Physical ageing in systems without detailed balance

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Overview:

1. Ageing phenomena
2. Interface growth (KPZ universality class)
3. Form of the scaling functions & Local Scale-Invariance (LSI)
4. Logarithmic conformal & ageing invariance (LLSI)
5. Numerical experiments (KPZ and DP classes in 1D, majority voter in 2D)
6. Outlook: growth on semi-infinite substrates
7. Conclusions
1. Ageing phenomena

Known & practically used since prehistoric times (metals, glasses) systematically studied in physics since the 1970s

**Discovery**: ageing effects reproducible & universal! Occur in widely different systems (structural glasses, spin glasses, polymers, simple magnets, ...)

Three **defining properties** of **ageing**:

1. Slow relaxation (non-exponential!)
2. **No** time-translation-invariance (TTI)
3. Dynamical scaling without fine-tuning of parameters

Most existing studies on ‘magnets’: relaxation towards equilibrium

**Question**: what can be learned about intrinsically irreversible and/or complex systems by studying their ageing behaviour?
\( t = t_1 \) \hspace{1cm} \( t = t_2 > t_1 \)

magnet \( T < T_c \) \hspace{2cm} \rightarrow \text{ordered cluster}

magnet \( T = T_c \) \hspace{2cm} \rightarrow \text{correlated cluster}

critical contact process \hspace{2cm} \Longrightarrow \text{cluster dilution}

common feature: growing length scale \( L(t) \sim t^{1/z} \)

\( z \): dynamical exponent
Two-time observables: analogy with ‘magnets’

time-dependent order-parameter $\phi(t, r)$

two-time **correlator** $C(t, s) := \langle \phi(t, r) \phi(s, r) \rangle - \langle \phi(t, r) \rangle \langle \phi(s, r) \rangle$

two-time **response** $R(t, s) := \frac{\delta \langle \phi(t, r) \rangle}{\delta h(s, r)} \bigg|_{h=0} = \langle \phi(t, r) \tilde{\phi}(s, r) \rangle$

$t$: observation time, $s$: waiting time

a) system at equilibrium: fluctuation-dissipation theorem

$$R(t - s) = \frac{1}{T} \frac{\partial C(t - s)}{\partial s} , \quad T: \text{temperature}$$

b) far from equilibrium: $C$ and $R$ independent!

The **fluctuation-dissipation ratio** (FDR)

$$X(t, s) := \frac{TR(t, s)}{\partial C(t, s)/\partial s}$$

measures the distance with respect to equilibrium: $X_{eq} = X(t - s) = 1$
Scaling regime: \( t, s \gg \tau_{\text{micro}} \) and \( t - s \gg \tau_{\text{micro}} \)

\[
C(t, s) = s^{-b} f_C \left( \frac{t}{s} \right), \quad R(t, s) = s^{-1-a} f_R \left( \frac{t}{s} \right)
\]

Asymptotics: \( f_{C,R}(y) \sim y^{-\lambda_{C,R}/z} \) for \( y \gg 1 \)

\( \lambda_C \): autocorrelation exponent, \( \lambda_R \): autoresponse exponent,
\( z \): dynamical exponent, \( a, b \): ageing exponents

\[ \lambda_C = \lambda_R = d + z + \frac{\beta}{\nu_{\perp}} , \quad b = \frac{2\beta'}{\nu_{\parallel}} \]

\( \rightarrow \) stationary-state critical exponents \( \beta, \beta', \nu_{\perp}, \nu_{\parallel} = z \nu_{\perp} \)

Example: critical particle-reaction model (contact process),
initial particle density > 0

Baumann & Gambassi 07
2. Interface growth

deposition (evaporation) of particles on a substrate \( \rightarrow \) **height profile** \( h(t, r) \)
generic situation : RSOS (**restricted solid-on-solid**) model

\[ p = \text{deposition prob.} \]
\[ 1 - p = \text{evap. prob.} \]

here \( p = 0.98 \)

some universality classes :

