The condensation phase transition in random graph coloring

Victor Bapst

Goethe University, Frankfurt

Joint work with Amin Coja-Oghlan, Samuel Hetterich, Felicia Rassmann and Dan Vilenchik

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Outline

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Random graph coloring

- Draw a random graph on N vertices by connecting any two vertices with probability d/N at random.
- Is this graph k-colorable ?
- How many k-colorings can we find ? For a given graph: $Z(G)$.
  In general: either zero or exponentially many.
- Taking the average over the choice of the graph and $N \to \infty$:
  - Average number of colorings: $[EZ(G)]^{1/N} \to k(1-1/k)^{d/2}$.
  - Typical number of colorings: $E\left[Z(G)^{1/N}\right] \xrightarrow{?} \Phi_k(d) =??$.

Phase transitions

- Phase transition (informel): discontinuity in some "macroscopic" quantity describing a problem. For instance:
  - the size of the largest connected component for Erdős-Rényi random graphs, upon increasing the average degree,
  - the density when freezing water,
  - the derivative of the magnetization when heating a magnet.

- Phase transition (here): non analyticity of $\Phi_k(d)$.
  For instance it is conjectured that there exists $d_{\mathrm{col}}(k)$ such that:
  - for $d < d_{\mathrm{col}}(k)$, $\Phi_k(d) > 0$, and $\lim_{d \nearrow d_{\mathrm{col}}(k)} \Phi_k(d) > 0$.
  - for $d > d_{\mathrm{col}}(k)$, $\Phi_k(d) = 0$.

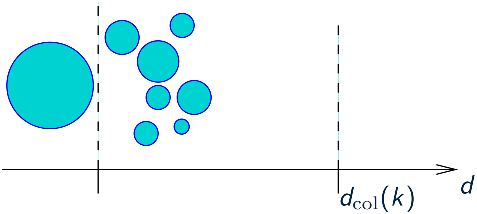
- Here we look at another phase transition that happens for $d < d_{\mathrm{col}}(k)$.

Outline

The physics picture

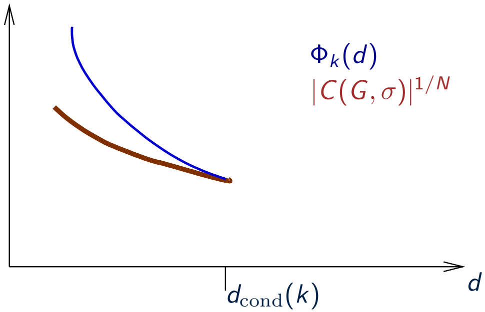
- A powerful tool to study random optimization problems: the cavity method.
  - Introduced by Mézard and Parisi in 2000.
  - General overview for random optimization problems: Krzakala, Montanari, Ricci-Tersenghi, Semerjian, Zdeborová in PNAS 2007.
  - Application to coloring: Krzakala, Pagnani, Weigt, Zdeborová ...

- Upon increasing d, solutions tend to group into clusters.
  $C(G,\sigma) = \{$colorings τ that can be reached from σ by altering at most $N/(k \log k)$ vertices at a time$\}$
  (Proofs: [Achlioptas - Coja-Oghlan 2008, Molloy 2012])



The physics picture [Zdeborová - Krzakala 2007]

Compare the cluster size with the total number of colorings.



Number of clusters: $\Phi_k(d) - |C(G,\sigma)|^{1/N}$.
What happens for $d > d_{\mathrm{cond}}(k)$?

Interlude: a broader view of condensation.

- A similar phenomenon appears when cooling too fast some liquids.
- This is the famous Kauzmann paradox:

THE NATURE OF THE GLASSY STATE AND THE BEHAVIOR OF LIQUIDS AT LOW TEMPERATURES

WALTER KAUZMANN

Department of Chemistry, Princeton University, Princeton, New Jersey

Received March 1, 1948

- .

