Distinct and Common Sites visited by N random Walkers

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M. V. Tamm (Moscow University, Russia)

Refs:

- S.M. and M. V. Tamm, Phys. Rev. E 86, 021135 (2012)
- A. Kundu, S.M. and G. Schehr, Phys. Rev. Lett. 110, 220602 (2013)

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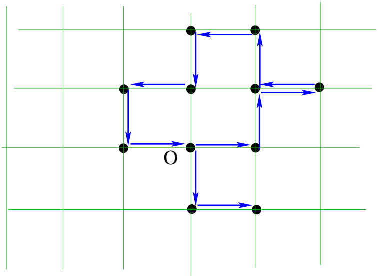
- Summary and Conclusion

Distinct sites visited by a *single* random walker



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$\rightarrow$ sites visited atleast once

physics, chemistry, metallurgy, ecology,...

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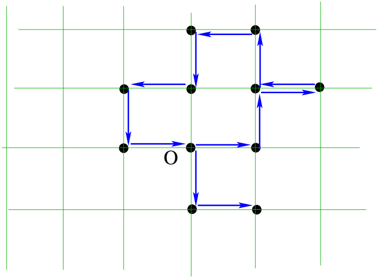

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For large t

$$\langle D_1(t)\rangle \sim \left(\sqrt{t}\right)^d, \qquad d < 2$$
$$\sim t/\ln t, \qquad d = 2$$
$$\sim \gamma_d\, t, \qquad d > 2$$

Random walk

recurrent for $d < 2$

transient for $d > 2$

[Dvoretzky & Erdös ('51), Vineyard ('63), Montrol & Weiss ('65), ...]

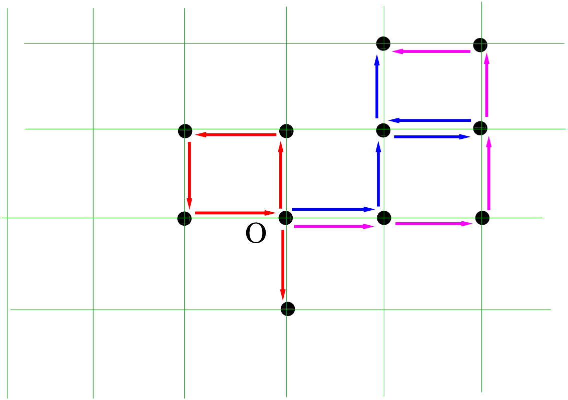
Territory covered by N diffusing particles

Hernan Larralde*, Paul Trunfio*, Shlomo Havlin*†, H. Eugene Stanley* & George H. Weiss†

* Center for Polymer Studies and Department of Physics, Boston University, Boston, Massachusetts 02215, USA
† Physical Sciences Laboratory, Division of Computer Research and Technology, National Institutes of Health, Bethesda, Maryland 20892, USA

THE number of distinct sites visited by a random walker after t steps is of great interest[1–21], as it provides a direct measure of the territory covered by a diffusing particle. Thus, this quantity appears in the description of many phenomena of interest in ecology[13–16], metallurgy[5–7], chemistry[17,18] and physics[19–22]. Previous analyses have been limited to the number of distinct sites visited by a single random walker[19–22], but the (nontrivial) generalization to the number of distinct sites visited by N walkers is particularly relevant to a range of problems—for example, the classic problem in mathematical ecology of defining the territory covered by N members of a given species[13–16]. Here we present an analytical solution

Distinct sites visited by N indep. random walkers



$D_N(t) \rightarrow$ no. of distinct sites visited by N indep. walkers each of t steps

$\rightarrow$ sites visited atleast once by any of the walkers

[Larralde, Trunfio, Havlin, Stanley, & Weiss, Nature, 355, 423 (1992)]

Number of distinct sites: growth with time t

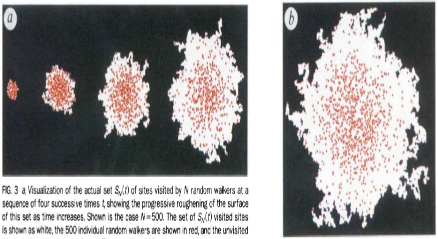
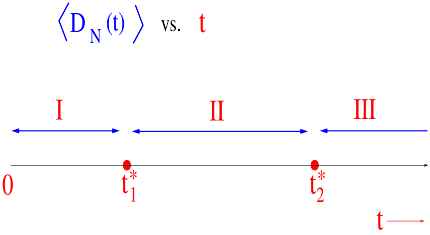


FIG. 3 *a*, Visualization of the actual set $S_N(t)$ of sites visited by N random walkers at a sequence of four successive times t, showing the progressive roughening of the surface of this set as time increases. Shown is the case $N=500$. The set of $S_N(t)$ visited sites is shown as white, the 500 individual random walkers are shown in red, and the unvisited 'virgin territory' is shown in black. *b*, The case $N=1{,}000$ at late time, which demonstrates the part played by only a few individual random walkers in causing the roughening of the interface of the set $S_N(t)$.



Asymptotic late time ($t >> t_2^*$) growth:

$$\langle D_N(t) \rangle \sim (\ln N)^{d/2} \left(\sqrt{t}\right)^d \qquad d < 2$$

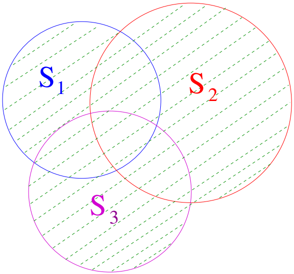
$$\sim N\, t/\ln t \qquad d = 2$$

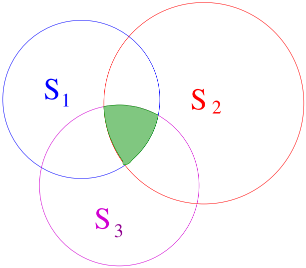
$$\sim N\, t \qquad d > 2$$

[Larralde et. al. ('92)]

