Taxi walks and the hard-Core model on $\mathbb{Z}^2$

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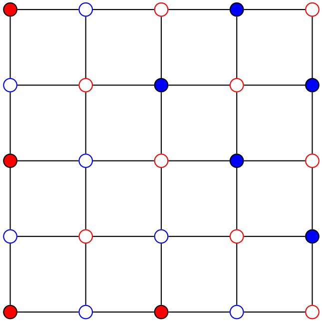
The hard-core model on a box in $\mathbb{Z}^2$

Model of occupation of space by particles with non-negligible size



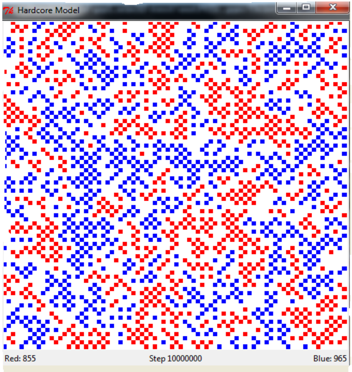
An independent set/hard-core configuration on a box in $\mathbb{Z}^2$

Density parameter $\lambda > 0$ Each I occurs with probability proportional to $\lambda^{|I|}$

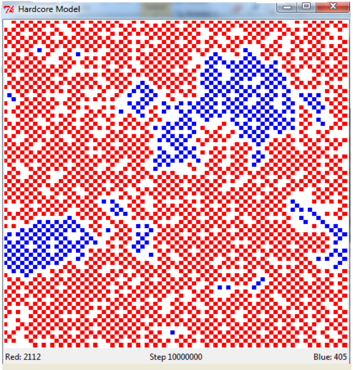
Phase transition

- Small λ: typical configuration disordered
- Large λ: typical configuration mostly in either red or blue sublattice

Simulations on a wrapped-around box in $\mathbb{Z}^2$



80×80, $\lambda = 2$

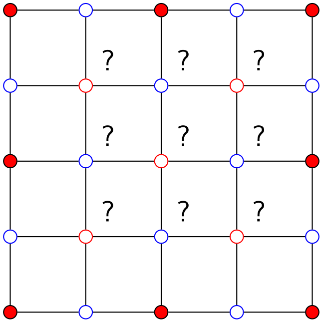


80×80, $\lambda = 5$

(Simulations by Justin Hilyard)

Conjecture: Model on boxes in $\mathbb{Z}^2$ flips from disorder to order around some λ_{crit}

Dealing with infinite graphs

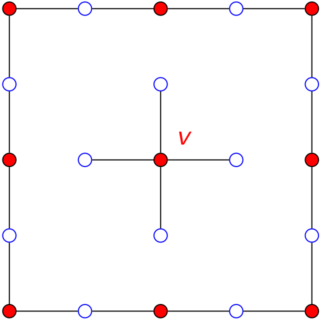


Gibbs measures à la Dobrushin, Lanford, Ruelle

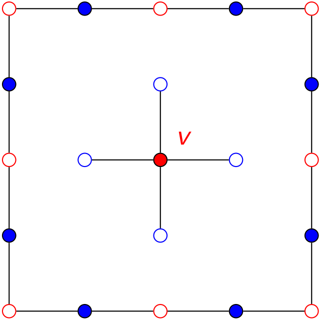
- Hardwire a boundary condition on a finite piece, and extend inside
- *Gibbs measure*: any limit measure as the finite pieces grow

Can different boundary conditions lead to different Gibbs measures?

The picture for large λ on $\mathbb{Z}^2$



$\mu^{\text{red}}(v \in I)$ likely



$\mu^{\text{blue}}(v \in I)$ unlikely

For large λ "influence of boundary" should persist

Enough to show $\mu^{\text{blue}}(v \in I)$ small (this forces $\mu^{\text{red}}(v \in I)$ large)

FKG this is necessary, too

A precise conjecture

Conjecture (folklore, 1950's): There is $\lambda_{\rm crit} \approx 3.796$ such that

- for $\lambda < \lambda_{\rm crit}$, hard-core model on $\mathbb{Z}^2$ has unique Gibbs measure
- for $\lambda > \lambda_{\rm crit}$, there is phase coexistence (multiple Gibbs measures)

