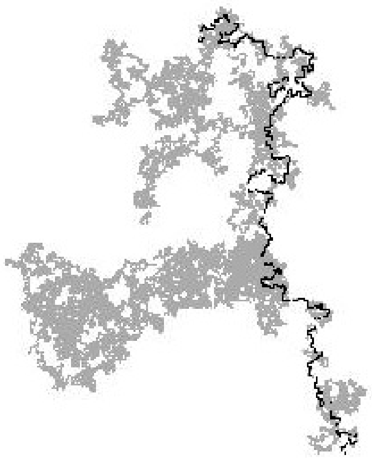


LERW SCALING IN $\mathbb{Z}^d$

Consider LERW as a process $(L_n)_{n\geq 0}$ (assume original random walk is transient).

In $\mathbb{Z}^d$, $d \geq 5$, L rescales diffusively to BM [Lawler 1980].

In $\mathbb{Z}^4$, with logarithmic corrections rescales to BM [Lawler].



Picture: Ariel Yadin

In $\mathbb{Z}^3$, $\{L_n : n \in [0, \tau]\}$ has a scaling limit [Kozma 2007].

In $\mathbb{Z}^2$, $\{L_n : n \in [0, \tau]\}$ has SLE(2) scaling limit, UST peano curve has SLE(8) scaling limit [Lawler/Schramm/Werner 2004]. Growth exponent is 5/4 [Kenyon, Masson, Lawler].

Let $M_n = |LERW(0, B_E(0,n))|$ be the length of a LERW run from 0 to $B_E(0,n)^c$.

Theorem.(d=2)

[Kenyon 2000] $\lim_{n\to\infty} \frac{\log E^0 M_n}{log n} = 5/4$

[Lawler 2013] $c_1 n^{5/4} \leq E^0 M_n \leq c_2 n^{5/4}$

Now consider random walk on the UST.

RW on random graphs: General theory.

Let $\mathcal{G}(\omega)$ be a random graph on $(\Omega, \mathbb{P})$. Assume $\exists 0 \in \mathcal{G}(\omega)$.

Let $D \geq 1$. For $\lambda \geq 1$, we sat that $B(0,R)$ in $\mathcal{G}(\omega)$ is λ-**good** if

$$\lambda^{-1} R^D \leq |B(0,R)| \leq \lambda R^D,$$
$$\lambda^{-1} R \leq R_{\text{eff}}(0, B(0,R)^c) \leq R + 1.$$

λ-good is a nice control of the volume and resistance for $B(0,R)$.

Theorem. [Barlow/Jarai/K/Slade 2008, K/Misumi 2008]

Suppose $\exists p > 0$ such that

$$\mathbb{P}(\{\omega : B(0,R) \text{ is } \lambda\text{-good.}\}) \geq 1 - \lambda^{-p} \qquad \forall R \geq R_0, \forall \lambda \geq \lambda_0.$$

Then $\exists \alpha_1, \alpha_2 > 0$ and $N(\omega), R(\omega) \in \mathbb{N}$ s.t. the following holds for $\mathbb{P}$-a.e. ω:

$$(\log n)^{-\alpha_1} n^{-\frac{D}{D+1}} \le p_{2n}^{\omega}(0,0) \le (\log n)^{\alpha_1} n^{-\frac{D}{D+1}}, \qquad \forall n \ge N(\omega),$$
$$(\log R)^{-\alpha_2} R^{D+1} \le E_{\omega}^{0} \tau_{B(0,R)} \le (\log R)^{\alpha_2} R^{D+1}, \qquad \forall R \ge R(\omega).$$

In particular,

$$d_s(G) := \lim_{n \to \infty} \frac{\log p_{2n}^{\omega}(0,0)}{\log n} = \frac{2D}{D+1}$$

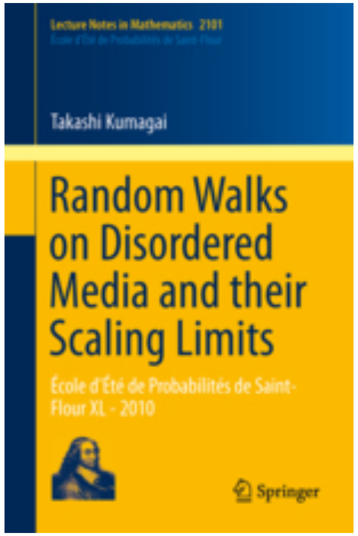
Examples.