Union and Intersection of the visited sites

$S_i(t) \rightarrow$ the set of sites visited by the i-th walker up to t



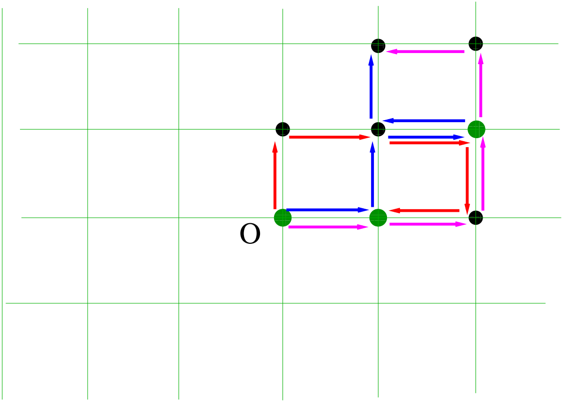


Distinct sites $D_N(t) \equiv$ size of $S_1(t) \cup S_2(t) \cup S_3(t) \ldots \cup S_N(t) \rightarrow$ Union

A natural counterpart:

Common sites $C_N(t) \equiv$ size of $S_1(t) \cap S_2(t) \cap S_3(t) \ldots \cap S_N(t)$
$\rightarrow$ Intersection

Common sites visited by N indep. random walkers

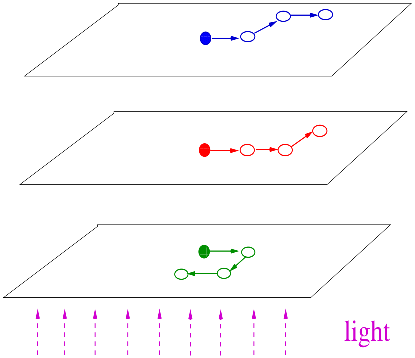


$C_N(t) \to$ no. of common sites visited by N indep. walkers each of t steps

$\to$ sites visited by all the walkers (green sites)

[S.M. & M. V. Tamm, Phys. Rev. E, 86, 021135 (2012)]

Common Sites



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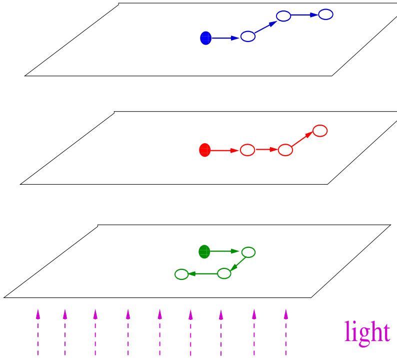

light transmits from bottom to top provided the same site in each of the N planes is visited by the corresponding walker

Intensity of transmitted light $I_N(t) \propto C_N(t)$

$\longrightarrow$ no. of common sites visited by N independent walkers in a single plane

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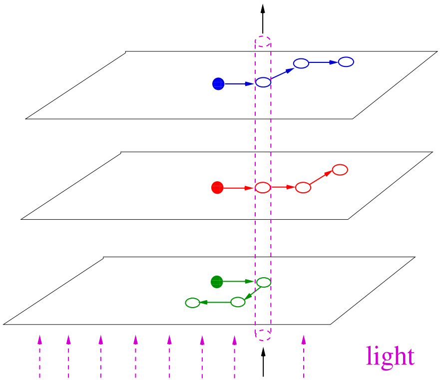

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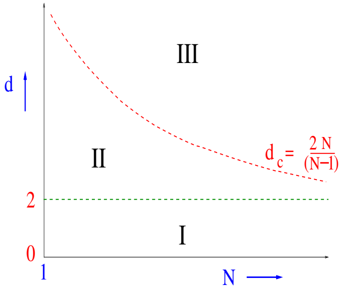
Asymptotic growth of $\langle C_N(t) \rangle$



$$\langle C_N(t) \rangle \sim \begin{cases} b_N\, t^{d/2} & d < 2 \\ t/(\ln t)^N & d = 2 \\ t^{\nu = N - d(N-1)/2} & 2 < d < d_c(N) \\ \ln t & d = d_c(N) \\ \text{const.} & d > d_c(N) \end{cases}$$

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Nontrivial exponent $\nu = N - d(N-1)/2$ in the intermediate Phase II

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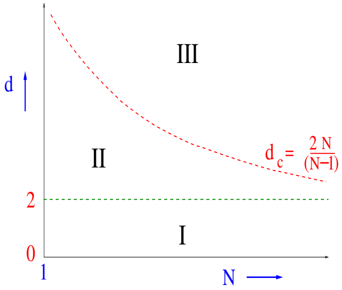

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Few Examples:

- $N = 2 \to d_c = 4$; $\langle C_2(t) \rangle \sim t^{1/2}$ in $2 < d = 3 < 4$

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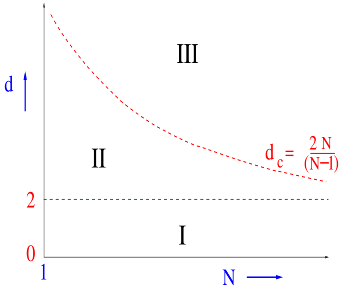

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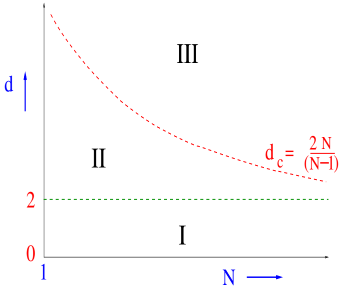

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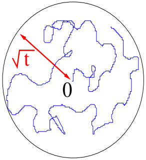
- $N = 4 \rightarrow d_c = 2$; $\langle C_4(t) \rangle \sim t/(\ln t)^4$ in $d = 2$

Heuristic scaling argument



- In time t, a single walker explores a volume $V(t) \sim t^{d/2}$
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  $\rightarrow$ prob. that a site is visited by the walker

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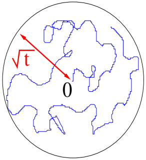

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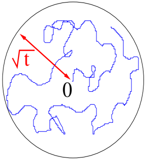

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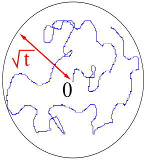

