Some combinatorics of random tensor models

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arXiv:1301.1535[hep-th], *Annales Henri Poincaré* (in press)

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arXiv:1310.3132[hep-th], *Annales Henri Poincaré* (in press)

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work in progress

11th of April 2014

Plan

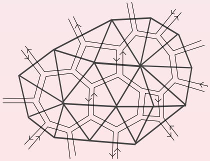
- Introduction
- Matrix models and their large N expansion (dominant graphs)
- 3-dimensional tensor models; the colored and the multi-orientable QFT simplifications
- Classification of Feynman tensor graphs
- Some combinatorial and topological tools
- Large N expansion - leading order graphs
- Large N expansion - next-to-leading graphs
- Large N expansion - the general term
- Perspectives

Introduction

Tensor models were introduced already in the 90's - replicate in dimensions higher than 2 the success of **random matrix models**:

J. Ambjorn *et. al.*, *Mod. Phys. Lett.* ('91), N. Sasakura, *Mod. Phys. Lett.* ('91), M. Gross *Nucl. Phys. Proc. Suppl.* ('92)

The Feynman graphs arising from the perturbative expansion of the partition function of matrix models are **dual graphs to triangulated 2D surfaces**.



- The model defines a certain statistical ensemble over discrete geometries - connection with 2D quantum gravity.
- An important technique is the **large N expansion**, which is controlled by the **genus** of the ribbon Feynman graphs; the leading order contribution to the partition function is given by **planar graphs** (pave the *2D sphere* S^2).
- By simultaneous scaling of N and the coupling constant, the **double-scaling limit** allowed to define a continuum limit, where all topologies contribute, connected to 2D gravity.

the Kontsevich matrix model (the Witten conjecture): rigorous approach to the moduli space of punctured Riemann surfaces

E. Witten, *Nucl. Phys.* **B** (1990), M. Kontsevich, *Commun. Math. Phys.* (1992)

QCD with a large number of colors

't Hooft *Nucl. Phys.* **B** (1974)

Matrix models

Ph. Di Francesco *et. al.*, *Phys. Rept.* (1995), hep-th/9306153

M - $N \times N$ matrix

the partition function:

$$Z = e^F = \int dM e^{-\frac{1}{2}\mathrm{Tr} M^2 + \frac{g}{\sqrt{N}}\mathrm{Tr} M^3}.$$

diagrammatic expansion - Feynman ribbon graphs

generates random triangulations

sum over random triangulations - discretized analogue of the integral over all possible geometries

0−dimensional string theory (a pure theory of surfaces with no coupling to matter on the string worldsheet)

Large N expansion of matrix models

the matrix amplitude can be combinatorially computed - in terms of number of vertices (V), edges and faces (F) of the graph
change of variables: $M \to M\sqrt{N}$ (easy to count powers of N)

$$\mathcal{A} = \lambda^V N^{-\frac{1}{2}V+F} = \lambda^V N^{2-2g}$$

(since $E = \frac{3}{2}V$)
the partition function (and the free energy) supports a $1/N$ expansion:

$$Z = N^2 Z_0(\lambda) + Z_1(\lambda) + \ldots = \sum_g N^{2-2g} Z_g(\lambda)$$

Z_g gives the contribution from surfaces of genus g

large N limit, only planar surfaces survive - dominant graphs (triangulations of the sphere $\mathcal{S}^2$)

V. A. Kazakov, *Phys. Lett.* **B** ('85), F. David, *Nucl. Phys.* **B** ('85), E. Brézin *et al., Commun. Math. Phys.* ('78)

The double scaling limit for matrix models

The successive coefficient functions $Z_g(\lambda)$ as well diverge at the same critical value of the coupling $\lambda = \lambda_c$
the leading singular piece of Z_g:

$$Z_g(\lambda) \propto f_g(\lambda_c - \lambda)^{(2-\gamma_{\rm str})\chi/2} \text{ with } \gamma_{\rm str} = -\frac{1}{2} \text{ (pure gravity)}$$

contributions from higher genera ($\chi < 0$) are enhanced as $\lambda \to \lambda_c$
$\kappa^{-1} := N(\lambda - \lambda_c)^{(2-\gamma_{\rm str})/2}$
the partition function expansion:

$$Z = \sum_g \kappa^{2g-2} f_g$$

double scaling limit: $N \to \infty$, $\lambda \to \lambda_c$ while holding fixed κ
coherent contribution from all genus surfaces

