Random 3-colourings and a phase transition for the 3-state Potts antiferromagnet on a class of planar lattices

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Antiferromagnetic Potts models

Let $G = (V, E)$ be a locally finite graph with vertex set V and edge set E.
For each *spin configuration* $\sigma : V \to \{1, \ldots, q\}$, define a *Hamiltonian*

$$H(\sigma) := \sum_{\{x,y\} \in E} 1_{\{\sigma(x) = \sigma(y)\}}.$$

If V is finite, define finite volume *Gibbs measures*

$$\mu_\beta(\sigma) := \frac{1}{Z_\beta} e^{-\beta H(\sigma)},$$

where $Z_\beta := \sum_\sigma e^{-\beta H(\sigma)}$ is the *partition sum*.
If V is infinite, then define infinite-volume Gibbs measures through the *DLR conditions*.

In the *zero temperature limit* $\beta \to \infty$, the measure μ_β converges to the uniform distribution on q-colourings of G (if there exist any).

A critical number of colours

'Facts' believed to be true.

For each infinite quasi-transitive graph G, there is a q_{c} such that:

- For $q < q_{\mathrm{c}}$, the model has a phase transition at some $0 < \beta_{\mathrm{c}} < \infty$ between disorder and long-range order.

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- For $q = q_c$, the zero-temperature model is critical (?).

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On $\mathbb{Z}^d$ with nearest-neighbour bounds, it is believed that $q_c = 3$ in dimension $d = 2$ and $q_c \to \infty$ as $d \to \infty$.

Bipartite lattices

Recall that a graph $G = (V, E)$ is *bipartite* if the vertices can be partitioned into two subsets $V = V_0 \cup V_1$ such that each edge has one endvertex in V_0 and the other in V_1.

It is often useful to view the sublattices $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ as graphs in their own right by connecting vertices that are at distance 2 in G.

On bipartite graphs (like $\mathbb{Z}^d$), boundary conditions where one sublattice (G_0, say) is uniformly coloured in a single colour (k, say) may lead to long-range order, in particular, the percolation of colour k on G_0.

Effectively, the second sublattice G_1 induces a ferromagnetic interaction on the first sublattice G_0.

An external forcing

It is instructive to add an extra interaction

$$H'(\sigma) := \sum_{\{x,y\}\in E_0} 1_{\{\sigma(x)=\sigma(y)\}} - \sum_{\{x,y\}\in E_1} 1_{\{\sigma(x)=\sigma(y)\}}.$$

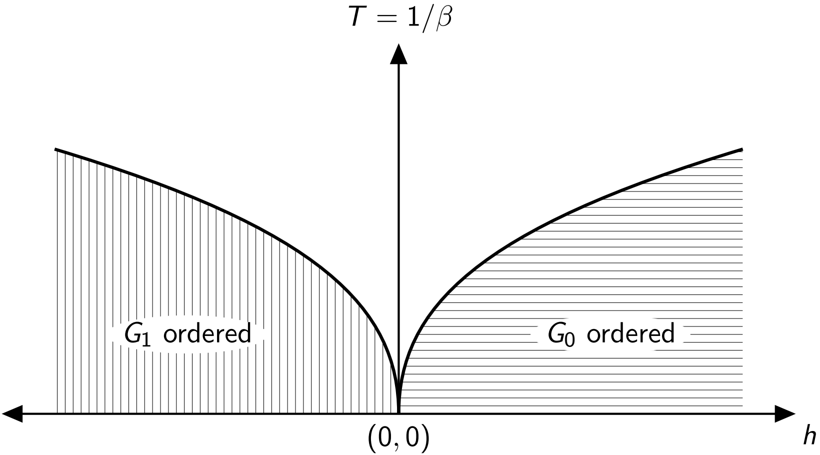
And define Gibbs measures

$$\mu_{\beta,h}(\sigma) := \frac{1}{Z_{\beta,h}} e^{-\beta H(\sigma) + hH'(\sigma)}.$$

If $h > 0$ (resp. $h < 0$), this favours uniform colourings of the sublattice G_0 (resp. G_1). In particular, for $G = \mathbb{Z}^d$, this breaks the symmetry between the sublattices G_0 and G_1.