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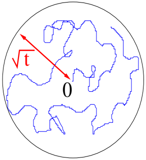

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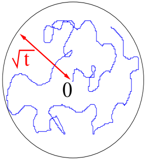

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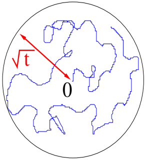

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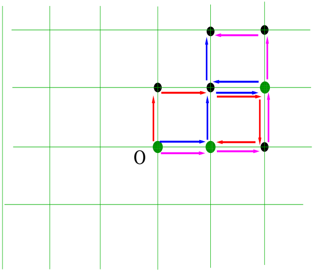
$\sim \mathrm{const.}$ if $\nu < 0$, i.e., for $d > d_c$

Exact computation



$$\sigma(\vec{x}, t) = 1 \quad \text{if } \vec{x} \text{ visited by all } t\text{-step walkers (green)}$$
$$= 0 \quad \text{otherwise}$$

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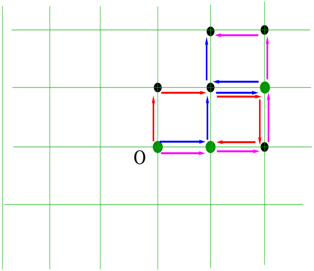

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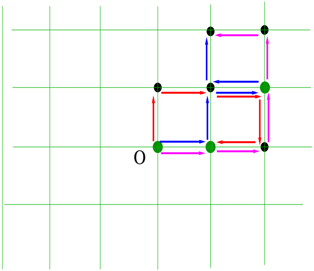

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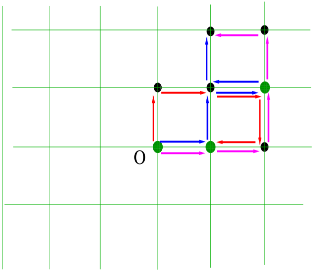
- $$\langle C_N(t)\rangle = \sum_{\vec{x}} \langle \sigma(\vec{x},t)\rangle = \sum_{\vec{x}} [p(\vec{x},t)]^N$$

$p(\vec{x},t) \rightarrow$ prob. that $\vec{x}$ is visited by a single t-step walker



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  $p(\vec{x},t) \to$ prob. that $\vec{x}$ is visited by a single t-step walker

- Also $\left[1-p(\vec{x},t)\right]^N \to$ prob. that $\vec{x}$ is NOT visited by any of the walkers

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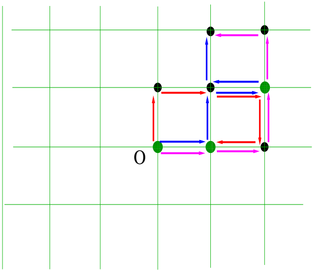

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- Also $[1 - p(\vec{x}, t)]^N \rightarrow$ prob. that $\vec{x}$ is NOT visited by any of the walkers

$$\Longrightarrow \langle D_N(t) \rangle = \sum_{\vec{x}} \left[1 - [1 - p(\vec{x}, t)]^N\right]$$

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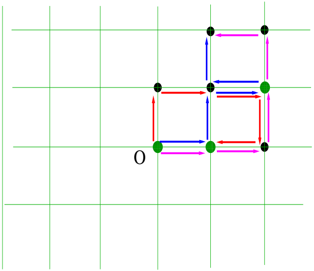

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Then $C_N(t) = \sum_{\vec{x}} \sigma(\vec{x}, t)$

- $$\langle C_N(t) \rangle = \sum_{\vec{x}} \langle \sigma(\vec{x}, t) \rangle = \sum_{\vec{x}} [p(\vec{x}, t)]^N$$

  $p(\vec{x}, t) \to$ prob. that $\vec{x}$ is visited by a single t-step walker

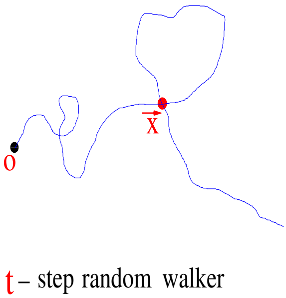
- Also $[1 - p(\vec{x}, t)]^N \to$ prob. that $\vec{x}$ is NOT visited by any of the walkers

$$\implies \langle D_N(t) \rangle = \sum_{\vec{x}} \left[1 - [1 - p(\vec{x}, t)]^N \right]$$

$p(\vec{x}, t) \longrightarrow$ central quantity

The probability $p(\vec{x}, t)$

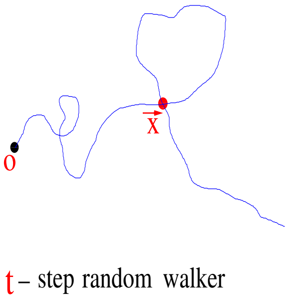
$p(\vec{x}, t) \rightarrow$ prob. that $\vec{x}$ is visited by a single t-step walker



Let $\tau \longrightarrow$ last time before t the site $\vec{x}$ was visited

The probability $p(\vec{x}, t)$

$p(\vec{x}, t) \rightarrow$ prob. that $\vec{x}$ is visited by a single t-step walker

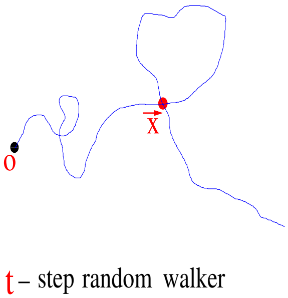


Let $\tau \longrightarrow$ last time before t the site $\vec{x}$ was visited

Then
$$p(\vec{x}, t) = \int_0^t G(\vec{x}, \tau)\, q(t - \tau)\, d\tau$$

The probability $p(\vec{x}, t)$

$p(\vec{x}, t) \to$ prob. that $\vec{x}$ is visited by a single t-step walker



Let $\tau \longrightarrow$ last time before t the site $\vec{x}$ was visited

Then $$p(\vec{x}, t) = \int_0^t G(\vec{x}, \tau)\, q(t-\tau)\, d\tau$$

- $G(\vec{x}, t) = (4\pi D\tau)^{-d/2} \exp\left[-x^2/(4D\tau)\right] \to$ diffusion Green's function
- $q(\tau) \to$ prob. of no return to the starting pt. in time τ