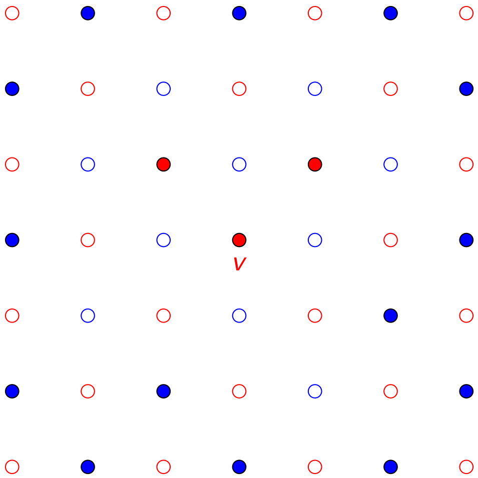
What's known (if $\lambda_{\rm crit}$ exists)

- Dobrushin (1968): $\lambda_{\rm crit} > .25$
  (meaning: for $\lambda \leq .25$ there is unique Gibbs measure)
- Vera-Vigoda-Yang (2013): $\lambda_{\rm crit} > 2.48$ [building on Restrepo-Shin-Tetali-Vigoda-Yang (2011), Weitz (2006); earlier work by Dobrushin-Kolafa-Shlosman (1985), Kirillov-Radulescu-Styer (1989), van den Berg-Steif (1994), Radulescu (1997)]

- Dobrushin (1968): $\lambda_{\rm crit} < C$ for some large C
  (meaning: for $\lambda \geq C$ there are multiple Gibbs measures)
- Borgs-G. (2002-2011): $\lambda_{\rm crit} < 300$, with 80 as theoretical limit

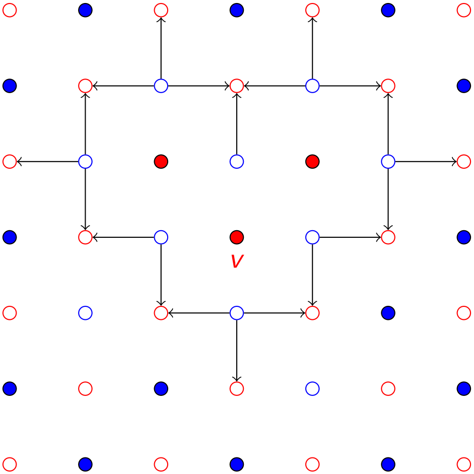
Theorem (Blanca-G.-Randall-Tetali 2012+): $\lambda_{\rm crit} < 5.3646$

The Peierls argument for phase coexistence



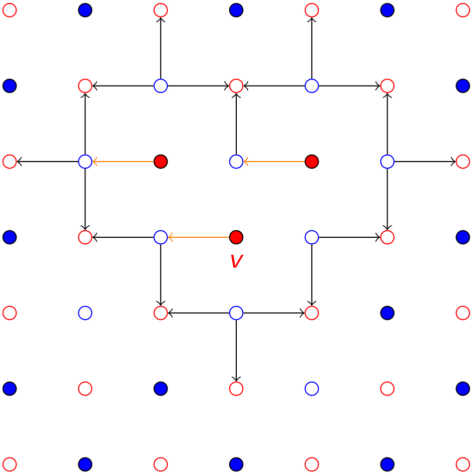
Blue boundary, red center ...

The Peierls argument for phase coexistence



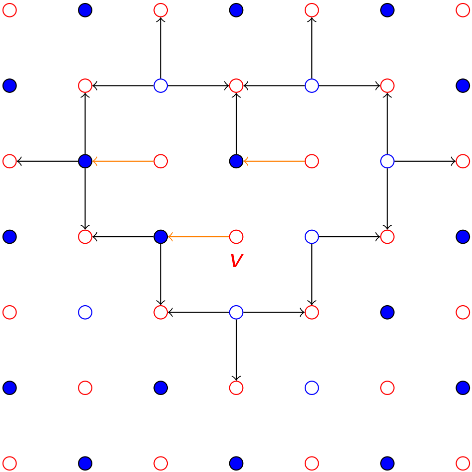
... leads to separating *contour* ...

The Peierls argument for phase coexistence



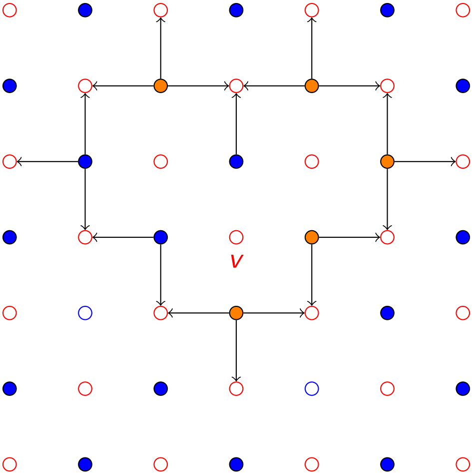
... shifting inside contour creates a more ordered independent set ...