(a) **KPZ**  \[ \partial_t h = \nu \nabla^2 h + \frac{\mu}{2} (\nabla h)^2 + \eta \]

(b) **EW**  \[ \partial_t h = \nu \nabla^2 h + \eta \]

(c) **MH**  \[ \partial_t h = -\nu \nabla^4 h + \eta \]

\( \eta \) is a gaussian white noise with \( \langle \eta(t, r)\eta(t', r') \rangle = 2\nu T \delta(t - t')\delta(r - r') \)

\( \nu, T \) are constants
**Family-Viscek** scaling on a spatial lattice of extent $L^d$:

$$\overline{h}(t) = L^{-d} \sum_j h_j(t)$$

Family & Viscek 85

$$w^2(t; L) = \frac{1}{L^d} \sum_{j=1}^{L^d} \left\langle (h_j(t) - \overline{h}(t))^2 \right\rangle = L^{2\zeta} f\left(tL^{-z}\right) \sim \begin{cases} L^{2\zeta} & \text{if } tL^{-z} \gg 1 \\ t^{2\beta} & \text{if } tL^{-z} \ll 1 \end{cases}$$

$\beta$: growth exponent, $\zeta$: roughness exponent, $\zeta = \beta z$

**two-time correlator**:

$$C(t, s; r) = \left\langle (h(t, r) - \langle \overline{h}(t) \rangle) (h(s, 0) - \langle \overline{h}(s) \rangle) \right\rangle = s^{-b} F_C\left(\frac{t}{s}, \frac{r}{s^{1/z}}\right)$$

with ageing exponent: $b = -2\beta$

Kallabis & Krug 96

expect for $y = t/s \gg 1$: $F_C(y, 0) \sim y^{-\lambda_C/z}$ autocorrelation exponent
1D relaxation dynamics, starting from an initially flat interface

observe all 3 properties of ageing:
- slow dynamics
- no TTI
- dynamical scaling

confirm simple ageing for the 1D KPZ universality class

Pars pro toto

Kallabis & Krug 96; Krech 97; Bustingorry et al. 07-10; Chou & Pleimling 10; D’Aquila & Täuber 11/12; h.n.p. 12
extend Family-Viscek scaling to two-time responses:
analogue: TRM integrated response in magnetic systems

two-time integrated response:
* sample A with deposition rates \( p_i = p \pm \epsilon_i \), up to time \( s \),
* sample B with \( p_i = p \) up to time \( s \);
then switch to common dynamics \( p_i = p \) for all times \( t > s \)

\[
\chi(t, s; r) = \int_0^s du \, R(t, u; r) = \frac{1}{L} \sum_{j=1}^L \left\langle \frac{h_{j+r}^{(A)}(t; s) - h_{j+r}^{(B)}(t)}{\epsilon_j} \right\rangle = s^{-a} F_{\chi} \left( \frac{t}{s}, \frac{|r|^z}{s} \right)
\]

with \( a \): ageing exponent

expect for \( y = t/s \gg 1 \): \( F_R(y, 0) \sim y^{-\lambda_R/z} \); autoresponse exponent

? Values of these exponents ?
**Effective action** of the KPZ equation:

\[
\mathcal{J} [\phi, \tilde{\phi}] = \int dt dr \left[ \tilde{\phi} \left( \partial_t \phi - \nu \nabla^2 \phi - \frac{\mu}{2} (\nabla \phi)^2 \right) - \nu T \tilde{\phi}^2 \right]
\]

\[\Rightarrow\] **Very special properties of KPZ in** \(d = 1\) **spatial dimension!**

**Exact critical exponents** \(\beta = 1/3, \ \zeta = 1/2, \ z = 3/2, \ \lambda_C = 1\) \[kpz \ 86; \ Krech \ 97\]

related to precise symmetry properties:

A) tilt-invariance (Galilei-invariance)

kept under renormalisation!

\[\Rightarrow\] exponent relation \(\zeta + z = 2\) \(\text{(holds for any dimension } d)\)