Scaling form of $p(\vec{x}, t)$:

$$p(\vec{x}, t) \sim \begin{cases} f_< \left(\frac{x}{\sqrt{4\,\pi\,D\,t}} \right) & (d < 2) \\ \frac{1}{\ln t}\, f_2 \left(\frac{x}{\sqrt{4\,\pi\,D\,t}} \right) & (d = 2) \\ t^{1-d/2}\, f_> \left(\frac{x}{\sqrt{4\,\pi\,D\,t}} \right) & (d > 2) \end{cases}$$

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$f_<(z) \sim \mathrm{const.}$ as $z \to 0$

$\sim z^{-d}\, e^{-z^2}$ as $z \to \infty$

$f_>(z) \sim z^{2-d}$ as $z \to 0$

$\sim z^{-2}\, e^{-z^2}$ as $z \to \infty$

Scaling form of $p(\vec{x}, t)$:

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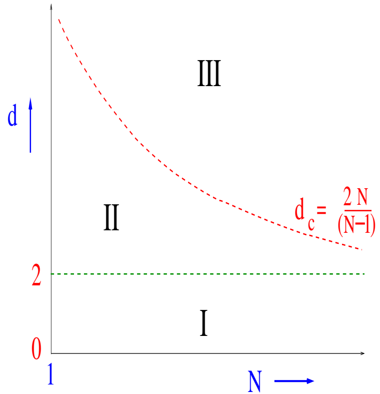
$$\sim z^{-2}\, e^{-z^2} \quad \text{as } z \to \infty$$

$$\boxed{\langle C_N(t)\rangle = \int d\vec{x}\, [p(\vec{x}, t)]^N \sim \int_1^\infty dx\, x^{d-1}\, [p(\vec{x}, t)]^N} \Longrightarrow$$

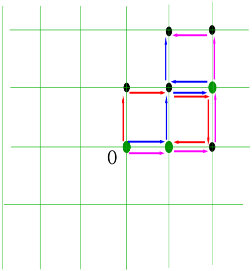
Exact asymptotic results



$$\langle C_N(t)\rangle \sim \begin{cases} b_N\, t^{d/2} & d<2 \\ t/(\ln t)^N & d=2 \\ t^{\nu=N-d(N-1)/2} & 2<d<d_c(N) \\ \ln t & d=d_c(N) \\ \text{const.} & d>d_c(N) \end{cases}$$

[S.M. & M. V. Tamm, Phys. Rev. E, 86, 021135 (2012)]

Distribution of Distinct and Common sites

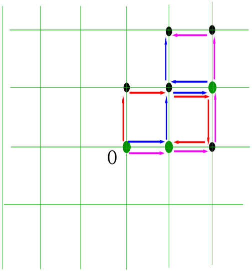


$D_N(t)$, $C_N(t)$ $\rightarrow$ no. of distinct/common sites visited by N indep. walkers each of t steps

$D_N(t)$ and $C_N(t)$ $\rightarrow$ random variables

What about the full distribution of $D_N(t)$ and $C_N(t)$?

Distribution of Distinct and Common sites



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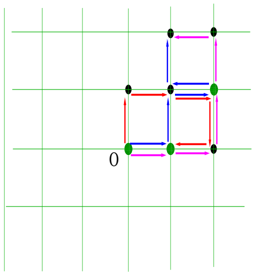
What about the full distribution of $D_N(t)$ and $C_N(t)$?

Focus on $d = 1$:

Distribution of Distinct and Common sites



$D_N(t)$, $C_N(t)$ $\to$ no. of distinct/common sites visited by N indep. walkers each of t steps

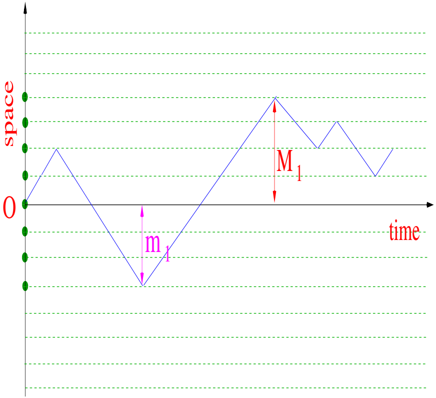
$D_N(t)$ and $C_N(t)$ $\to$ random variables

What about the full distribution of $D_N(t)$ and $C_N(t)$?

Focus on $d = 1$:

- maximum overlap between walkers in $d = 1$ $\longrightarrow$ nontrivial
- interesting link to Extreme Value Statistics $\longrightarrow$ exactly solvable
- Various applications in $d = 1$:
  biological applications $\Rightarrow$ proteins diffusing along DNA
  environmental applications $\Rightarrow$ diffusion of river pollutants

A single walker $N = 1$ in one dimension



$$D_1(t) = C_1(t) = D_1^+(t) + D_1^-(t)$$

$\longrightarrow$ span of the walk

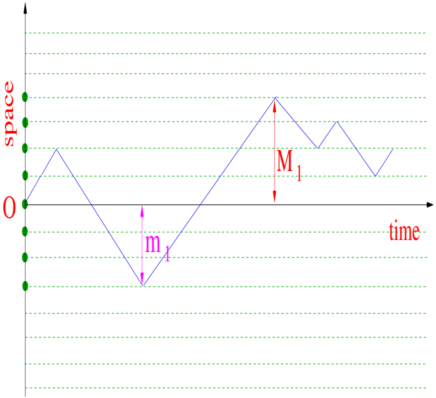
where

$$D_1^+(t) = M_1(t) = \max_{0 \leq \tau \leq t} \{x_1(\tau)\}$$

$$D_1^-(t) = m_1(t) = -\min_{0 \leq \tau \leq t} \{x_1(\tau)\}$$

$\Longrightarrow$ link to Extreme Value Statistics

A single walker $N = 1$ in one dimension



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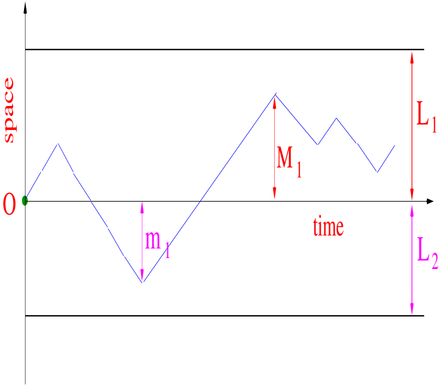
$\Longrightarrow$ link to Extreme Value Statistics