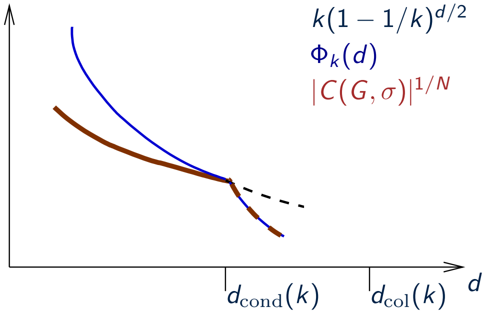
When such an extrapolation is applied to the observed entropy *vs.* temperature curves of several substances (most strikingly with glucose and lactic acid), a rather startling result is obtained.

the extrapolated entropy of the liquid becomes less than that of the crystalline solid

The physics picture [Zdeborová - Krzakala 2007]

Physics prediction: $d_{\mathrm{cond}}(k)$ marks a phase transition:

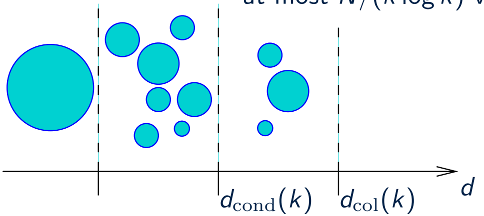
- for $d < d_{\mathrm{cond}}(k)$, $|C(G,\sigma)|^{1/N} < \Phi_k(d) = k(1-1/k)^{d/2}$,
- for $d > d_{\mathrm{cond}}(k)$, $|C(G,\sigma)|^{1/N} = \Phi_k(d) < k(1-1/k)^{d/2}$,
- the second derivative of $\Phi_k(d)$ is discontinuous at $d_{\mathrm{cond}}(k)$.



The physics picture [Zdeborová - Krzakala 2007]

Upon increasing d, the geometry of the set of solutions dramatically changes.

$C(G,\sigma) = \{$colorings τ that can be reached from σ by altering at most $N/(k \log k)$ vertices at a time$\}$



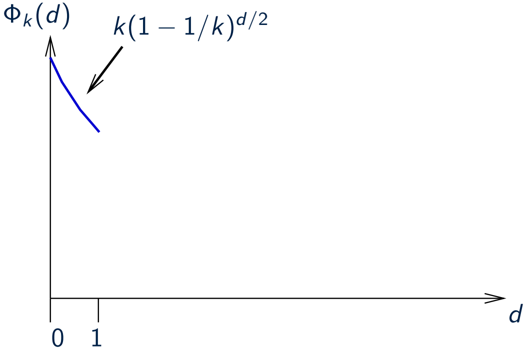
Condensation: when the number of clusters becomes sub-exponential.

$\Leftrightarrow$ when the cluster size $|C(G,\sigma)|^{1/N}$ equals $\Phi_k(d)$.

Outline

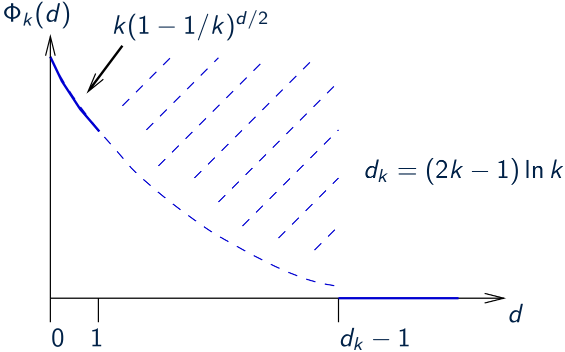
A first transition: the satisfiability transition

The number of colorings is easily understood when $d < 1$.



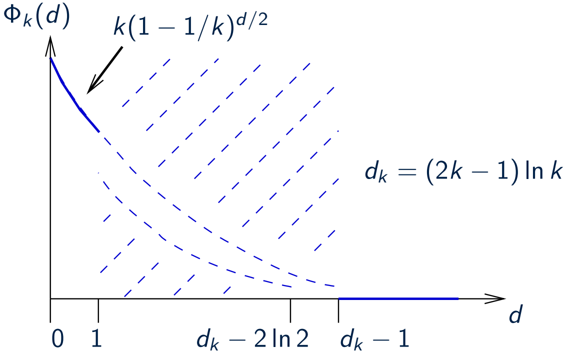
Upper bounds

Upper bound on the typical number of colorings: first moment method.
Can be improved from the naive result [Coja-Oghlan 2013].