B) time-reversal invariance

special property in 1D, where also \(\zeta = \frac{1}{2}\)
Special KPZ symmetry in $1D$ : let $v = \frac{\partial \phi}{\partial r}$, $\tilde{\phi} = \frac{\partial}{\partial r} (\tilde{p} + \frac{v}{2T})$

$$J = \int dtdr \left[ \tilde{p} \partial_t v - \frac{v}{4T} (\partial_r v)^2 - \frac{\mu}{2} v^2 \partial_r \tilde{p} + \nu T (\partial_r \tilde{p})^2 \right]$$

is invariant under time-reversal

$$t \mapsto -t \ , \ v(t, r) \mapsto -v(-t, r) \ , \ \tilde{p} \mapsto +\tilde{p}(-t, r)$$

$\Rightarrow$ fluctuation-dissipation relation for $t \gg s$

$$TR(t, s; r) = -\partial_r^2 C(t, s; r)$$

distinct from the equilibrium FDT $TR(t - s) = \partial_s C(t - s)$

Combination with ageing scaling, gives the ageing exponents :

$$\lambda_R = \lambda_C = 1 \quad \text{and} \quad 1 + a = b + \frac{2}{z}$$
1D relaxation dynamics, starting from an initially flat interface

confirm simple ageing in the autocorrelator
confirm expected exponents $b = -2/3$, $\lambda_C/z = 2/3$

**N.B.** : this confirmation is out of the stationary state

Kallabis & Krug 96; Krech 97; Bustingorry *et al.* 07-10; Chou & Pleimling 10; D’Aquila & Täuber 11/12; h.n.p. 12
relaxation of the integrated response, 1D

observe all 3 properties of \textbf{ageing}:
- slow dynamics
- no TTI
- dynamical scaling

exponents $a = -1/3$, $\lambda_R/z = 2/3$, as expected from FDR

\textbf{N.B.} : numerical tests for 2 models in KPZ class
Simple ageing is also seen in \textit{space-time} observables

\begin{align*}
\text{correlator } & C(t, s; r) = s^{2/3} F_C \left( \frac{t}{s}, \frac{r^{3/2}}{s} \right) \\
\text{integrated response } & \chi(t, s; r) = s^{1/3} F_\chi \left( \frac{t}{s}, \frac{r^{3/2}}{s} \right) \quad \text{confirm } z = \frac{3}{2}
\end{align*}
Values of some growth and ageing exponents in 1D

<table>
<thead>
<tr>
<th>model</th>
<th>$z$</th>
<th>$a$</th>
<th>$b$</th>
<th>$\lambda_R = \lambda_C$</th>
<th>$\beta$</th>
<th>$\zeta$</th>
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<tbody>
<tr>
<td>KPZ exp 1</td>
<td>$3/2$</td>
<td>$-1/3$</td>
<td>$-2/3$</td>
<td>$1$</td>
<td>$1/3$</td>
<td>$1/2$</td>
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<tr>
<td></td>
<td>$\approx -2/3$†</td>
<td>$\approx 1$†</td>
<td></td>
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<tr>
<td></td>
<td>$1.5(2)$</td>
<td></td>
<td></td>
<td>$0.336(11)$</td>
<td>$0.50(5)$</td>
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<tr>
<td>KPZ exp 2</td>
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<tr>
<td>EW</td>
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<td>MH</td>
<td>$4$</td>
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<td>$-3/4$</td>
<td>$1$</td>
<td>$3/8$</td>
<td>$3/2$</td>
</tr>
</tbody>
</table>

Liquid crystals

Cancer cells

Takeuchi, Sano, Sasamoto, Spohn 10/11/12

Huergo, Pasquale, Gonzalez, Bolzan, Arvia 12

† scaling holds only for flat interface

Two-time space-time responses and correlators consistent with simple ageing for 1D KPZ

Similar results known for EW and MH universality classes

Roethlein, Baumann, Pleimling 06
3. Form of the scaling functions & LSI

**Question**: Are there model-independent results on the form of universal scaling functions?

‘Natural’ starting point: try to draw analogies with conformal invariance at equilibrium

* Equilibrium critical phenomena: **scale-invariance**
* For sufficiently **local** interactions: extend to conformal invariance
  
  **space**-dependent re-scaling (angles conserved) \( r \mapsto r/b(r) \)

In **two** dimensions: \( \infty \) many conformal transformations
  
  \( w \mapsto \beta(w) \) complex analytic

\( \Rightarrow \) exact predictions for critical exponents, correlators, . . .