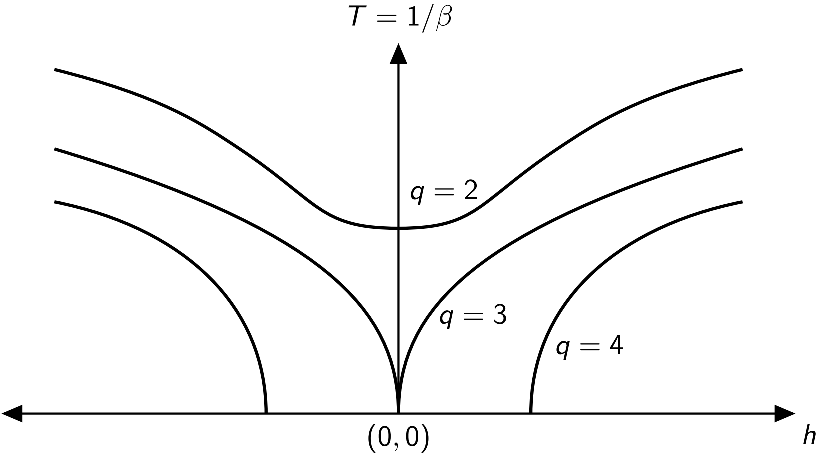
Phase diagram - speculative!

My guess for $\mathbb{Z}^2$ with $q = 3$.



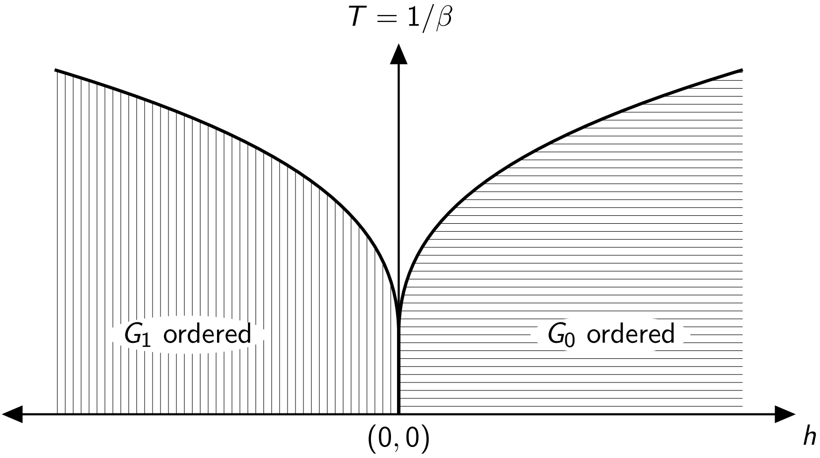
Phase diagram

My guess for $\mathbb{Z}^2$ with $q = 2, 3, 4, \ldots$.



Phase diagram

My guess for $\mathbb{Z}^d$ with $d \geq 3$ and $q = 3$.



At zero temperature, the case $q = 3$ is special:

Let $h : \mathbb{Z}^d \to \mathbb{Z}$ satisfy

$$|h(x) - h(y)| = 1 \quad \text{if} \quad |x - y| = 1.$$

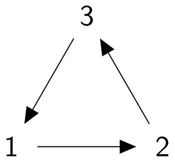
Then

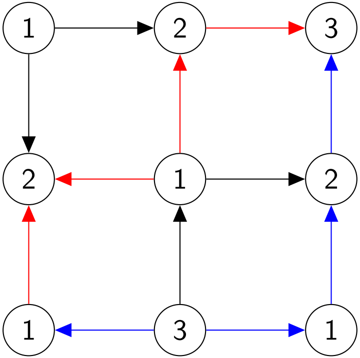
$$\sigma(x) := h(x) \mod(3)$$

is a 3-colouring.

Fact: If we fix $h(x_0)$ in one reference point x_0, then the mapping $h \mapsto \sigma$ is a *bijection*, i.e., we can recover h from σ.