Note that $\{M_1(t), m_1(t)\} \longrightarrow$ correlated random variables

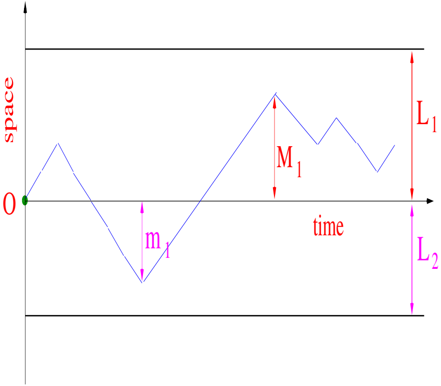
Joint distribution of $M_1(t)$ and $m_1(t)$



$\text{Prob.}[M_1(t) \leq L_1, m_1(t) \leq L_2]$
$\longrightarrow$ prob. that the walker stays inside the box $[-L_2, L_1]$ up to time t

$$\xrightarrow[t\to\infty]{} g\left(\frac{L_1}{\sqrt{4\ D\ t}} = l_1, \frac{L_2}{\sqrt{4\ D\ t}} = l_2\right)$$

Joint distribution of $M_1(t)$ and $m_1(t)$



Prob.$[M_1(t) \leq L_1, m_1(t) \leq L_2]$

$\longrightarrow$ prob. that the walker stays inside the box $[-L_2, L_1]$ up to time t

$\xrightarrow[t\to\infty]{} g\left(\frac{L_1}{\sqrt{4\ D\ t}} = l_1, \frac{L_2}{\sqrt{4\ D\ t}} = l_2\right)$

Solving Fokker-Planck equation with absorbing b.c. at L_1 and $-L_2$ gives

$$g(l_1, l_2) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi l_2}{l_1 + l_2}\right) \exp\left[-\frac{(2n+1)^2\pi^2}{4(l_1+l_2)^2}\right]$$

Prob. density of distinct/common sites for $N = 1$

Joint distribution:
$\mathrm{Prob.}[M_1(t) \leq L_1, m_1(t) \leq L_2] \to g\left(\frac{L_1}{\sqrt{4\ D\ t}} = l_1, \frac{L_2}{\sqrt{4\ D\ t}} = l_2\right)$

No. of distinct/common sites $D_1(t) = C_1(t) = M_1(t) + m_1(t)$

Prob. density: $\mathrm{Prob.}[D_1(t)] = \mathrm{Prob.}[C_1(t)] \to \frac{1}{\sqrt{4Dt}}\, P_1\left(\frac{D_1(t)}{\sqrt{4Dt}}\right)$

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$$\boxed{P_1(x) = \int_0^\infty \int_0^\infty \frac{\partial^2 g}{\partial l_1\, \partial l_2}\, \delta(x - l_1 - l_2)\, dl_1\, dl_2}$$

Prob. density of distinct/common sites for $N = 1$

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$$P_1(x) = \frac{8}{\sqrt{\pi}} \sum_{m=1}^{\infty} (-1)^{m+1}\, m^2\, e^{-m^2 x^2}$$

Joint distribution:

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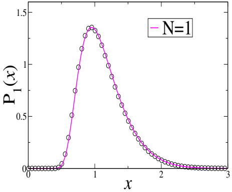
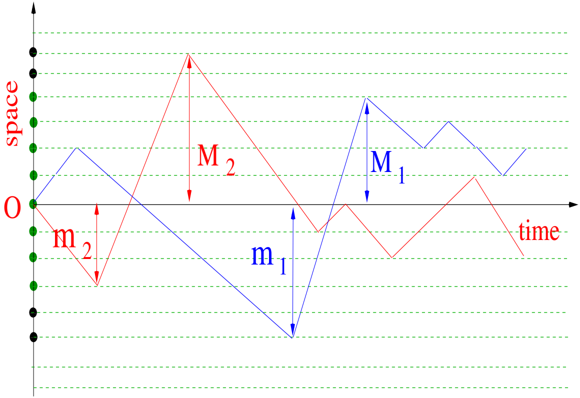
$$P_1(x) \to \begin{cases} 2\,\pi^2\, x^{-5}\, e^{-\pi^2/4\,x^2} & x \to 0 \\ \frac{8}{\sqrt{\pi}}\, e^{-x^2} & x \to \infty \end{cases}$$



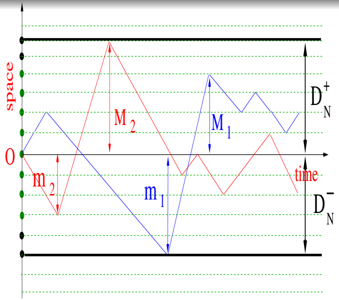
Multiple ($N > 1$) t-step walkers



$M_i(t) \rightarrow$ maximum of the i-th walker

$-m_i(t) \rightarrow$ minimum of the i-th walker

Distinct sites $D_N(t)$ for $(N > 1)$ walkers



$M_i(t) \rightarrow$ maximum of the i-th walker

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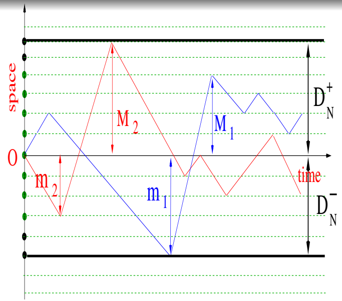
Distinct sites $D_N(t)$ for $(N > 1)$ walkers