$D = 2$ and $d_s = 4/3$

- Critical percolation on regular trees conditioned to survive forever. (Barlow/K '06)
- Infinite incipient cluster (IIC) for spread out oriented percolation for $d \geq 6$ (Barlow/Jarai/K/Slade '08)
- Invasion percolation on a regular tree. (Angel/Goodman/den Hollander/Slade '08)
- IIC for percolation on $\mathbb{Z}^d$, $d \geq 19$ (Kozma/Nachmias '09)

More general

- α-stable Galton-Watson trees conditioned to survive forever (Croydon/K '08) $d_s = 2\alpha/(2\alpha - 1)$



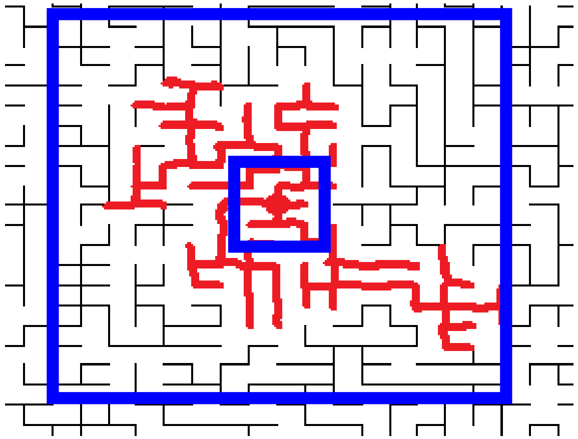
(Advertisement)

Takashi Kumagai

Random Walks on Disordered Media and their Scaling Limits.

Lecture Notes in Mathematics, Vol. 2101, École d'Été de Probabilités de Saint-Flour XL–2010. Springer, New York (2014).

VOLUME AND RESISTANCE ESTIMATES [BARLOW/MASSON 2010,2011]



With high probability,

$$B_E(x, \lambda^{-1}R) \subseteq B_{\mathcal{U}}(x, R^{5/4}) \subseteq B_E(x, \lambda R),$$

as $R \to \infty$ then $\lambda \to \infty$.

It follows that with high probability,

$$\mu_{\mathcal{U}}\left(B_{\mathcal{U}}(x, R)\right) \asymp R^{8/5}.$$

Also with high probability,

$$\text{Resistance}(x, B_{\mathcal{U}}(x, R)^c) \asymp R.$$

$\Rightarrow$ Exit time for intrinsic ball radius R is $R^{13/5}$,
HK bounds $p^{\mathcal{U}}_{2n}(0,0) \asymp n^{-8/13}$. ($D = 8/5, d_s = 16/13$)

(Q) How about scaling limit for UST?

Barlow/Masson obtained further detailed properties.

Theorem.[Barlow/Masson 2010]

$$\begin{aligned} \mathbb{P}(M_n > \lambda E M_n) &\leq 2e^{-c_1\lambda}, \\ \mathbb{P}(M_n < \lambda^{-1} E M_n) &\leq 2e^{-c_2\lambda^{c_3}} \end{aligned}$$

Theorem.[Barlow/Masson 2011]

$$\begin{aligned} \mathbb{P}(B_{\mathcal{U}}(0, R^{5/4}/\lambda) \not\subset B_E(0,R)) &\leq c_4 e^{-\lambda^{2/3}}, \\ \mathbb{P}(B_E(0,R) \not\subset B_{\mathcal{U}}(0, \lambda R^{5/4})) &\leq c_\epsilon \lambda^{-4/15-\epsilon}. \end{aligned}$$

While for most points $x \in \mathbb{Z}^2$, the balls $B_E(0, R)$ and $B_{\mathcal{U}}(0, R^{5/4})$ will be comparable, there are neighboring points in $\mathbb{Z}^2$ which are far in $\mathcal{U}$.