Lower bounds

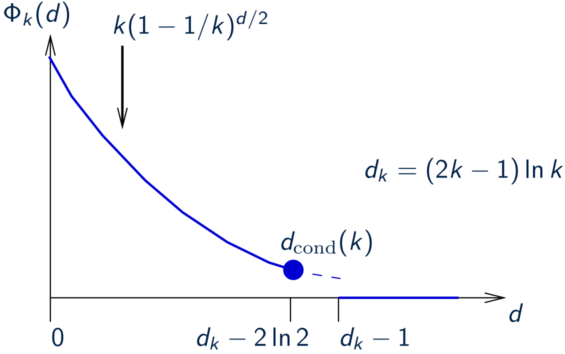
Lower bound on the typical number of colorings: second moment method [Achlioptas - Naor 2005, Coja-Oghlan - Vilenchik 2010].



The condensation transition

Theorem (1/2): for k large enough there exists $d_{\mathrm{cond}}(k)$ such that:

- there is a phase transition at $d_{\mathrm{cond}}(k)$,
- for $d < d_{\mathrm{cond}}(k)$: $\Phi_k(d) = k(1-1/k)^{d/2}$,
- for $d > d_{\mathrm{cond}}(k)$: $\Phi_k(d) < k(1-1/k)^{d/2}$ (or does not exist).



The condensation transition

Theorem (2/2): $d_{\mathrm{cond}}(k)$ is given by the formula predicted by the cavity method [Zdeborová - Krzakala 2007]. That is:

- $\Omega = \{$probability distributions on $\{1, \ldots, k\}\}$,
- $f : \bigcup_{\gamma \geq 0} \Omega^\gamma \to \Omega$,

$$f(\mu_1, \ldots, \mu_\gamma)(i) = \frac{\prod_{j=1}^{\gamma} 1 - \mu_j(i)}{\sum_{h \in [k]} \prod_{j=1}^{\gamma} 1 - \mu_j(h)}.$$

- $\mathcal{P} = \{$probability distributions on $\Omega\}$,
- $\mathcal{F}_{k,d} : \mathcal{P} \to \mathcal{P}$

$$\mathcal{F}_{k,d}(\pi) = \sum_{\gamma=0}^{\infty} \frac{\gamma^d \exp(-d)}{\gamma! \cdot Z_\gamma(\pi)} \int_{\Omega^\gamma} \left[\sum_{h=1}^{k} \prod_{j=1}^{\gamma} 1 - \mu_j(h) \right] \cdot \delta_{f[\mu_1, \ldots, \mu_\gamma]} \bigotimes_{j=1}^{\gamma} \mathrm{d}\pi(\mu_j).$$

where $Z_\gamma(\pi) = \sum_{h=1}^{k} \left(1 - \int_\Omega \mu(h) \mathrm{d}\pi(\mu)\right)^\gamma$

- $\Sigma_{k,d} : \mathcal{P} \to \mathbb{R}$ ("Complexity"). $\Sigma_{k,d}(\pi) = \ldots$
- $d_{\mathrm{cond}}(k)$ is the unique solution of $\Sigma_{k,d}(\pi^\star_{k,d}) = 0$ in $[d_k - 2, d_k]$, where $\pi^\star_{k,d}$ is a particular fixed point of $\mathcal{F}_{k,d}$.

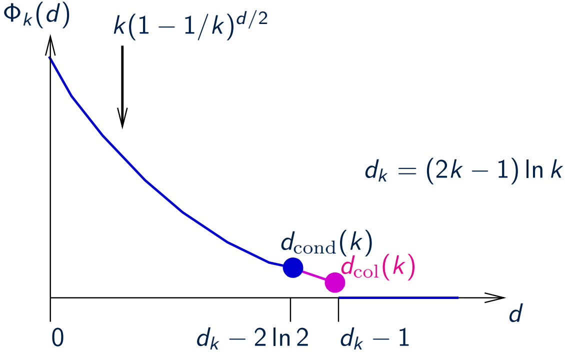
Conjectures: the satisfiability transition

Conjecture 1: $\Phi_k(d)$ exists for all d.