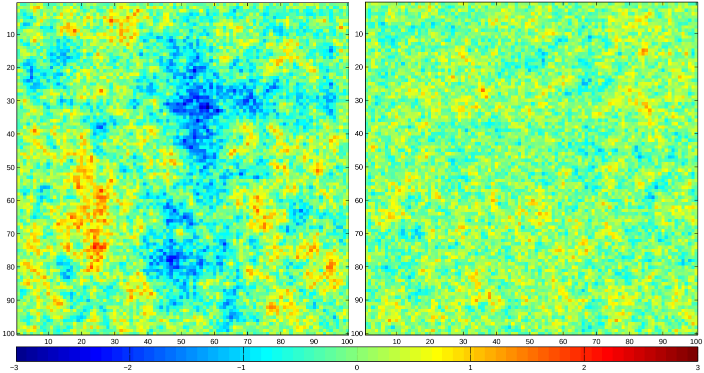
Height mapping





The red path can be deformed into the blue path so that the height difference between the endpoints stays the same.

Height mapping



Simulations by Ron Peled of a random height mapping on a 100×100 square and the middle layer of a $100 \times 100 \times 100$ cube. Simulated using Propp-Wilson's coupling from the past.

High dimension versus dimension two

Ron Peled (preprint 2010) has proved that for sufficiently high d, a typical height-configuration is flat.
This implies (some form of) long-range order for the zero-temperature, 3-state antiferromagnetic Potts model on $\mathbb{Z}^d$.

On the other hand, on $\mathbb{Z}^2$, the fluctuations of the height model are believed to be of order log(system size). This is similar to wat is known for dimer models (R. Kenyon).

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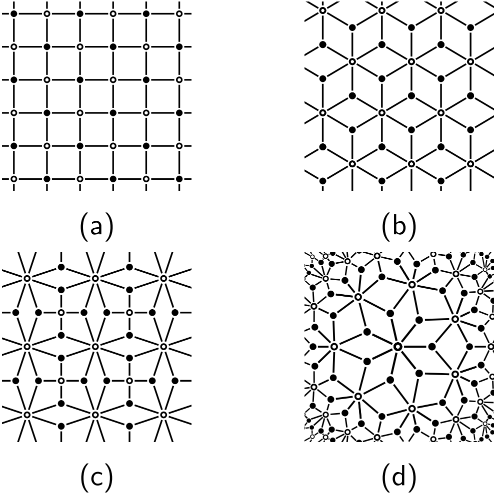
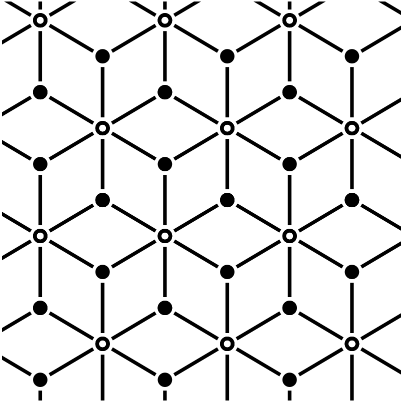
Is this behavior universal in two-dimensional AF 3-state Potts models?

Quadrangulations



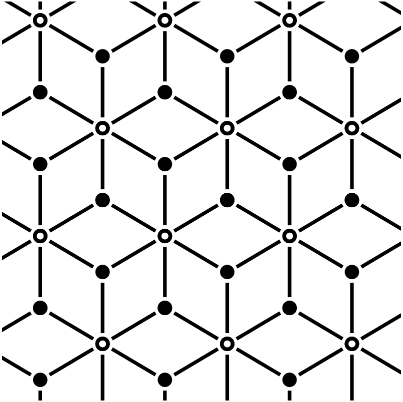
Four examples of quasi-transitive quadrangulations of the plane.

The diced lattice



Theorem (R. Kotecký, J. Salas & A.D. Sokal, 2008): The 3-state antiferromagnetic Potts model on the diced lattice has long-range order for β sufficiently large.

The diced lattice



The diced lattice:

- Is bipartite.
- Is a quadrangulation.
- Admits a height representation.