Lemma. [Benjamini et. al. 2001]

The box $[-n, n]^2$ contains with probability 1 neighbouring points $x, y \in \mathbb{Z}^2$ with $d_{\mathcal{U}}(x, y) \geq n$.

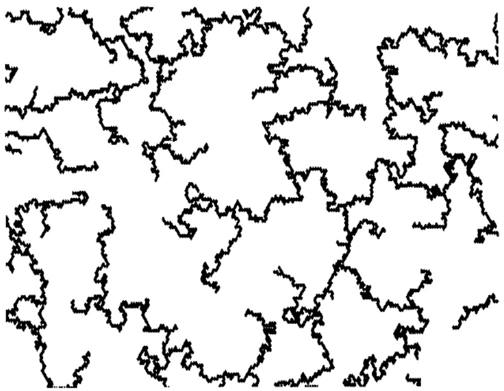
Proof. Consider the path (in $\mathbb{Z}^2$) of length $8n$ around the box $[-n, n]^2$: If each neiboring pair were connected by a path in $\mathcal{U}$ of length less than n, then this path would not contain 0. So we would obtain a loop around 0 — which is impossible since $\mathcal{U}$ is a tree.

UST SCALING [SCHRAMM 2000]

Consider $\mathcal{U}$ as an ensemble of paths:

$$\mathfrak{U} = \left\{(a, b, \pi_{ab}) : \ a, b \in \mathbb{Z}^2\right\},$$

where π_{ab} is the unique arc connecting a and b in $\mathcal{U}$.
cf. [Aizenman/Burchard/Newman/Wilson 1999].



Picture: Oded Schramm

Scaling limit $\mathfrak{T}$ a.s. satisfies:

- each pair $a, b \in \dot{\mathbb{R}}^2$ connected by a path;
- if $a \neq b$, then this path is simple;
- if $a = b$, then this path is a point or a simple loop;
- the trunk, $\cup_{\mathfrak{T}} \pi_{ab} \backslash \{a, b\}$, is a dense topological tree with degree at most 3.

ISSUE : This topology does not carry information about intrinsic distance, volume, or resistance.

GENERALISED GROMOV-HAUSDORFF TOPOLOGY (cf. [GROMOV, LE GALL/DUQUESNE])

Define $\mathbb{T}$ to be the collection of measured, rooted, spatial trees, i.e.

$$(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}),$$

where:

- $(\mathcal{T}, d_{\mathcal{T}})$ is a locally compact real tree;
- $\mu_{\mathcal{T}}$ is a Borel measure on $(\mathcal{T}, d_{\mathcal{T}})$;
- $\phi_{\mathcal{T}}$ is a cont. map from $(\mathcal{T}, d_{\mathcal{T}})$ into $\mathbb{R}^2$;
- $\rho_{\mathcal{T}}$ is a distinguished vertex in $\mathcal{T}$.

On $\mathbb{T}_c$ (compact trees only), define a distance Δ_c by

$$\inf_{\substack{Z,\psi,\psi',\mathcal{C}:\\(\rho_{\mathcal{T}},\rho'_{\mathcal{T}})\in\mathcal{C}}} \left\{ d_P^Z(\mu_{\mathcal{T}}\circ\psi^{-1}, \mu'_{\mathcal{T}}\circ\psi'^{-1}) + \sup_{(x,x')\in\mathcal{C}} \left(d_Z(\psi(x),\psi'(x')) + \left|\phi_{\mathcal{T}}(x) - \phi'_{\mathcal{T}}(x')\right| \right) \right\}.$$

Can be extended to locally compact case.