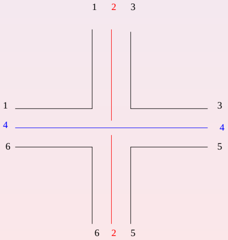
M. Douglas and S. Shenker, *Nucl. Phys.* **B** ('90), E. Brézin and V. Kazakov, *Phys. Lett.* **B**, *Nucl. Phys.* **B** ('90),

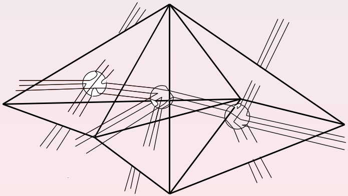
D. Gross and M. Migdal, *Phys. Rev. Lett.*, *Nucl. Phys.* **B** ('90)

$3D$ tensor models

natural generalization of matrix models

matrix $\rightarrow$ rank three tensor





QFT-inspired simplification - the colored tensor models

highly non-trivial combinatorics
$\rightarrow$ a QFT simplification of these models - colored tensor models

(R. Gurău, Commun. Math. Phys. (2011), arXiv:0907.2582)

a quadruplet of complex fields $\left(\phi^0, \phi^1, \phi^2, \phi^3\right)$;

$$S[\{\phi^i\}] = S_0[\{\phi^i\}] + S_{int}[\{\phi^i\}]$$

$$S_0[\{\phi^i\}] = \frac{1}{2}\sum_{p=0}^{3}\sum_{i,j,k,=1}^{N} \overline{\phi^p_{ijk}}\phi^p_{ijk} \tag{1}$$

$$S_{int}[\{\phi^i\}] = \frac{\lambda}{4}\sum_{i,j,k,i',j',k'=1}^{N} \phi^0_{ijk}\phi^1_{i'j'k}\phi^2_{i'jk'}\phi^3_{k'j'i} + \text{ c. c.},$$

the indices $0, \dots, 3$ - color indices.
extra property: the faces of the Feynman graphs of this model have always exactly two (alternating) colors.

Various QFT developments for colored tensor models

- large N expansion

  R. Gurau, Annales Henri Poincare (2011), [arXiv:1011.2726 [gr-qc]]

- large N expansion in any dimension
  R. Gurau and V. Rivasseau, Europhys. Lett. (2011), arXiv:1101.4182[gr-qc],

  R. Gurău, Annales Henri Poincaré (2012) [arXiv:1102.5759 [gr-qc]] .

- continuum phase transition and computation of critical exponents

  V. Bonzom *et. al.*, *Nucl. Phys.* **B** (2011) arXiv:1105.3122[hep-th]
- double-scale limit G. Schaeffer and R. Gurău, arXiv:1307.5279,

  S. Dartois *et. al.*, JHEP (2013)

- Noether currents

  J. Ben Gelon, *J. Math. Phys.* (2012), [arXiv:1107.3122 [hep-th]]

- renormalizable tensor models
  J. Ben Geloun and V. Rivasseau, Commun. Math. Phys. (2013), arXiv:1111.4997 [hep-th].

  S. Carrozza *et. al.*, arXiv:1207.6734 [hep-th].

- Connes-Kreimer algebraic reformulation of tensor renormalizability M. Raasakka and A. T., *Sém. Loth. Comb.* (in press)

Question:

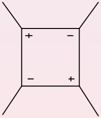
How much of these large N scaling properties of colored tensor models generalize to larger family of tensor graphs?

A (Moyal) QFT-inspired simplification of tensor models

highly non-trivial combinatorics
$\rightarrow$ a QFT simplification of these models - multi-orientable models

A. Tanasă, J. Phys. **A** (2012)

proposal made within the Group Field Theory framework (quantum gravity approach related to Loop Quantum Gravity)

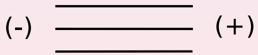


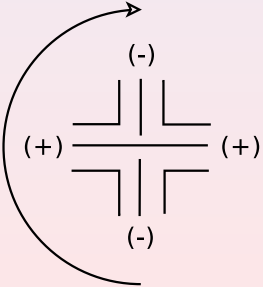
edge going from a $+$ to a $-$ corner

The action: the propagator and the vertex

$$S[\phi] = S_0[\phi] + S_{int}[\phi], \tag{2}$$

$$S_0[\phi] = \frac{1}{2}\sum_{i,j,k=1}^{N} \bar{\phi}_{ijk}\phi_{ijk}, \quad S_{int}[\phi] = \frac{\lambda}{4}\sum_{i,j,k,i',j',k'=1}^{N} \phi_{ijk}\bar{\phi}_{kj'i'}\phi_{k'ji'}\bar{\phi}_{k'j'i}.$$