$\Rightarrow$ There exist a colorability threshold $d_{\mathrm{col}}(k)$.

$\Rightarrow$ There is a phase transition at $d_{\mathrm{col}}(k)$.

Conjecture 2: $d_{\mathrm{cond}}(k) < d_{\mathrm{col}}(k)$. There are exactly two phase transitions.



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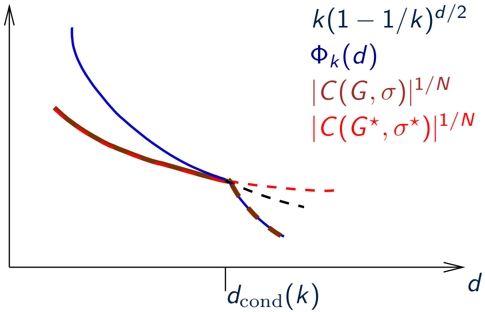
The planted model

- The condensation corresponds to the point where the cluster size $|C(G,\sigma)|^{1/N}$ equals (w.h.p.) $k(1-1/k)^{d/2}$.
- However it is hard to compute the cluster size:
  given a random graph, how do we even find a coloring ?
- Planting: first pick a configuration $\sigma^\star$ at random.
  Then generate a graph $G^\star$ by adding edges independently and uniformly at random such that:
  - $G^\star$ has as many vertices as G (in average),
  - $\sigma^\star$ is a coloring of this graph.

  Generating the pair ($G^\star$, $\sigma^\star$) is easy.
- The cluster size $|C(G^\star,\sigma^\star)|^{1/N}$ is also easier to compute.

Condensation and clusters sizes



- Physics intuition: [Krzakala - Zdeborová 2009]

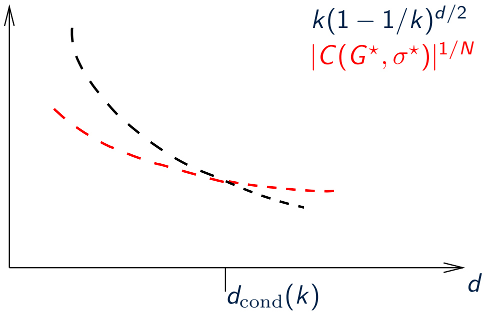
  – if $d < d_{\mathrm{cond}}(k)$,

  $$|C(G,\sigma)|^{1/N} = |C(G^\star,\sigma^\star)|^{1/N} < \Phi_k(d) = k(1-1/k)^{d/2},$$

  – if $d > d_{\mathrm{cond}}(k)$,

  $$|C(G,\sigma)|^{1/N} = \Phi_k(d) < k(1-1/k)^{d/2} < |C(G^\star,\sigma^\star)|^{1/N}.$$

Condensation and clusters sizes



- We use the following result: $\forall\epsilon>0$ [Coja-Oghlan - Vilenchik 2010]
  - if $|C(G^\star,\sigma^\star)|^{1/N} < k(1-1/k)^{d/2}-\epsilon$, then $d<d_{\rm cond}(k)$,
  - if $|C(G^\star,\sigma^\star)|^{1/N} > k(1-1/k)^{d/2}+\epsilon$, then $d>d_{\rm cond}(k)$.
- Therefore it is enough to understand the cluster size in the planted model.