$M_i(t) \rightarrow$ maximum of the i-th walker

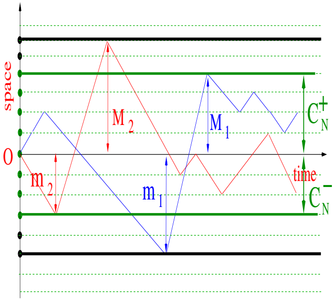
$-m_i(t) \rightarrow$ minimum of the i-th walker

Distinct: $D_N(t) = D_N^+(t) + D_N^-(t)$ where

$$D_N^+(t) = \max\left[M_1(t), M_2(t), \ldots, M_N(t)\right]$$

$$D_N^-(t) = \max\left[m_1(t), m_2(t), \ldots, m_N(t)\right]$$

Common sites $C_N(t)$ for $(N > 1)$ walkers



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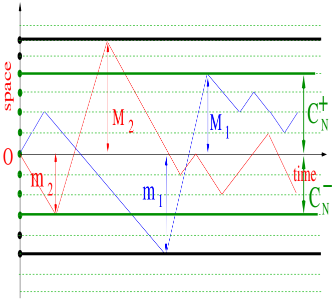
Common sites $C_N(t)$ for $(N > 1)$ walkers



$M_i(t) \rightarrow$ maximum of the i-th walker

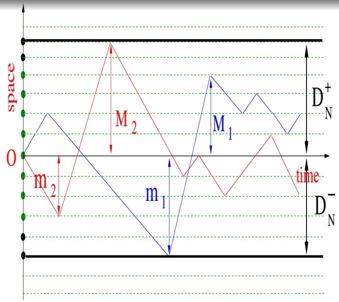
$-m_i(t) \rightarrow$ minimum of the i-th walker

Common: $C_N(t) = C_N^+(t) + C_N^-(t)$ where

$$C_N^+(t) = \min\left[M_1(t), M_2(t), \ldots, M_N(t)\right]$$

$$C_N^-(t) = \min\left[m_1(t), m_2(t), \ldots, m_N(t)\right]$$

Distribution of $D_N(t)$

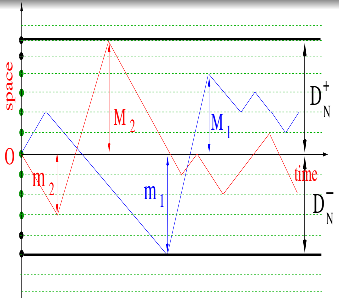


$$D_N(t) = D_N^+(t) + D_N^-(t)$$

$$D_N^+(t) = \max_{1 \le i \le N} \{M_i(t)\}$$

$$D_N^-(t) = \max_{1 \le i \le N} \{m_i(t)\}$$

Distribution of $D_N(t)$



$$D_N(t) = D_N^+(t) + D_N^-(t)$$

$$D_N^+(t) = \max_{1\le i\le N} \{M_i(t)\}$$

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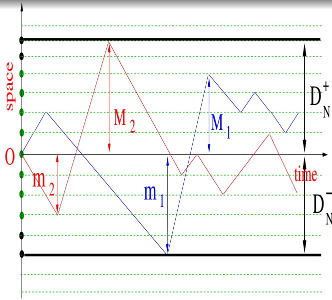
$$\text{Prob.}\left[D_N^+(t) \le L_1, D_N^-(t) \le L_2\right] = \prod_{i=1}^{N} \text{Prob.}\left[M_i(t) \le L_1, m_i(t) \le L_2\right]$$

$$= \left[g\left(l_1, l_2\right)\right]^N; \qquad l_1 = \frac{L_1}{\sqrt{4Dt}},\ \ l_2 = \frac{L_2}{\sqrt{4Dt}}$$

Distribution of $D_N(t)$



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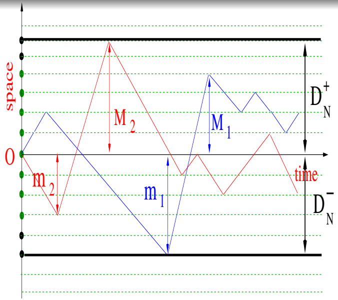
Prob. density: $\mathrm{Prob.}[D_N(t)] \to \frac{1}{\sqrt{4Dt}} P_N\left(\frac{D_N(t)}{\sqrt{4Dt}}\right)$

Distribution of $D_N(t)$



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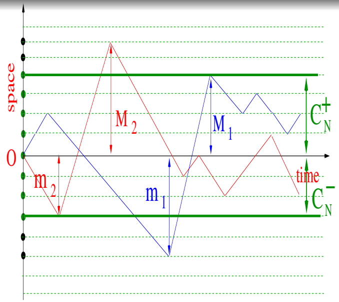
$$P_N(x) = \int_0^\infty \int_0^\infty \frac{\partial^2 g^N}{\partial l_1 \, \partial l_2} \, \delta(x - l_1 - l_2) \, dl_1 \, dl_2$$

Distribution of $C_N(t)$

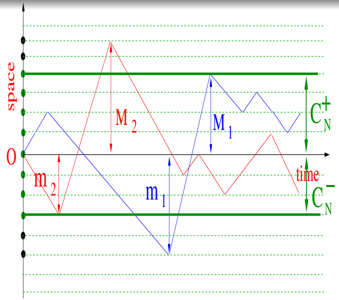


$$C_N(t) = C_N^{+}(t) + C_N^{-}(t)$$

$$C_N^{+}(t) = \min_{1 \le i \le N} \{M_i(t)\}$$

$$C_N^{-}(t) = \min_{1 \le i \le N} \{m_i(t)\}$$

Distribution of $C_N(t)$



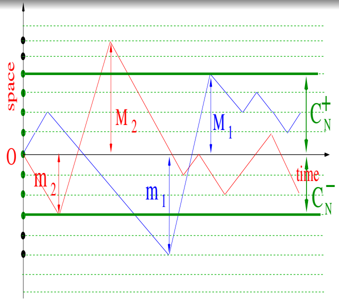
$$C_N(t) = C_N^+(t) + C_N^-(t)$$

$$C_N^+(t) = \min_{1\le i\le N}\{M_i(t)\}$$

$$C_N^-(t) = \min_{1\le i\le N}\{m_i(t)\}$$

$$\text{Prob.}\left[C_N^+(t) \ge L_1, C_N^-(t) \ge L_2\right] = \prod_{i=1}^{N} \text{Prob.}\left[M_i(t) \ge L_1, m_i(t) \ge L_2\right]$$
$$= \left[h\left(l_1, l_2\right)\right]^N$$