Multi-orientable tensor Feynman graphs

no twists on the propagators $\rightarrow$ one-to-one correspondence between multi-orientable tensor Feynman graphs and embedded graphs

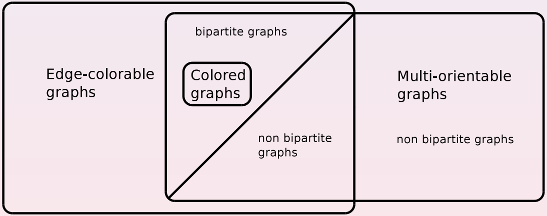
A *four-edge colorable graph* is a graph for which the edge chromatic number is equal to four.

Proposition

The set of Feynman graphs generated by the colored action (1) *is a strict subset of the set of Feynman graphs generated by the MO action* (2).

Proposition

A bipartite graph is four-edge colorable.



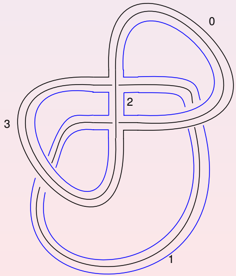
Example of tensor graphs

A **tadface** is a face "going" several times through the same edge.

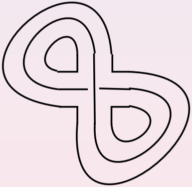
The condition of multi-orientability discards tadfaces

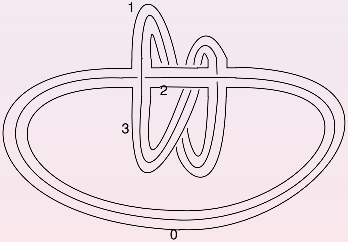
(A. T., J. Phys. **A** (2012), arXiv:1109.0694).

example of a graph with a tadface which is edge-colorable

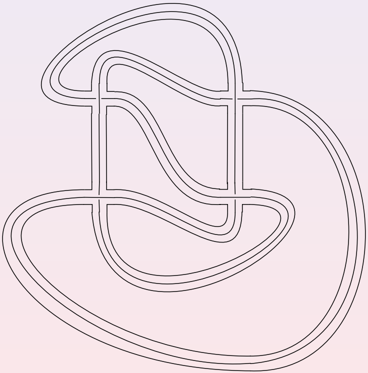


the planar double tadpole as an example of a MO graph which is not colorable. On the right, an example of a MO graph which is 4-edge colorable but does not occur in colorable tensor models.

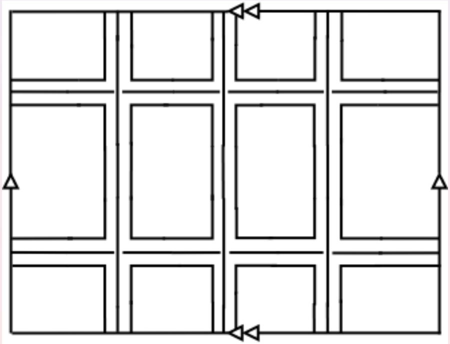




A 4-edge colorable MO graph which is not bipartite



A graph without tadfaces which is not m.o. Edges of the box are identified so that the graph is drawn on the torus

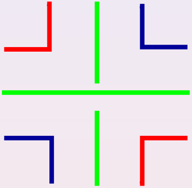


Combinatorial and topological tools - jacket ribbon subgraphs

In the colored case the $1/N$ expansion relies on the notion of **jacket ribbon subgraphs**, which are associated to the cycle of colors up to orientation.