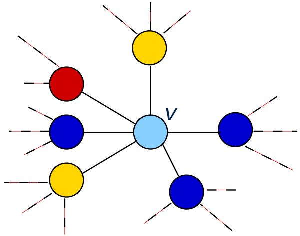
Outline

Frozen vertices

- We need to compute $|C(G^\star, \sigma^\star)|^{1/N}$.
  Remember that we defined $C(G^\star, \sigma^\star) = \{$colorings τ that can be reached from $\sigma^\star$ by altering at most $N/(k \log k)$ vertices at a time$\}$.
- Close to $d_{\mathrm{cond}}(k)$ *most of* the vertices are *frozen* : they take the same value for all $\tau \in C(G^\star, \sigma^\star)$. *Most of*: all but a fraction $1/k$.
- We need to identify the frozen vertices.

Frozen vertices

- Close to $d_{\mathrm{cond}}(k)$ most of the vertices are frozen: they take the same value for all $\tau \in C(G^\star, \sigma^\star)$.
- Intuition for that: a vertex v typically has *many* neighbors of each color.



Frozen vertices

- Close to $d_{\mathrm{cond}}(k)$ most of the vertices are frozen: they take the same value for all $\tau \in C(G^\star, \sigma^\star)$.
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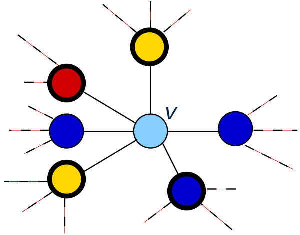
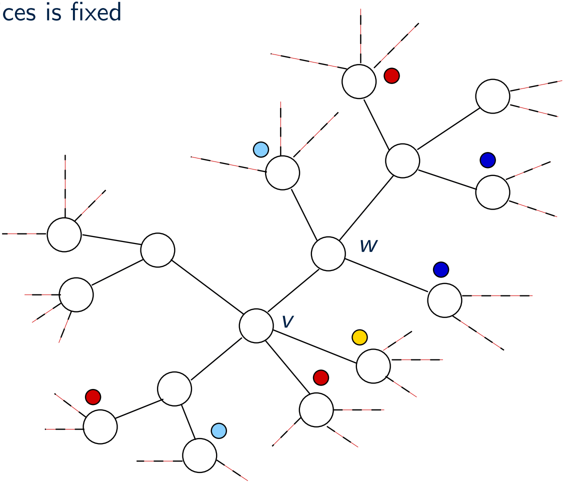


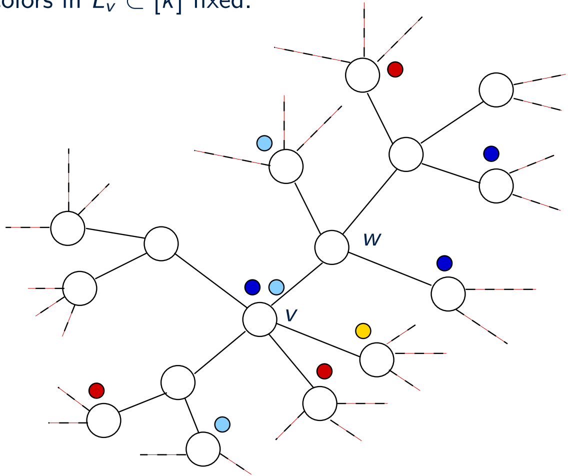
- If in addition to that, most of the neighbors of v are frozen, then so is v.
- Technically: "Warning Propagation" + existence *a priori* of a large set of frozen vertices + convergence of local neighborhoods to trees.

Outline

- Outcome of the previous analysis: coloring a graph where the color of some vertices is fixed



- Outcome of the previous analysis: coloring a graph where each vertex v can take colors in $L_v \subset [k]$ fixed.



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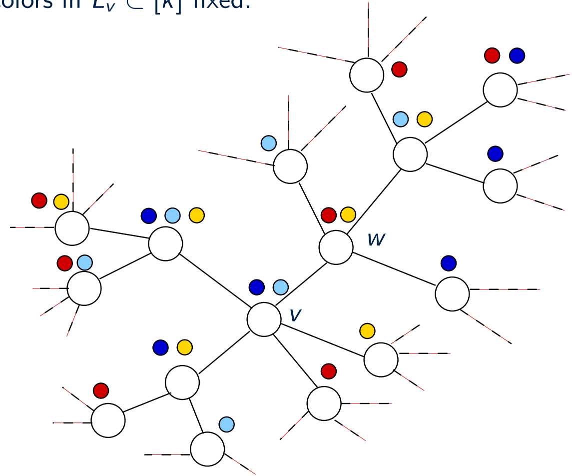