The Peierls argument for phase coexistence



… shifting inside contour creates a more ordered independent set …

The Peierls argument for phase coexistence



... and frees up some vertices (in orange) that can be added

The Peierls argument for phase coexistence

Facts about contours

- Unoccupied edge cutset separating v from boundary
- Interior-exterior edges always from blue sublattice to red
- Length 4ℓ for some $\ell \geq 3$, with ℓ edges in each direction
- Shift in any direction frees up ℓ vertices to be (potentially) added

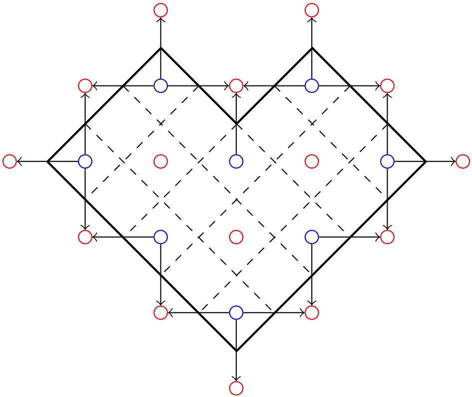
Using contours

- One-to-many map with image weight $(1+\lambda)^\ell$ times larger than input
- Overlap of images controlled by number of possible contours
- The Peierls bound:

$$\mu^{\text{blue}}(v \in I) \leq \sum_{\ell \geq 3} \frac{f_{\text{contour}}(\ell)}{(1+\lambda)^\ell}$$

where $f_{\text{contour}}(\ell)$ is number of contours of length 4ℓ

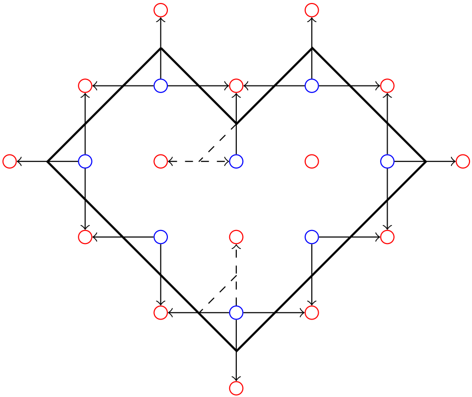
Contours are polygons —



Contours are *simple polygons* in a rotated, dilated copy of $\mathbb{Z}^2$ where

- vertices the midpoints of edges of $\mathbb{Z}^2$
- vertices adjacent if their corresponding edges meet perpendicularly

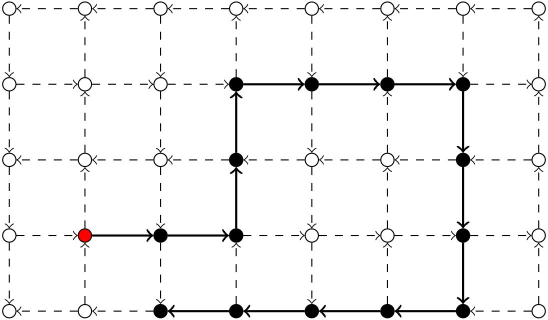
Contours are polygons — with significant restrictions



Contours are *simple polygons* in a rotated, dilated copy of $\mathbb{Z}^2$ where

- two consecutive turns not allowed
- turn direction forced by parity of length of straight segments

Taking a taxi around the Manhattan lattice



A length 14 *taxi walk* on $\mathbb{Z}^2$ (Manhattan lattice)

- Contours are closed, self-avoiding, taxi walks!
- $f_{\text{contour}}(\ell) \leq \text{poly}(\ell) 2^{4\ell}$
- $f_{\text{contour}}(\ell) = \text{subexp}(\ell) \mu_t^{4\ell}$, where μ_t is *taxi walk connective constant*

Upper bounds on $\lambda_{\rm crit}$ using Peierls

An easy bound

- For $\lambda > 40$ have $\mu^{\rm blue}(v \in I) \leq \sum_{\ell \geq 3} \frac{f_{\rm contour}(\ell)}{(1+\lambda)^\ell} < 1/100$
- Theoretical limit of this argument for $\lambda_{\rm crit}$ is $\mu_t^4 - 1 + \varepsilon$