TIGHTNESS OF UST

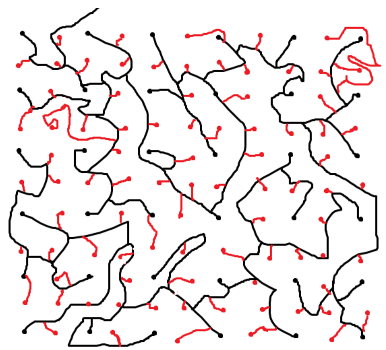
Theorem. If $\mathbf{P}_\delta$ is the law of the measured, rooted spatial tree

$$\left(\mathcal{U}, \delta^{5/4} d_{\mathcal{U}}, \delta^2 \mu_{\mathcal{U}}(\cdot), \delta\phi_{\mathcal{U}}, 0\right)$$

under $\mathbf{P}$, then the collection $(\mathbf{P}_\delta)_{\delta\in(0,1)}$ is tight in $\mathcal{M}_1(\mathbb{T})$.

Proof involves:

- strengthening estimates of [Barlow/Masson],
- comparison of Euclidean and intrinsic distance along paths.



UST LIMIT PROPERTIES

If $\tilde{\mathbf{P}}$ is a subsequential limit of $(\mathbf{P}_\delta)_{\delta\in(0,1)}$, then for $\tilde{\mathbf{P}}$-a.e. $(\mathcal{T}, d_\mathcal{T}, \mu_\mathcal{T}, \phi_\mathcal{T}, \rho_\mathcal{T})$ it holds that:

(i) $\mu_\mathcal{T}$ is non-atomic, supported on the leaves of $\mathcal{T}$, i.e. $\mu_\mathcal{T}(\mathcal{T}^o) = 0$, where $\mathcal{T}^o := \mathcal{T}\backslash\{x \in \mathcal{T} : \deg_\mathcal{T}(x) = 1\}$;

(ii) for any $R > 0$,

$$\liminf_{r\to 0} \frac{\inf_{x\in B_\mathcal{T}(\rho_\mathcal{T},R)} \mu_\mathcal{T}\left(B_\mathcal{T}(x,r)\right)}{r^{8/5}(\log r^{-1})^{-c}} > 0,$$

(iii) $\phi_\mathcal{T}$ is a homeo. between $\mathcal{T}^o$ and $\phi_\mathcal{T}(\mathcal{T}^o)$ (dense in $\mathbb{R}^2$);

(iv) $\max_{x\in\mathcal{T}} \deg_\mathcal{T}(x) = 3$;

(v) $\mu_\mathcal{T} = \mathcal{L} \circ \phi_\mathcal{T}$.

To prove this, we need the following 'uniform control':

$$\lim_{\eta\to 0}\liminf_{\delta\to 0}\mathbf{P}\left(\sup_{\substack{x,y\in B_{\mathcal{U}}(0,c_1\delta^{-5/4}r):\\ d_{\mathcal{U}}(x,y)\leq c_2\delta^{-5/4}\eta}} d^S_{\mathcal{U}}(x,y) > \delta^{-1}\varepsilon\right) = 0,$$

$$\lim_{\varepsilon\to 0}\limsup_{\delta\to 0}\mathbf{P}\left(\inf_{\substack{x,y\in B_{\mathcal{U}}(0,\delta^{-5/4}r):\\ d_{\mathcal{U}}(x,y)\geq \delta^{-5/4}\eta}} d^S_{\mathcal{U}}(x,y) < \delta^{-1}\varepsilon\right) = 0,$$

where $d^S_{\mathcal{U}} = \mathrm{diam}(\gamma(x,y))$ (Euclidean diameter of the LERW between x and y; Schramm's distance).

$\Rightarrow$ This involves uniform control and requires more detailed estimates than those of Barlow/Masson.

As a by-product of the detailed estimates, we can sharpen some HK estimates.

Proposition. For each $q > 0$, there exist $c_q, C_q > 0$ such that the following holds

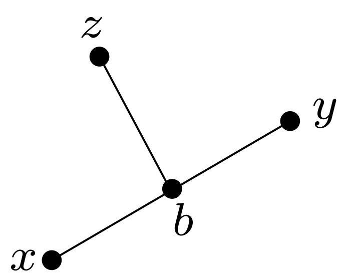
$$c_q n^{5q/13} \leq \mathbb{E}\left(d_{\mathcal{U}}(0, X_n)^q\right) \leq C_q n^{5q/13} \quad \forall n \geq 1.$$

NB. Marlow/Masson's estimates include $(\log n)^{\pm c}$.