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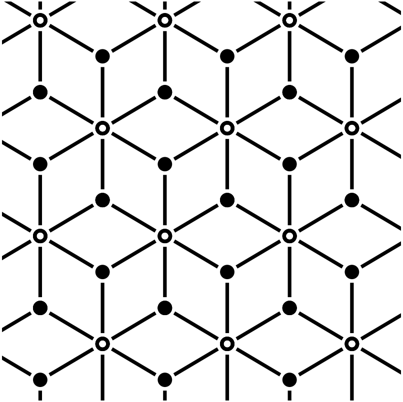

The diced lattice:

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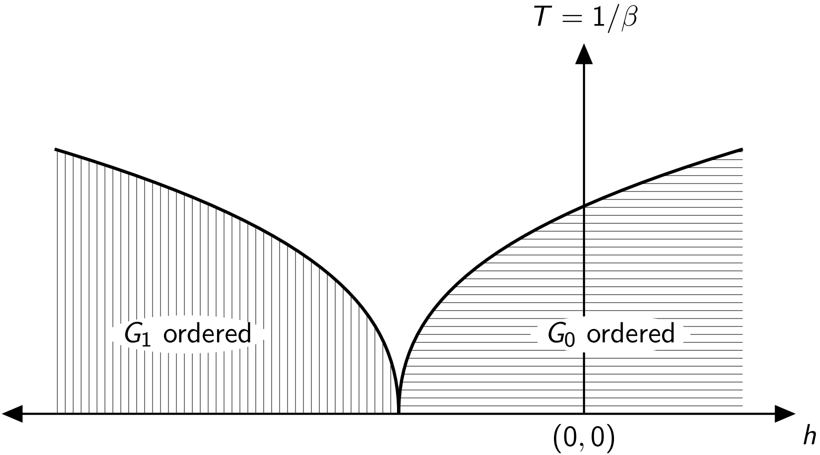
So why is it different from $\mathbb{Z}^2$*?*

Explanation 1: broken symmetry

My guess for the phase diagram on the diced lattice.



Explanation 1: broken symmetry

On $\mathbb{Z}^2$, the sublattices G_0 and G_1 play a symmetric role, but on the diced lattice, this symmetry is broken.

For the diced lattice, the spatial density of points of G_1 is *twice as high* as for G_0. Therefore, it is entropically favourable to paint G_0 in one colour and reserve the other two colours to G_1. (Certainly compared to doing things the other way around!)

Effectively, this is like applying an external forcing that favours ferromagnetic order on G_0 and disfavours it on G_1.

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Is it possible to counter this with an external forcing of exactly the right strength h?

Theorem [Kotecký, Sokal, S.] Let $G = (V, E)$ be a quadrangulation of the plane, and let $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$ be its sublattices, connected through bonds along the diagonals of quadrilaterals. Assume that G_0 is a locally finite, 3-connected, quasi-transitive *triangulation* with one end. Then there exist $\beta_0, C < \infty$ and $\varepsilon > 0$ such that for each inverse temperature $\beta \in [\beta_0, \infty]$ and each $k \in \{1, 2, 3\}$, there exists an infinite-volume Gibbs measure $\mu_{k,\beta}$ for the 3-state Potts antiferromagnet on G satisfying:

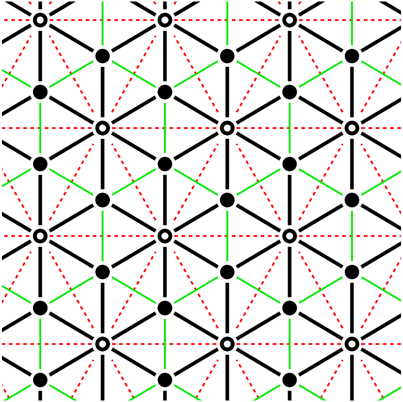
(a) For all $v_0 \in V_0$, we have $\mu_{k,\beta}(\sigma_{v_0} = k) \geq \frac{1}{3} + \varepsilon$.

(b) For all $v_1 \in V_1$, we have $\mu_{k,\beta}(\sigma_{v_1} = k) \leq \frac{1}{3} - \varepsilon$.

(c) For all $\{u, v\} \in E$, we have $\mu_{k,\beta}(\sigma_u = \sigma_v) \leq Ce^{-\beta}$.