**Bateman & Cunningham 1909/10, Polyakov 70**

**BPZ 84**
Hidden assumptions:

1) extension scale-invariance $\quad\rightarrow\quad$ conformal invariance $\quad?$
   formally: energy-momentum tensor symmetric & traceless $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\qua
What about **time**-dependent critical phenomena?

Characterised by **dynamical exponent** $z : t \mapsto tb^{-z}, r \mapsto rb^{-1}$

Can one extend to **local** dynamical scaling, with $z \neq 1$?

If $z = 2$, the **Schrödinger group** is an example: 

$$t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta}, \quad r \mapsto \frac{D r + vt + a}{\gamma t + \delta}; \quad \alpha \delta - \beta \gamma = 1$$

⇒ study **ageing** phenomena as paradigmatic example

**essential**: (i) absence of TTI & (ii) **Galilei**-invariance

Transformation $t \mapsto t'$ with $\beta(0) = 0$ and $\dot{\beta}(t') \geq 0$ and

$$t = \beta(t'), \quad \phi(t) = \left(\frac{d\beta(t')}{dt'}\right)^{-x/z} \left(\frac{d \ln \beta(t')}{dt'}\right)^{-2\xi/z} \phi'(t')$$

**out of equilibrium**, have 2 **distinct** scaling dimensions, $x$ and $\xi$.

mean-field for **magnets** : expect \(\xi = 0\) in ordered phase $T < T_c$

$\xi \neq 0$ at criticality $T = T_c$

**NB**: if TTI (equilibrium criticality), then $\xi = 0$. 

**Cardy 85**

**Jacobi 1842, Lie 1881**

**MH et. al. 06**
**physical requirement:**

**co-variance of response functions** under local scaling!

**why:** certain extended scaling symmetries **predict causality** for co-variant \(n\)-point functions!

⇒ set of linear differential equations for \(R(t, s)\)

\[
R(t, s) = \langle \phi(t) \tilde{\phi}(s) \rangle = s^{-1-a} f_R \left( \frac{t}{s} \right)
\]

\[
f_R(y) = f_0 y^{1+a'-\lambda_R/z} (y-1)^{-1-a'} \Theta(y-1)
\]

causality

\[
a = \frac{1}{z} (x + \tilde{x}) - 1 , \quad a' - a = \frac{2}{z} \left( \xi + \tilde{\xi} \right) , \quad \frac{\lambda_R}{z} = x + \xi
\]

**magnetic example:** 1D Glauber-Ising model at \(T = T_c = 0\):

\[
a = 0 , \quad a' - a = -\frac{1}{2} , \quad \lambda_R = 1 , \quad z = 2
\]
Particle models: comparison of $R(t, s)$ with LSI-prediction:

- Contact process (CP): $A \rightarrow 2A, A \rightarrow \emptyset$
- Nonequilibrium kinetic Ising (PC): $A \leftrightarrow 3A, 2A \rightarrow \emptyset$
- Voter Potts-3 (VP3)

**CP**: $a' - a \simeq 0.27$

**PC**: $a' - a \simeq 0.00(1)$

**VP3**: $a' - a \simeq -0.1$

**Observation**: the hidden assumption $a = a'$, uncritically taken over from equilibrium, is often invalid out of equilibrium. Observables cannot always be identified with scaling operators.
4. Logarithmic conformal & ageing invariance

generalise conformal invariance \( \rightarrow \) doubletts \( \Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix} \)