Distribution of $C_N(t)$



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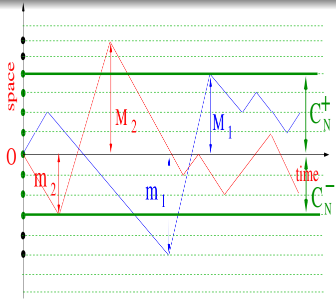
where $h(l_1, l_2) = 1 - \mathrm{erf}(l_1) - \mathrm{erf}(l_2) + g(l_1, l_2)$

Distribution of $C_N(t)$



$$C_N(t) = C_N^+(t) + C_N^-(t)$$

$$C_N^+(t) = \min_{1\le i\le N} \{M_i(t)\}$$

$$C_N^-(t) = \min_{1\le i\le N} \{m_i(t)\}$$

$$\mathrm{Prob.}\left[C_N^+(t) \ge L_1, C_N^-(t) \ge L_2\right] = \prod_{i=1}^{N} \mathrm{Prob.}\left[M_i(t) \ge L_1, m_i(t) \ge L_2\right]$$

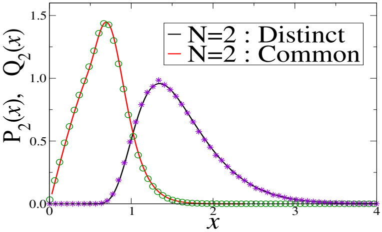
$$= \left[h\left(l_1, l_2\right)\right]^N$$

where $h(l_1, l_2) = 1 - \mathrm{erf}(l_1) - \mathrm{erf}(l_2) + g(l_1, l_2)$

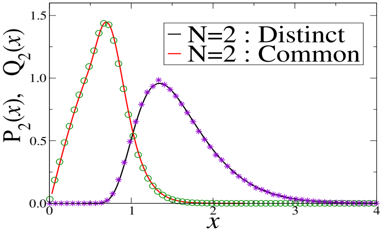
Prob. density: $\mathrm{Prob.}[C_N(t)] \to \frac{1}{\sqrt{4Dt}}\, Q_N\left(\frac{C_N(t)}{\sqrt{4Dt}}\right)$

$$Q_N(x) = \int_0^\infty \int_0^\infty \frac{\partial^2 h^N}{\partial l_1\, \partial l_2}\, \delta(x - l_1 - l_2)\, dl_1\, dl_2$$

Distributions for fixed N



Distributions for fixed N



Asymptotics for fixed $N \geq 2$:

Distinct:

$$P_N(x) \to \begin{cases} a_N\, x^{-5}\, e^{-N\,\pi^2/4\,x^2} & x \to 0 \\ b_N\, e^{-x^2/2} & x \to \infty \end{cases}$$

Common:

$$Q_N(x) \to \begin{cases} c_N\, x & x \to 0 \\ d_N\, x^{1-N}\, e^{-N\,x^2} & x \to \infty \end{cases}$$

[Kundu, S.M. & Schehr, PRL, 110, 220602 (2013)]

First and second moments for large N:

First moment:

Distinct: $$\frac{\langle D_N(t)\rangle}{\sqrt{4Dt}} = 2\int_0^\infty x\, \frac{d}{dx}[\mathrm{erf}(x)]^N\, dx$$

Common: $$\frac{\langle C_N(t)\rangle}{\sqrt{4Dt}} = 2\int_0^\infty [\mathrm{erfc}(x)]^N\, dx$$

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Large N:

Distinct:

$$\frac{\langle D_N(t)\rangle}{\sqrt{4Dt}} \approx 2\sqrt{\ln N} + \frac{\gamma}{\sqrt{\ln N}}; \qquad \gamma = 0.577216... \to \text{Euler const.}$$

$$\mathrm{Var}\left[\frac{D_N(t)}{\sqrt{4Dt}}\right] \approx \frac{2\alpha}{\ln N}; \qquad \alpha = \gamma + \pi^2/6$$

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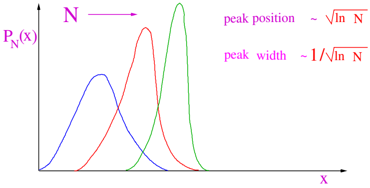
Common:

$$\frac{\langle C_N(t)\rangle}{\sqrt{4Dt}} \approx \frac{\sqrt{\pi}}{N}$$

decreases with N !!

$$\mathrm{Var}\left[\frac{C_N(t)}{\sqrt{4Dt}}\right] \approx \frac{\pi}{2N^2}$$

Scaling form for the distribution of $D_N(t)$



Suggests a scaling form:

$$P_N(x) \sim 2\sqrt{\ln N}\, \mathcal{D}\left(2\sqrt{\ln N}\, \left(x - 2\sqrt{\ln N}\right)\right)$$

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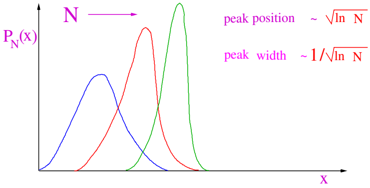

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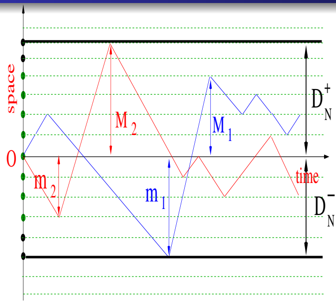
Exact scaling analysis gives:

$$\boxed{\mathcal{D}(y) = 2\, e^{-y}\, \mathrm{K}_0\left(2\, e^{-y/2}\right)}$$ where $\mathrm{K}_0(z) \to$ modified Bessel function

$$\mathcal{D}(y) \to \begin{cases} y\, e^{-y} & y \to \infty \\ \sqrt{\pi}\, e^{-3y/4}\, \exp\left[-2\, e^{-y/2}\right] & y \to -\infty \end{cases}$$