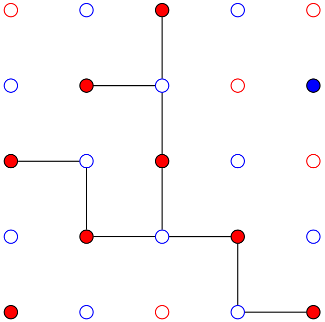
Easy estimate of the connective constant μ_t

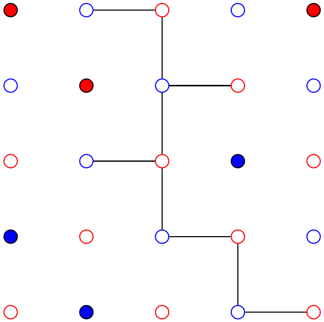
- Taxi walk encoded by $\{s, t\}-$string, no tt, so $\mu_t \leq 1.618\ldots$

A better estimating

- Fix $m < n$, form matrix with ij-entry the number of taxi walks of length n that begin with the ith walk of length m and end with the jth walk of length m; largest eigenvalue bounds μ_t (Alm 1993)
- Estimating eigenvalues of 10057×10057 matrix, get $\mu_t < 1.59$
- $\lambda_{\rm crit} < 11$, theoretical limit 5.3646

Improving things – crosses and fault lines





Theorem (Randall 2006) Every independent set in a box has *one* of

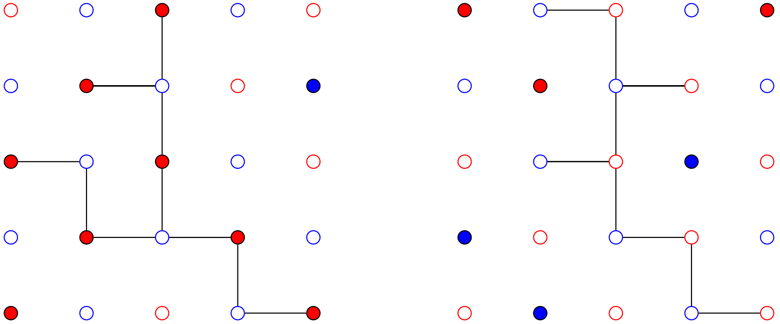
- red cross
- blue cross
- fault line

Improving things – long contours

A new event that distinguishes between μ^{blue} and μ^{red}

- Old: E is set of independent sets with particular red vertex occupied
- New: E is set of independent sets with either a red cross or fault line in some fixed $m \times m$ box
- Under blue boundary condition on $n \times n$ box, build *long* ($\Omega(m)$) contour around red cross or fault line in $m \times m$ box

Improving things – long contours



Peierls bound

$$\mu^{\text{blue}}(E) \le \sum_{\ell \ge \Omega(m)} \frac{f_{\text{contour}}(\ell)}{(1+\lambda)^{\ell}}$$

Theorem: For all $\varepsilon > 0$, $\lambda_{\text{crit}} < \mu_t^4 - 1 + \varepsilon < 5.3646$

A rapid slide on slow mixing

How quickly does Glauber dynamics converge on an $n \times n$ box?

Conjecture (folklore) Slowly, if $\lambda > \lambda_{\mathrm{crit}}$

Using conductance

- Old: Independent sets with equal numbers of red, blue vertices form bottleneck
- New: Independent sets with fault line forms bottleneck

Theorem (Blanca-G.-Randall-Tetali 2012):
Glauber dynamics mixes exponentially slowly for

$$\lambda > 7.1031 \quad \text{on } n \times n \text{ box}$$
$$\lambda > 5.3646 \quad \text{on } n \times n \text{ torus}$$

A problem in need of three new ideas

Lower bounds on the critical density

- $\lambda_{\text{crit}} > 2.48$
- SSM on $T_{\text{SAW}}(\mathbb{Z}^2)$ stops before $\lambda = 3.4$ (Vera-Vigoda-Wang 2013)

Upper bounds on the critical density

- $\lambda_{\text{crit}} < \mu_t^4 - 1 < 5.37$
- $\mu_t^4 - 1 > 4.22$ (Blanca-G-Randall-Tetali 2013+)

Existence of the critical density

- Hard-core model not always monotone (Brightwell-Häggström-Winkler 1998)
- Problem hard for a reason?

Future work?

THANK YOU!