Given such generalized G-H convergence of trees, we can prove convergence of the process on the trees (generalization of the theory due to Crodon (2008)).

On the (limiting) real treee $(\mathcal{T}, d_{\mathcal{T}}, \mu^{\mathcal{T}})$ s.t. $\mu^{\mathcal{T}}$ has full support, one can define a 'Brownian motion' $X^{\mathcal{T}} = (X_t^{\mathcal{T}})_{t\geq 0}$.

- For $x, y, z \in \mathcal{T}$,



$$P_z^{\mathcal{T}}(\tau_x < \tau_y) = \frac{d_{\mathcal{T}}(b(x,y,z),y)}{d_{\mathcal{T}}(x,y)}.$$

- Mean occupation density when started at x and killed at y,

$$2d_{\mathcal{T}}(b(x,y,z),y)\mu^{\mathcal{T}}(dz).$$

Requirement :

$$\liminf_{r\to 0} \frac{\inf_{x\in\mathcal{T}} \mu_{\mathcal{T}}(B_{\mathcal{T}}(x,r))}{r^{\kappa}} > 0. \qquad \exists\kappa > 0.$$

LIMITING PROCESS FOR SRW ON UST

Suppose $(\mathbf{P}_{\delta_i})_{i\geq 1}$, the laws of

$$\left(\mathcal{U}, \delta_i^{5/4} d_{\mathcal{U}}, \delta_i^2 \mu_{\mathcal{U}}, \delta_i \phi_{\mathcal{U}}, 0\right),$$

form a convergent sequence with limit $\tilde{\mathbf{P}}$.

Let $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}}, \rho_{\mathcal{T}}) \sim \tilde{\mathbf{P}}$.

It is then the case that $\mathbb{P}_{\delta_i}$, the annealed laws of

$$\left(\delta_i X^{\mathcal{U}}_{\delta_i^{-13/4} t}\right)_{t\geq 0},$$

converge to $\tilde{\mathbb{P}}$, the annealed law of

$$\left(\phi_{\mathcal{T}}(X^{\mathcal{T}}_t)\right)_{t\geq 0},$$

as probability measures on $C(\mathbb{R}_+, \mathbb{R}^2)$.

HEAT KERNEL ESTIMATES FOR SRW LIMIT

Let $R>0$. For $\tilde{\mathbf{P}}$-a.e. realisation of $(\mathcal{T},d_{\mathcal{T}},\mu_{\mathcal{T}},\phi_{\mathcal{T}},\rho_{\mathcal{T}})$, there exist random constants $c_1,c_2,c_3,c_4,t_0\in(0,\infty)$ and deterministic constants $\theta_1,\theta_2,\theta_3,\theta_4\in(0,\infty)$ such that the heat kernel associated with the process $X^{\mathcal{T}}$ satisfies:

$$p_t^{\mathcal{T}}(x,y)\leq c_1 t^{-8/13}\ell(t^{-1})^{\theta_1}\exp\left\{-c_2\left(\frac{d_{\mathcal{T}}(x,y)^{13/5}}{t}\right)^{5/8}\ell(d_{\mathcal{T}}(x,y)/t)^{-\theta_2}\right\},$$

$$p_t^{\mathcal{T}}(x,y)\geq c_3 t^{-8/13}\ell(t^{-1})^{-\theta_3}\exp\left\{-c_4\left(\frac{d_{\mathcal{T}}(x,y)^{13/5}}{t}\right)^{5/8}\ell(d_{\mathcal{T}}(x,y)/t)^{\theta_4}\right\},$$

for all $x,y\in B_{\mathcal{T}}(\rho_{\mathcal{T}},R)$, $t\in(0,t_0)$, where $\ell(x):=1\vee\log x$.