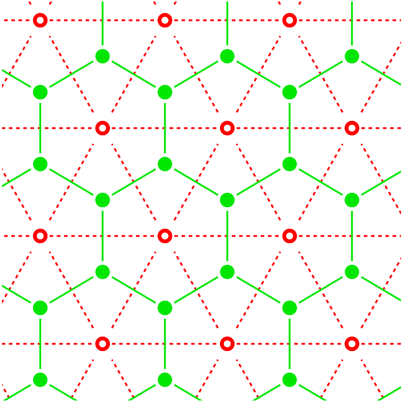
In particular, for each inverse temperature $\beta \in [\beta_0, \infty]$, the 3-state Potts antiferromagnet on G has at least three distinct extremal infinite-volume Gibbs measures.

The diced lattice



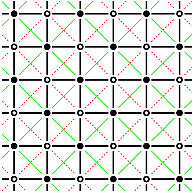
Connecting vertices in the sublattices at distance 2 in the original lattice. . .

The diced lattice

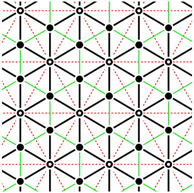


...yields two graphs G_0 and G_1 that are dual in the sense of planar graph duality. By assumption, G_0 is a triangulation.

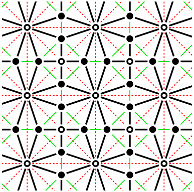
Examples



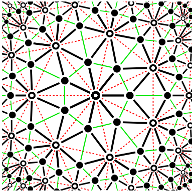
(a)



(b)



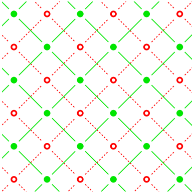
(c)



(d)

Our theorem applies to the lattices (b)–(d), but not to (a).

Examples



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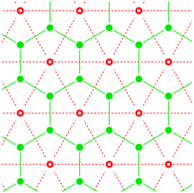
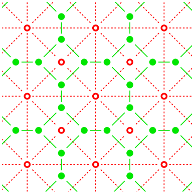
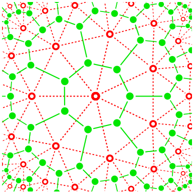


(b)



(c)



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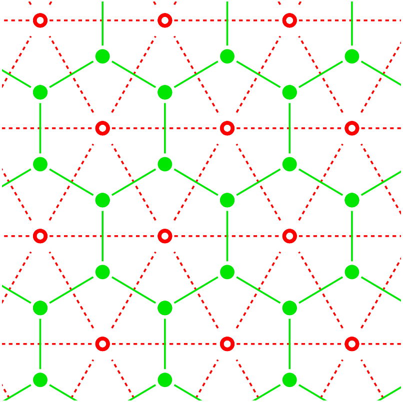
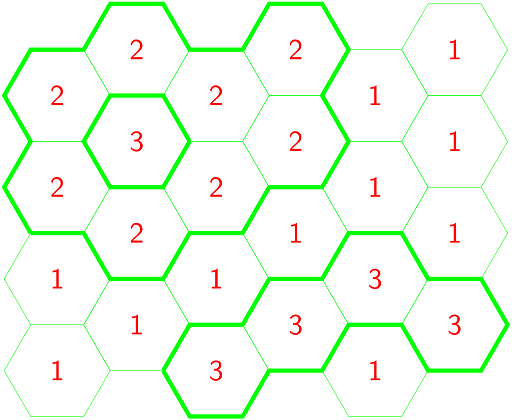
Contour model

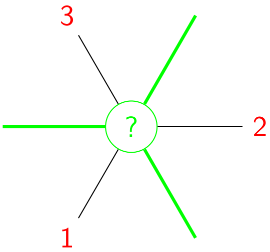


Recall the dual planar graphs G_0 and G_1
By assumption, G_0 is a triangulation.



We separate vertices with different colours in G_0 by contours in G_1.

Contour model



At zero temperature, contours are collections of simple cycles, since vertices in G_1 cannot be surrounded by three different types in G_0.