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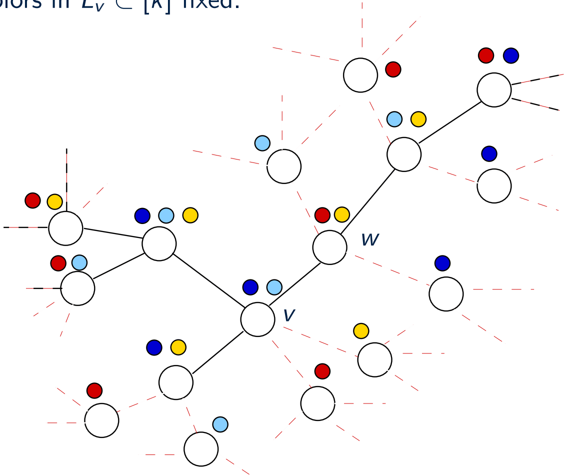

- If $|L_v| = 1$, we can remove v.

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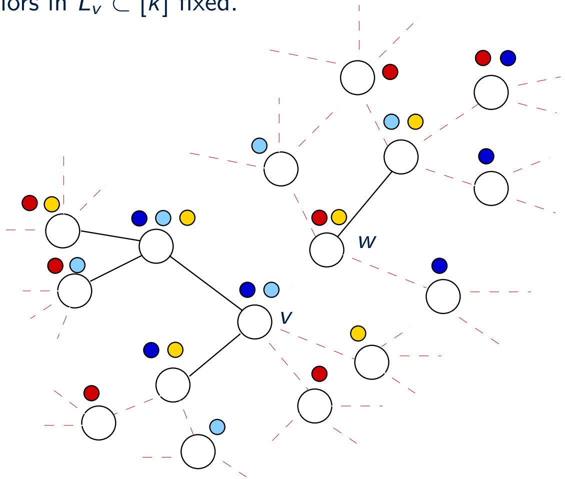


- If $|L_v| = 1$, we can remove v.
- If $L_v \cap L_w = \{\emptyset\}$, we can disconnect v and w
  $\longrightarrow$ problem over finite trees: easy.

Outline

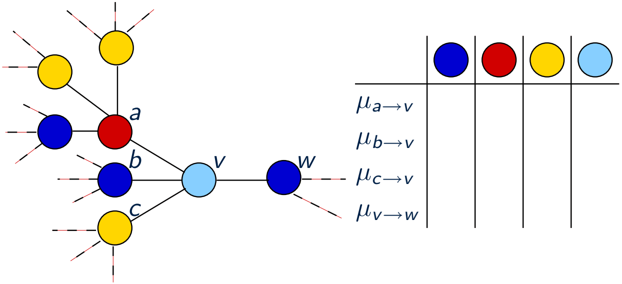
Conclusions

- Rigorous proof of the existence of the condensation transition for (Erdős-Rényi) random graphs coloring.
  The transition point is a number (does not depend on N).
- Confirms the prediction of the cavity method.
  Condensation for a model with fluctuating degrees.
- Some directions for future work: what about
  - the colorability threshold ?
  - finite temperature ?
  - models where the non-condensed phase is non-trivial ?