Generalization of the notion of jackets for MO graphs

three pairs of opposite corner strands



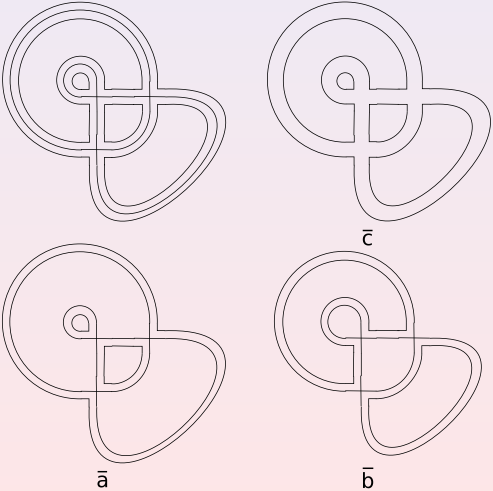
Definition

A **jacket of an MO graph** is the graph made by excluding one type of strands throughout the graph. The *outer* jacket $\bar{c}$ is made of all outer strands, or equivalently excludes the inner strands (the green ones); jacket $\bar{a}$ excludes all strands of type a (the red ones) and jacket $\bar{b}$ excludes all strands of type b (the blue ones).

$\hookrightarrow$ such a splitting is always possible

Example of jacket subgraphs

A MO graph with its three jackets $\bar{a}$, $\bar{b}$, $\bar{c}$

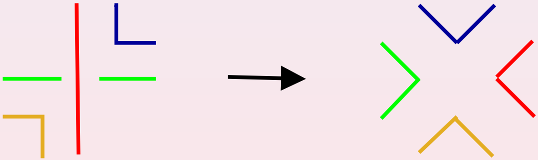


Is such a jacket subgraph a ribbon subgraph?

Proposition

Any jacket of a MO graph is a (connected vacuum) ribbon graph (with uniform degree 4 at each vertex).

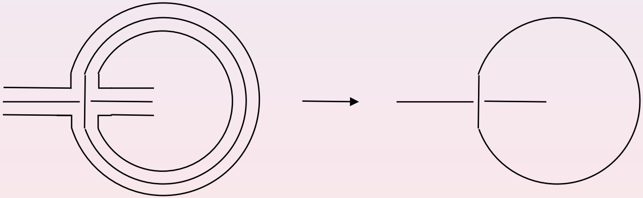
untwisting vertex procedure:



may introduce twists on the edges

this does not hold for any, non-m. o., tensor graph

Example: Deleting a pair of opposite corner strands in this tadpole (which has tadfaces), does not lead to a ribbon graph.



Euler characteristic & degree of MO tensor graphs

ribbon graphs can represent orientable or non-orientable surfaces.

Euler characteristic formula:

$$\chi(\mathcal{J}) = V_{\mathcal{J}} - E_{\mathcal{J}} + F_{\mathcal{J}} = 2 - k_{\mathcal{J}},$$

$k_{\mathcal{J}}$ is the non-orientable genus,
$V_{\mathcal{J}}$ is the number of vertices,
$E_{\mathcal{J}}$ the number of edges and
$F_{\mathcal{J}}$ the number of faces.

If the surface is orientable, k is even and equal to twice the orientable genus g

Given an MO graph $\mathcal{G}$, its degree $\varpi(\mathcal{G})$ is defined by

$$\varpi(\mathcal{G}) = \sum_{\mathcal{J}} \frac{k_{\mathcal{J}}}{2} = 3 + \frac{3}{2} V_{\mathcal{G}} - F_{\mathcal{G}},$$

the sum over $\mathcal{J}$ running over the three jackets of $\mathcal{G}$.

Large N expansion of the MO tensor model

Feynman amplitude calculation - each tensor graph face contributes with a factor N, N being the size of the tensor

$\Longrightarrow$ one needs to count the number of faces of the tensor graph

this can be achieved using the graph's jackets (ribbon subgraphs)

The Feynman amplitude of a general MO tensor graph $\mathcal{G}$ writes:

$$A(\mathcal{G}) = \lambda^{V_{\mathcal{G}}} N^{3-\varpi(\mathcal{G})}.$$

The free energy writes as a formal series in $1/N$:

$$F(\lambda, N) = \sum_{\varpi \in \mathbb{N}/2} C^{[\varpi]}(\lambda) N^{3-\varpi},$$

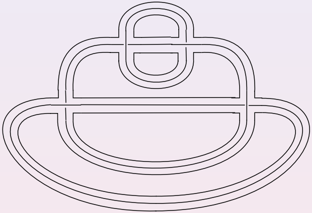
$$C^{[\varpi]}(\lambda) = \sum_{\mathcal{G}, \varpi(\mathcal{G})=\varpi} \frac{1}{s(\mathcal{G})} \lambda^{v_{\mathcal{G}}}.$$