Peierls argument

For vertices on a contour, only one type is available, while for vertices that are not on a contour, 2 types are available. As a result, for each configuration in which a given cycle is present, we can find $2^{|\gamma|}$ configurations where this contour has been removed, with $|\gamma| =$ the length of γ. Thus, the probability of a given cycle γ being present is less or equal than $2^{-|\gamma|}$ and the expected number of cycles surrounding a given vertex can be estimated by

$$\sum_{L=6}^{\infty} N(L) 2^{-L},$$

where $N(L)$ denotes the number of cycles of length L surrounding a given vertex.

Peierls argument

Let $\tilde{N}(L)$ denotes the number of self-avoiding paths of length L in the honeycomb lattice. Duminil-Copin and Smirnov (2010) have proved that

$$\lim_{L\to\infty} \tilde{N}(L)^{1/L} = \sqrt{2+\sqrt{2}},$$

i.e., the connective constant of the honeycomb lattice is $\sqrt{2+\sqrt{2}}$. Since $N(L) \leq L\tilde{N}(L)$, it follows that

$$N(L) \leq \mathrm{constant} \times L \times \Big(\sqrt{2+\sqrt{2}}\Big)^L.$$

Note that $\sqrt{2+\sqrt{2}} < 2$ and hence the expected number of cycles surrounding a given vertex is bounded by

$$\mathrm{constant} \times \sum_{L=6}^{\infty} 2^{-L} L \Big(\sqrt{2+\sqrt{2}}\Big)^L < \infty.$$

Peierls argument

Using moreover explicit counting of cycles up to length 140 due to Jensen (2006), Kotecký, Salas & Sokal (2008) were able to prove that for any vertex v_0 in G_0

$$\mathbb{P}[v_0 \text{ is surrounded by a cycle}] < \frac{2}{3}.$$

Using 1-boundary conditions on G_0 and letting the box size to infinity, it follows that there exists a zero-temperature infinite-volume Gibbs measure μ_∞ such that

$$\mu_\infty\big(\sigma(v_0) = 1\big) > \frac{1}{3}.$$

In particular, this 'positive magnetization' proves Gibbs state multiplicity and long range order.

Improved Peierls argument

For our general class of lattices:

$$G_0 \text{ triangulation} \quad \Rightarrow \quad \text{every vertex in } G_1 \text{ has degree } 3$$

$$\Rightarrow \quad \text{connective constant of } G_1 \text{ is } < 2,$$

which implies that the Peierls sum is *finite*, though possibly large. It follows that for every $\varepsilon > 0$ there exists a finite Δ_0 such that

$$\mathbb{P}\big[\Delta_0 \text{ is surrounded by a cycle} \,\big|\, \text{no contour intersects } \Delta_0\big] < \varepsilon.$$

and hence

$$\mu_\infty\big(\Delta_0 \text{ has colour } 1 \,\big|\, \Delta_0 \text{ is uniformly coloured }\big) > 1 - \varepsilon,$$

uniformly in the system size, which implies *Gibbs state multiplicity*. With some extra work, we can even prove positive magnetization.

Proof of positive magnetization

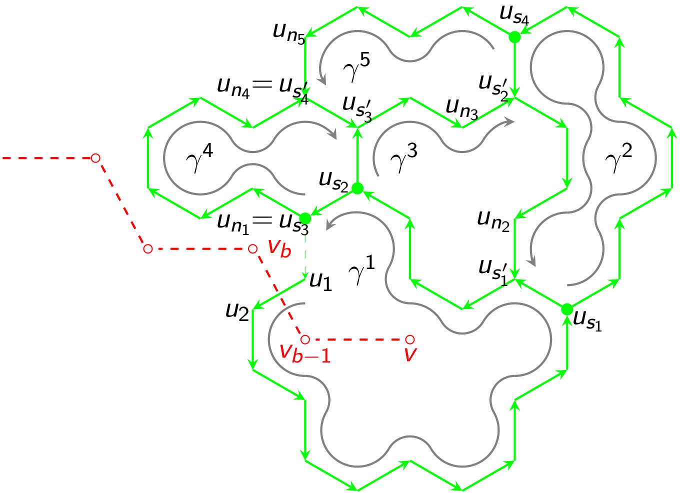
(Inspired by Wang-Swendsen-Kotecký algorithm)

Condition on the position of the 3's. Then the 1's and 2's on the remaining sublattice form an antiferromagnetic Ising model. Flipping one sublattice, we can make this ferromagnetic and construct it with the usual random-cluster representation. Then, under the conditional law:

$$\mathbb{P}[\Delta_0 \text{ uniformly } 1] - \mathbb{P}[\Delta_0 \text{ uniformly } 2]$$
$$= \mathbb{P}[\Delta_0 \text{ connected to the boundary}].$$

A finite energy argument shows that if this is positive for *some* finite Δ_0, then it is positive for *all* such Δ_0, in particular, singletons.