Rozansky & Saleur 92
Gurarie 93

generators: \( \ell_n = -w^{n+1} \partial_w - (n+1)w^n \begin{pmatrix} \Delta & 1 \\ 0 & \Delta \end{pmatrix} \)

two-point functions: have \( \Delta_1 = \Delta_2 \)

Gurarie 93, Rahimi Tabar et al. 97...

\[
F = \langle \phi_1(w_1)\phi_2(w_2) \rangle = 0
\]

\[
G = \langle \phi_1(w_1)\psi_2(w_2) \rangle = G_0|w|^{-2\Delta_1}
\]

\[
H = \langle \psi_1(w_1)\psi_2(w_2) \rangle = (H_0 - 2G_0 \ln |w|) |w|^{-2\Delta_1}
\]

\[
= w_2^{-2\Delta_1}(H_0 - 2G_0 \ln |y-1| - 2G_0\ln |w_2|) |y-1|^{-2\Delta_1}
\]

with \( w = w_1 - w_2 \) and \( y = w_1/w_2 \).

Simultaneous log corrections to scaling and modified scaling function

Logarithmic conformal invariance has been found in, e.g.

- critical 2D percolation
- Cardy 92, Watts 96, Mathieu & Ridout 07/08
- disordered systems
- Caux et al. 96
- sand-pile models
- Ruelle et al. 08-10
construct **logarithmic ageing-invariance** by the formal changes (generic case; $x' = 0$ or $x' = 1$):

$$x \mapsto \hat{x} = \begin{pmatrix} x & x' \\ 0 & x \end{pmatrix}, \quad \xi \mapsto \hat{\xi} = \begin{pmatrix} \xi & \xi' \\ 0 & \xi \end{pmatrix}$$

(must show: both dimension matrices $\hat{x}, \hat{\xi}$ are simultaneously Jordan!)

we find the **co-variant two-point functions** (with $y = t/s$):

$$\langle \phi(t) \bar{\phi}(s) \rangle = s^{-(x+\bar{x})/2} f(y)$$

$$\langle \phi(t) \bar{\psi}(s) \rangle = s^{-(x+\bar{x})/2} (g_{12}(y) + \ln s \cdot \gamma_{12}(y))$$

$$\langle \psi(t) \bar{\phi}(s) \rangle = s^{-(x+\bar{x})/2} (g_{21}(y) + \ln s \cdot \gamma_{21}(y))$$

$$\langle \psi(t) \bar{\psi}(s) \rangle = s^{-(x+\bar{x})/2} (h_0(y) + \ln s \cdot h_1(y) + \ln^2 s \cdot h_2(y))$$

all scaling functions explicitly known

**Question**: interesting models described by logarithmic **LSI**?
5. Numerical experiments

(A) Kardar-Parisi-Zhang (KPZ)
(B) directed percolation (DP)
(C) majority voter/Glauber models (MV) at $T = T_c$, triangular lattice

simple ageing of the correlators and responses, especially

\[ C(t, s) = s^{-b} f_C \left( \frac{t}{s} \right), \quad R(t, s) = s^{-1-a} f_R \left( \frac{t}{s} \right) \]

\[ f_C(y) \sim y^{-\lambda_C/z}, \quad f_R(y) \sim y^{-\lambda_R/z} \quad y \gg 1 \]

values of the non-equilibrium exponents & scaling relations

**KPZ in 1D**: $\lambda_C = \lambda_R = 1$, $1 + a = b + \frac{2}{z}$, $b = -2\beta = -\frac{2}{3}$, $z = \frac{3}{2}$

**DP**:

\[ \lambda_C = \lambda_R = d + z + \frac{\beta}{\nu_\perp}, \quad 1 + a = b = \frac{2\beta}{\nu_\parallel} \]

**MV in 2D**:

\[ \lambda_C = \lambda_R \approx 0.732 z, \quad a = b = \frac{2\beta}{\nu_\parallel}, \quad z \approx 2.17 \]

what can be said on the form of the scaling function of the auto-response?