Dominant graphs

dominant graphs:

$$\varpi = 0.$$

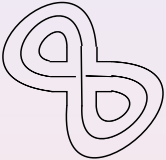
An example of a dominant tensor graph

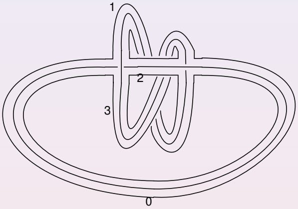


- outer jacket is orientable (always the case for the outer jacket), and it has genus $g_1 = 0$.
- the two remaining jackets also have vanishing genus $g_2 = g_3 = 0$ (can be directly computed using Euler's characteristic formula)

$\Longrightarrow$ vanishing degree ($\varpi = 0$) $\Leftrightarrow$ dominant graph

Two examples of non-dominant tensor graphs





double tadpole:
$\varpi = 0 + \frac{1}{2} + 0 = \frac{1}{2}$.

"twisted sunshine" (bipartite 4−edge colorable graph):
Its outer jacket is orientable (always the case for the outer jacket), and it has genus $g_1 = 1$.
The two remaining jackets are isomorphic and have non-orientable genus $k_2 = k_3 = 1$.
$\Longrightarrow \varpi = 2$.

General identification of dominant graphs

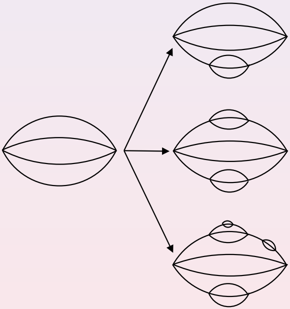
Theorem

Non-bipartite MO graphs have at least one non-orientable jacket and are thus non-dominant of degree

$$\varpi \geq \frac{1}{2}.$$

Proof. cycle parity (at least one odd cycle in a non-bipartite graph) & MO parity constraints on cycles

The only bipartite (and hence edge-colorable) MO tensor graphs of vanishing degree ($\varpi = 0$) are the graphs obtained from insertions of the "melon" graph.



series-parallel graphs in combinatorics

Dominant graphs of the large N expansion

Main result:

Theorem

The MO model admits a $1/N$ expansion whose dominant graphs are the "melonic" ones.

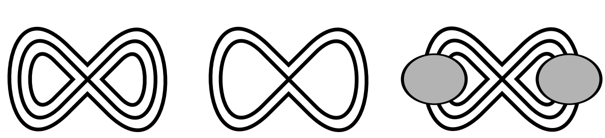
More on "melonic" tensor graphs

1. they maximize the number of faces for a given number of vertices.
2. they correspond to a particular class of triangulations of the sphere $\mathcal{S}^3$.

Next-to-leading order (NLO) in the $1/N$ expansion

M. Raasakka and A. T., arXiv:1310.3132, *Annales Henri Poincaré* (in press)

- The multi-orientable next-to-leading order sector is given by $\omega = 1/2$, because of non-orientable jackets, not $\omega = 1$ as for colored models!
- The simplest NLO graph is the double-tadpole:



- Any insertion of a melonic 2-point subgraph conserves the degree.

Main result:

All possible NLO ($\omega = 1/2$) graphs are given by melon insertions in the double-tadpole MO tensor graph.

LO series

V. Bonzom *et. al. Nucl. Phys.* **B** (2011), W. Kaminski *et. al.* arXiv:1304.6934 [hep-th]

$$G_{\rm LO} \propto {\rm const.} + \left(1 - \left(\frac{\lambda}{\lambda_c}\right)^2\right)^{1/2}$$

$G_{\rm LO}$ - the LO two-point function

λ_c - critical value of the coupling constant (radius of convergence of the $G_{\rm LO}$ series)

$$F_{\rm LO} \propto {\rm const.} + \left(1 - \frac{\lambda^2}{\lambda_c^2}\right)^{2-\gamma_{\rm LO}}, \qquad \gamma_{\rm LO} = \frac{1}{2}$$

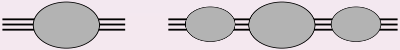
F - the free energy

γ - the susceptibility exponent (the entropy exponent)