Positive temperature



The argument can be extended to small positive temperatures by a careful counting of nonsimple contours.

Low temperature stability

Our Gibbs measures satisfy, for all $\beta \in [\beta_0, \infty]$:

(a) For all $v_0 \in V_0$, we have $\mu_{k,\beta}(\sigma_{v_0} = k) \geq \frac{1}{3} + \varepsilon$.

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(c) For all $\{u, v\} \in E$, we have $\mu_{k,\beta}(\sigma_u = \sigma_v) \leq Ce^{-\beta}$.

Letting $\beta \to \infty$, we can find a (sub-) sequence of positive temperature Gibbs measures converging to a zero-temperature limit that also has these properties.

In particular, by (c), such a zero-temperature Gibbs measure is concentrated on proper 3-colourings.

Low temperature stability.

Why is this important?

Low temperature stability

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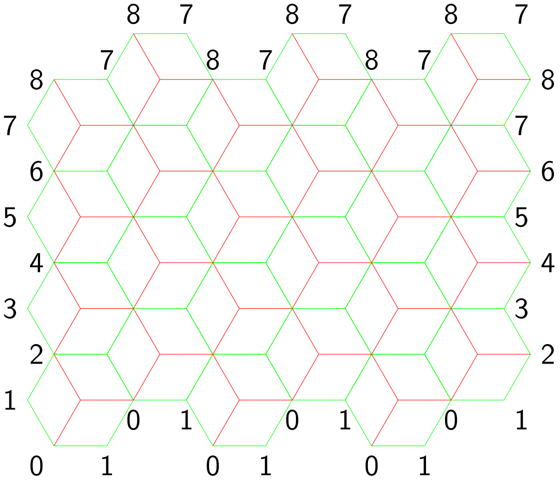
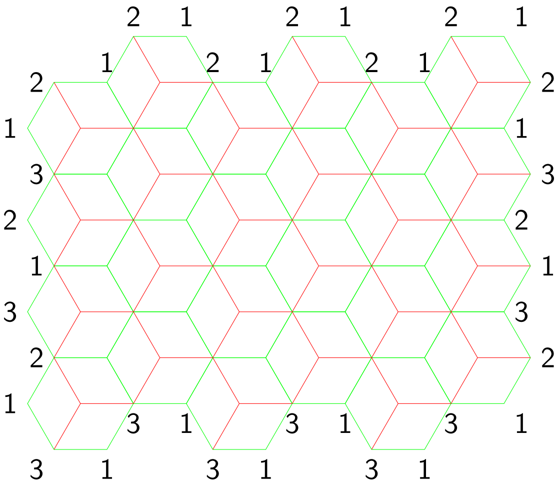
There are trivial counterexamples of zero-temperature Gibbs measures that are not stable, in the sense that they are not the limit of any positive-temperature Gibbs measures.

Low temperature stability



Stiff boundary conditions for the height model. . .

Low temperature stability



. . . translate into b.c. for the Potts model that correspond to a unique 3-colouring of the interior.

Low temperature stability

These Gibbs measures are *not* the limit of any positive temperature Gibbs measures as the temperature is sent to zero.

The reason is that at any $\beta < \infty$, we pay an energetic price of order L (surface effect) to change to more advantageous boundary conditions that lead to an entropic advantage of order L^2 (bulk effect).

Theorem
[Huang, Chen, Deng, Jacobsen, Koteck, Salas, Sokal & S.]
There exist quadrangulations of the plane with arbitrary high values of $q_{\rm c}$.

Long-range order with many colours

The essential feature seems to be a strong asymmetry between the sublattices.