Warning propagation

Consider the following process:

- associate to each pair of vertices (v, w) connected by an edge a sequence $\mu_{v \to w}(i \in [k], t \geq 0) \in \{0, 1\}$ defined by:
  - $\mu_{v \to w}(i, t = 0) = 1$ iff v has color i under $\sigma^\star$,



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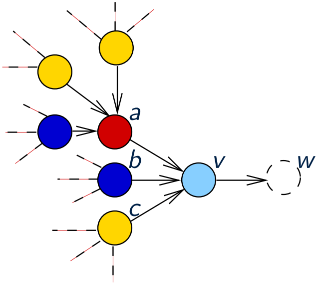


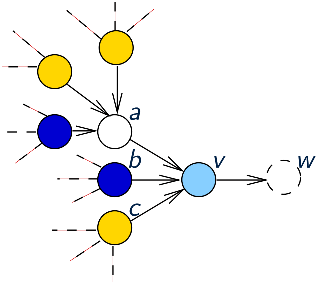
$t = 0$

$\mu_{a \to v}$	0	1	0	0
$\mu_{b \to v}$	1	0	0	0
$\mu_{c \to v}$	0	0	1	0
$\mu_{v \to w}$	0	0	0	1

Warning propagation

Consider the following process:

- associate to each pair of vertices (v, w) connected by an edge a sequence $\mu_{v\to w}(i \in [k], t \geq 0) \in \{0, 1\}$ defined by:
  - $\mu_{v\to w}(i, t = 0) = 1$ iff v has color i under $\sigma^\star$,
  - $\mu_{v\to w}(i, t + 1) = 1$ iff for all $j \neq i$, there is $u \in \partial v \setminus \{w\}$ such that $\mu_{u\to v}(j, t) = 1$ ("u warns v that it cannot take color j").



$t = 1$

	(blue)	(red)	(yellow)	(light blue)
$\mu_{a\to v}$	0	0	0	0
$\mu_{b\to v}$	1	0	0	0
$\mu_{c\to v}$	0	0	1	0
$\mu_{v\to w}$	0	0	0	1

Warning propagation

Consider the following process:

- associate to each pair of vertices (v, w) connected by an edge a sequence $\mu_{v\to w}(i \in [k], t \geq 0) \in \{0, 1\}$ defined by:
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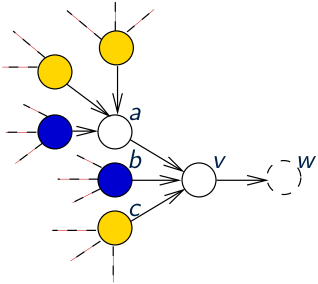

$t = 2$

$\mu_{a\to v}$	0	0	0	0
$\mu_{b\to v}$	1	0	0	0
$\mu_{c\to v}$	0	0	1	0
$\mu_{v\to w}$	0	0	0	0

Warning propagation

- The process is decreasing and converges.
- Define $L(v) = \{i \in [k], \forall \text{u} \in \partial v, \mu_{u \to v}(i, t = \infty) = 0\}$
  ("colors that v is allowed to take").
- Let $Z(G^\star, \sigma^\star)$ be the number of colorings of $G^\star$ such that $\sigma(v) \in L(v)$.
  Then $|C(G^\star, \sigma^\star)|^{1/N} = Z(G^\star, \sigma^\star)^{1/N}$ w.h.p.

Warning propagation (2d version)

Consider the following process:

- associate to each pair of vertices (v, w) connected by an edge a sequence $\mu_{v\to w}(i \in [k], t \geq 0) \in \{0, 1\}$ defined by:
  - $\mu_{v\to w}(i, t = 0) = 1$ iff v has color i under $\sigma^\star$ and v is in the core,
  - $\mu_{v\to w}(i, t + 1) = 1$ iff for all $j \neq i$, there is $u \in \partial v \setminus \{w\}$ such that
    $\mu_{u\to v}(j, t) = 1$ ("u warns v that it cannot take color j").

- The process is increasing and converges.
  Define $L_2(v) = \{i \in [k], \forall u \in \partial v, \mu_{u\to v}(i, t = \infty) = 0\}$
  ("colors that v is allowed to take").

- Let $Z_2(G^\star, \sigma^\star)$ be the number of colorings of $G^\star$ such that $\sigma(v) \in L_2(v)$.
  Then $Z_2(G^\star, \sigma^\star)$ is an upper bound on the cluster size (w.h.p).
  W.h.p. $\ln Z_1(G^\star, \sigma^\star) = \ln Z_2(G^\star, \sigma^\star) + o(N)$.