[Kundu, S.M. & Schehr, PRL, 110, 220602 (2013)]

Simple interpretation of the scaling function



$$D_N(t) = D_N^+(t) + D_N^-(t)$$

$$D_N^+(t) = \max_{1\le i\le N} \{M_i(t)\}$$

$$D_N^-(t) = \max_{1\le i\le N} \{m_i(t)\}$$

$\frac{M_i}{\sqrt{4Dt}} = z_i$'s $\rightarrow$ i.i.d variables each drawn from: $p(z) = \frac{2}{\sqrt{\pi}}\, e^{-z^2}$ with $z \ge 0$

$\Rightarrow D_N^+$ (centered and scaled) $\rightarrow$ Gumbel distributed:

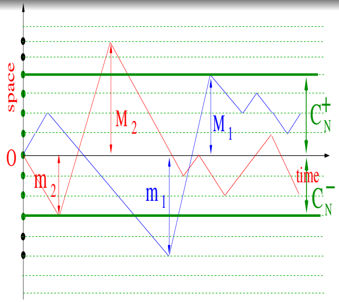
$$P_G(x) = e^{-x}\, \exp\left[-e^{-x}\right]$$

As $N \to \infty$, D_N^+ and D_N^- becomes uncorrelated

Thus, $D_N(t) \rightarrow$ sum of two independent Gumbel variables

$$\mathcal{D}(y) = \int_{-\infty}^{\infty} dx\, P_G(x)\, P_G(y-x) = 2\, e^{-y}\, \mathrm{K}_0\left(2\, e^{-y/2}\right)$$

Scaling form for the distribution of $C_N(t)$



$$C_N(t) = C_N^+(t) + C_N^-(t)$$

$$C_N^+(t) = \min_{1 \le i \le N} \{M_i(t)\}$$

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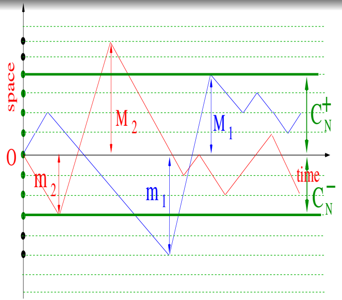

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Exact large N analysis gives: $Q_N(x) \sim N\,\mathcal{C}(N\,x)$ where

$$\mathcal{C}(y) = \frac{4}{\pi}\, y\, \exp\left[-\frac{2}{\sqrt{\pi}}\, y\right]$$

[Kundu, S.M. & Schehr, PRL, 110, 220602 (2013)]

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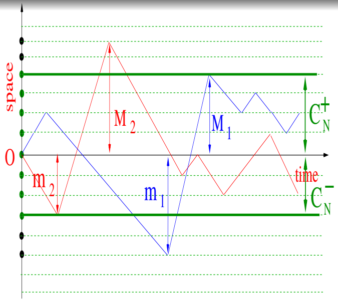

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$\mathcal{C}(y) = \frac{4}{\pi}\, y\, \exp\left[-\frac{2}{\sqrt{\pi}}\, y\right]$ [Kundu, S.M. & Schehr, PRL, 110, 220602 (2013)]

Interpretation: For large N, C_N^+ and C_N^- get uncorrelated

Thus, $C_N(t) \to$ sum of two independent Weibull variables each distributed with $P_W(x) = \frac{2}{\sqrt{\pi}}\, e^{-2x/\sqrt{\pi}}\,\theta(x)$

$$\mathcal{C}(y) = \int_0^\infty dx\, P_W(x)\, P_W(y-x) = \frac{4}{\pi}\, y\, \exp\left[-\frac{2}{\sqrt{\pi}}\, y\right]$$

Summary and Conclusions

• Mean number of common sites visited by N noninteracting random walkers in all dimensins d

As $t \to \infty$

$\langle C_N(t) \rangle \sim t^{\nu}$

$$\nu = \begin{cases} d/2 & d < 2 \\ N - \frac{d}{2}(N-1) & 2 < d < d_c(N) = \frac{2N}{N-1} \\ 0 & d > d_c(N) \end{cases}$$

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Open Questions:

• Distributions of $D_N(t)$ and $C_N(t)$ in higher dimensions $d > 1$

• Interacting walkers: e.g. Vicious walkers

Growth of the mean no. of distinct sites visited with time

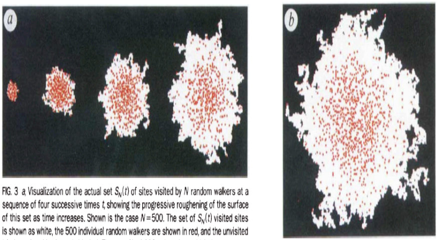
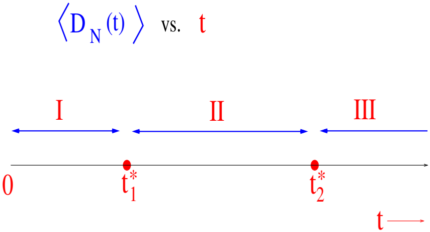


FIG. 3 a, Visualization of the actual set $S_N(t)$ of sites visited by N random walkers at a sequence of four successive times t, showing the progressive roughening of the surface of this set as time increases. Shown is the case $N=500$. The set of $S_N(t)$ visited sites is shown as white, the 500 individual random walkers are shown in red, and the unvisited 'virgin territory' is shown in black. b, The case $N=1{,}000$ at late time, which demonstrates the part played by only a few individual random walkers in causing the roughening of the interface of the set $S_N(t)$.



$\langle D_N(t) \rangle$

$\sim t^d \qquad t < t_1^*$

$\sim t^{d/2} \left[\ln \left(N \langle D_1(t) \rangle \, t^{-d/2} \right) \right]^{d/2} \qquad t_1^* < t < t_2^*$

$t_1^* \sim \ln N$ for all d

$t_2^* \sim \infty \qquad d < 2$

$\sim e^N \qquad d = 2$

$\sim N^{2/(d-2)} \qquad d > 2$