**N.B.**: Galilei-invariance for KPZ is kept under renormalisation, unusual form
(A) assumption: \( R(t, s) = \langle \psi(t)\bar{\psi}(s) \rangle \)  

1D KPZ equation/RSOS model

good collapse \( \Rightarrow \) no logarithmic corrections \( \Rightarrow \) \( x' = \tilde{x}' = 0 \)

no logarithmic factors for \( y \gg 1 \) \( \Rightarrow \) \( \xi' = 0 \)

\( \Rightarrow \) only \( \tilde{\xi}' = 1 \) remains

\[
f_R(y) = y^{-\lambda_R/z} \left( 1 - \frac{1}{y} \right)^{-1-a'} \left[ h_0 - g_0 \ln \left( 1 - \frac{1}{y} \right) - \frac{1}{2} f_0 \ln^2 \left( 1 - \frac{1}{y} \right) \right]
\]

use specific values of 1D KPZ class \( \frac{\lambda_R}{z} - a = 1 \)

find integrated autoresponse \( \chi(t, s) = \int_0^s du \ R(t, u) = s^{1/3} f_\chi(t/s) \)

\[
f_\chi(y) = y^{1/3} \left\{ A_0 \left[ 1 - \left( 1 - \frac{1}{y} \right)^{-a'} \right] \\
+ \left( 1 - \frac{1}{y} \right)^{-a'} \left[ A_1 \ln \left( 1 - \frac{1}{y} \right) + A_2 \ln^2 \left( 1 - \frac{1}{y} \right) \right] \right\}
\]

with free parameters \( A_0, A_1, A_2 \) and \( a' \)
non-log LSI with $a = a'$: deviations $\approx 20\%$

non-log LSI with $a \neq a'$: works up to $\approx 5\%$

log LSI: works better than $\approx 0.1\%$

<table>
<thead>
<tr>
<th>$R$</th>
<th>$a'$</th>
<th>$A_0$</th>
<th>$A_1$</th>
<th>$A_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle \phi\phi \rangle$ – LSI</td>
<td>$-0.500$</td>
<td>$0.662$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\langle \phi\psi \rangle$ – $L^1$LSI</td>
<td>$-0.500$</td>
<td>$0.663$</td>
<td>$-6 \cdot 10^{-4}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\langle \psi\psi \rangle$ – $L^2$LSI</td>
<td>$-0.8206$</td>
<td>$0.7187$</td>
<td>$0.2424$</td>
<td>$-0.09087$</td>
</tr>
</tbody>
</table>

logarithmic LSI fits data at least down to $y \approx 1.01$, with
$a' - a \approx -0.4873$ (can we make a conjecture?)
(B) assumption: \( R(t, s) = \langle \psi(t)\tilde{\psi}(s) \rangle \) 

1D critical contact process

good collapse \( \Rightarrow \) **no** logarithmic corrections \( \Rightarrow \) \( x' = \tilde{x}' = 0 \)

\[ h_R(y) = \left(1 - \frac{1}{y}\right)^{a-a'} \left[ h_0 - g_{12,0} \tilde{\xi}' \ln(1 - 1/y) - g_{21,0} \xi' \ln(y - 1) \right. \]

\[ \left. - \frac{1}{2} f_0 \tilde{\xi}'^2 \ln^2(1 - 1/y) + \frac{1}{2} f_0 \xi'^2 \ln^2(y - 1) \right] \]

find empirically:

very small amplitude of \( \ln^2 \)-terms

\( \Rightarrow f_0 = 0 \)

require both \( \xi \neq 0, \tilde{\xi}' \neq 0 \)

**BUT:** logarithmic factor for \( y \gg 1 \) ?

logar. LSI fit data, at least down to \( y \approx 1.002 \); with \( a' - a \approx -0.002 \).
(C) assumption: \( R(t, s) = \langle \psi(t) \tilde{\psi}(s) \rangle \) 2D majority voter/Glauber model (triangular lattice)

good collapse ⇒ no logarithmic corrections ⇒ \( x' = \tilde{x}' = 0 \)

\[
h_R(y) = \left(1 - \frac{1}{y}\right)^{a-a'} \left[ h_0 - g_{12,0} \ln(1 - 1/y) - \frac{1}{2} f_0 \ln^2(1 - 1/y) \right]
\]

no logarithmic terms for \( y \gg 1 \) ⇒ \( \xi' = 0 \)

can normalise \( \tilde{\xi}' = 1 \)