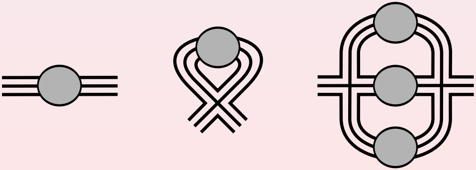
same behavior for the LO series of the MO model

Next-to-leading order series for the MO tensor model

- Following the combinatorics of NLO graphs, one may analyze the NLO series by relating it to the LO series.
  Let G and Σ be the connected and the 1PI 2-point functions.
- $G_{NLO} = G_{LO}^2 \Sigma_{NLO}$:



- One has: $\Sigma_{NLO} = \lambda G_{LO} + 3\lambda^2 G_{LO}^2 G_{NLO}$



- Substituting, one has

$$G_{NLO} = \frac{\lambda G_{LO}^3}{1 - 3\lambda^2 G_{LO}^4}$$

- Differentiating the LO two-point function relation $G_{LO} = 1 + \lambda^2 G_{LO}^4$ we get

$$\frac{\partial}{\partial \lambda} G_{LO} = \frac{2\lambda G_{LO}^4}{1 - 4\lambda^2 G_{LO}^3} = \frac{2\lambda G_{LO}^5}{1 - 3\lambda^2 G_{LO}^4},$$

- This leads to:

$$G_{NLO} = \frac{\lambda}{G_{LO}^2} \frac{\partial}{\partial \lambda^2} G_{LO},$$

which implies, together with $G_{LO} \propto \text{const.} + (1 - (\lambda^2/\lambda_c^2))^{1/2}$,

$$G_{NLO} \propto \left(1 - \frac{\lambda^2}{\lambda_c^2}\right)^{-1/2}.$$

From the Schwinger-Dyson equation relating the connected two-point function G_{NLO} to the free energy F_{NLO} one has:

Critical behavior of the NLO free energy:

$$F_{\mathrm{NLO}} \propto \left(1 - \frac{\lambda^2}{\lambda_c^2}\right)^{2-\gamma_{\mathrm{NLO}}}, \quad \text{where} \quad \gamma_{\mathrm{NLO}} = 3/2.$$

- same critical value of the coupling constant (radius of convergence) for the NLO series as for the LO series;
- distinct value for the NLO susceptibility exponent.

similar behaviour to the matrix model case

Some considerations on the general term of the expansion

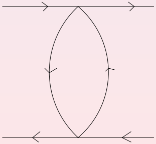
work in progress with E. Fusy

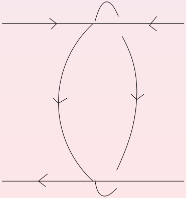
generalization of the Gurău-Schaeffer combinatorial approach

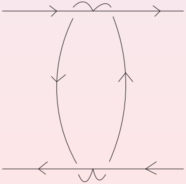
R. Gurău and G. Schaeffer, arXiv:1307.5279[math.CO]

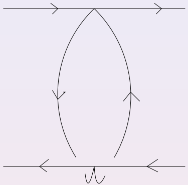
(S. Dartois *et. al.*, JHEP (2013))

generalization of the "colored" notion of 2-dipole:
A two-dipole is a subgraph formed by a couple of vertices connected by two parallel edges which has a face of length two.









some configurations (particular chains of $2-$dipoles) can be repeated without increasing the degree $\bar{\omega}$ - except for the tadpole case

the dominant configurations maximize the number of such chains

gluing together 1/2-degree building blocks leads to dominant configurations (even with respect to the dominant configuration of same degree of the colored model)

bijection with binary trees with $2\bar{\omega}+1$ leaves

Perspectives

- double-scaling limit of the MO tensor model; convergence of the series ?

  work in progress with Eric Fusy

- study the Noether currents of MO tensor models - generalization of J. Ben Geloun, *J. Math. Phys.* (2012)
- renormalizable MO models
- generalization of the matrix integral techniques to tensor integral techniques - *3D map counting*
- *Schaeffer bijection* G. Schaeffer, *Electronic J. Comb.* (1997)
  $3D$ geodesic length?
- enlarge the MO framework to include still larger classes of tensor graphs and check whether they admit a $1/N$ expansion and a double scaling limit.

Thank you for your attention!

"The amount of theoretical work one has to cover before being able to solve
problems of real practical value is rather large, but this circumstance is [...] likely to become more pronounced in the theoretical physics of the future."

P.A.M. Dirac, *"The principles of Quantum Mechanics"*, 1930