F. Sastre (2013) preliminary

logar. LSI fit data, at least down to \( y \approx 1.005 \).
6. Outlook: growth on semi-infinite substrates

properties of growing interfaces near to a boundary?
→ crystal dislocations, face boundaries . . .

Experiments: Family-Vicsek scaling not always sufficient
→ distinct global and local interface fluctuations

\[ \begin{align*}
\text{anomalous scaling, growth exponent } \beta \text{ larger than expected} \\
\text{grainy interface morphology, facetting}
\end{align*} \]

! analyse simple models on a semi-infinite substrate !
frame co-moving with average interface deep in the bulk
characterise interface by
\[ \begin{align*}
\text{height profile } & \langle h(t, r) \rangle \\
\text{width profile } & w(t, r) = \left\langle [h(t, r) - \langle h(t, r) \rangle]^2 \right\rangle^{1/2}
\end{align*} \]

\( h \to 0 \text{ as } |r| \to \infty \)
specialise to $d = 1$ space dimensions; boundary at $x = 0$, bulk $x \to \infty$

cross-over for the phenomenological growth exponent $\beta$ near to boundary

bulk behaviour $w \sim t^\beta$

‘surface behaviour’ $w_1 \sim t^{\beta_1}$  

cross-over, if causal interaction with boundary

experimentally observed, e.g. for semiconductor films

values of growth exponents (bulk & surface):

$\beta = 0.25 \quad \beta_{1,\text{eff}} \approx 0.32 \quad$ Edwards-Wilkinson class

$\beta \approx 0.32 \quad \beta_{1,\text{eff}} \approx 0.35 \quad$ Kardar-Parisi-Zhang class

Nascimento, Ferreira, Ferreira 11

EW-class

Allegra, Fortin, mh 13
need explicit boundary interactions in Langevin equation

\[ h_1(t) = \partial_x h(t, x)|_{x=0} \]

\[
(\partial_t - \nu \partial_x^2) h(t, x) - \frac{\mu}{2} (\partial_x h(t, x))^2 + \eta(t, x) = \nu (\kappa_1 + \kappa_2 h_1(t)) \delta(x)
\]

height profile \[ \langle h(t, x) \rangle = t^{1/\gamma} \Phi \left( xt^{-1/z} \right), \quad \gamma = \frac{z}{z - 1} = \frac{\zeta}{\zeta - \beta} \]

EW & exact solution, \( h(t, 0) \sim \sqrt{t} \) self-consistently

KPZ
Scaling of the width profile:

EW & exact solution $\lambda^{-1} = 4tx^{-2}$

**bulk**

**boundary**

**same** growth scaling exponents in the bulk and near to the boundary

large **intermediate scaling regime** with effective exponent (slopes)

agreement with **RG** for non-disordered, local interactions

**Lopéz, Castro, Gallego 05**

? ageing behaviour near to a boundary ?
7. Conclusions

- Physical ageing occurs naturally in many irreversible systems relaxing towards non-equilibrium stationary states considered here: absorbing phase transitions & surface growth.
- Scaling phenomenology analogous to simple magnets.
- But finer differences in relationships between non-equilibrium exponents.
- A major difference w/ equilibrium: intrinsic absence of time-translation-invariance ⇒ 2 scaling dimensions.
- Shape of scaling functions: logarithmic local scale-invariance?
- Performed numerical experiments on auto-response function:
  (i) 1D KPZ equation (ii) 1D critical directed percolation
  (iii) 2D majority voter/Glauber models
- Surprises in scaling near a boundary: height/width profiles.
- Studies of the ageing properties, via two-time observables, might become a new tool, also for the analysis of complex systems!