Convergence of cluster expansions: A review of the main strategies and their relations

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The basic setup

Goal: To study systems of objects constrained only by a "non-overlapping" condition

Countable family $\mathcal{P}$ of objects: polymers, animals, ..., characterized by

- An *incompatibility* constraint:

$$\begin{array}{ccc} \gamma \nsim \gamma' & \text{if } \gamma, \gamma' \in \mathcal{P} & \text{incompatible} \\ \gamma \sim \gamma' & & \text{compatible} \end{array}$$

  For simplicity: each polymer incompatible with itself ($\gamma \nsim \gamma$, $\forall \gamma \in \mathcal{P}$)

- A family of *activities* $\boldsymbol{z} = \{z_\gamma\}_{\gamma \in \mathcal{P}} \in \mathbb{C}^{\mathcal{P}}$.

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The basic ("finite-volume") measures

Defined, for each *finite* family $\mathcal{P}_\Lambda \subset \mathcal{P}$, by weights

$$W_\Lambda(\{\gamma_1, \gamma_2, \ldots, \gamma_n\}) = \frac{1}{\Xi_\Lambda(\boldsymbol{z})} z_{\gamma_1} z_{\gamma_2} \cdots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}}$$

for $n \geq 1$ $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathcal{P}_\Lambda$, and $W_\Lambda(\emptyset) = 1/\Xi_\Lambda$, where

$$\Xi_\Lambda(\boldsymbol{z}) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} z_{\gamma_1} z_{\gamma_2} \cdots z_{\gamma_n} \prod_{j<k} \mathbb{1}_{\{\gamma_j \sim \gamma_k\}}$$

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Polymer correlation functions

For $\gamma_1, \ldots, \gamma_k$ mutually compatible polymers in $\mathcal{P}_\Lambda$

$$\mathrm{Prob}_\Lambda\big(\{\gamma_1, \ldots, \gamma_k \text{ are present}\}\big) \;=\; z_{\gamma_1} \cdots z_{\gamma_k} \, \frac{\Xi_{\Lambda \setminus \{\gamma_1, \ldots, \gamma_k\}^*}}{\Xi_\Lambda}$$

where

$$\{\gamma_1, \ldots, \gamma_k\}^* = \text{polymers incompatible with } \gamma_1, \ldots, \gamma_k$$

The questions:

- Existence of the limit $\mathcal{P}_\Lambda \to \mathcal{P}$ ("thermodynamic limit")
- Properties of the resulting measure (mixing properties, dependency on parameters,...)
- Asymptotic behavior of Ξ_Λ (analyticity!)
- Control of correlation functions

Motivation

Immediate:

- *Physics:* Grand-canonical ensemble of polymer gas with activities z_γ and hard-core interaction
- *Statistics:* Invariant measure of point processes with not-overlapping grains and birth rates z_γ

Less immediate:

- Statistical mechanical models at high and low temperatures are mapped into such systems
- More generally: most perturbative arguments in physics involve maps of this type (choice of the "right" variables)
- Zeros of the partition functions Ξ_Λ (phase transitions, sphere packing, chromatic polynomials, Lovász lemma)

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Example zero: Hard-core lattice gases

Measures on configurations $\omega \in \mathbb{L}^E$ with

- $\mathbb{L}$ =vertices of a graph (eg. $\mathbb{Z}^d$),
- $E = \{0, 1\}$ ("1"=occupied)
- No occupied neighbors are allowed
- Allowed configurations have weights $\sim \exp(\mu\beta\,|\Gamma|)$ (μ=Gibbs chemical potential, β=inverse temperature)

This is a polymer model with

- $\mathcal{P} = \{\text{vertices of } \mathbb{L}\}$
- $x \not\sim y$ iff x and y are graph neighbors
- $z_x = e^{\beta u}$

(For Sokal-like people *all* polymer models are of this type)

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Single-call loss networks

Defined through the following dynamics:

- $\mathcal{P}$ = finite subsets of $\mathbb{Z}^d$ —the *calls*
- A call γ is attempted with Poissonian rates z_γ
- Call succeeds if it does not intercept existing calls
- Once established, calls have an exp(1) life span

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Low-T expansions

Ising model at low T:

- Polymers = connected closed surfaces (contours)
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- $z_\gamma = \exp\{-2\beta J\ |\gamma|\}$

LTE for Ising ferromagnets:

- $\mathcal{P}$ = connected families of (excited) bonds (contours)
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High-T expansions

General HTE:

- $\mathcal{P} = \{\text{connected finite subsets of bonds}\}$
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$$z_{\boldsymbol{B}} = \int_{\underline{\boldsymbol{B}}} \prod_{A \in \boldsymbol{B}} \left(\mathrm{e}^{-\beta\,\phi_A(\omega)} - 1\right) \bigotimes_{x \in \underline{\boldsymbol{B}}} \mu_E(d\omega_x)$$

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Random geometrical models

FK representation of Potts models:

- $\mathcal{P} = \{\gamma \subset\subset \mathbb{L}\}$
- $$z_\gamma \;=\; q^{-(|\gamma|-1)} \sum_{\substack{\boldsymbol{B} \subset \mathcal{B}_\gamma \\ (\gamma, \boldsymbol{B})\ \text{connected}}} \prod_{\{x,y\} \in \boldsymbol{B}} v_{x\,y}$$

  with $v_{x\,y} = \mathrm{e}^{\beta J_{x\,y}} - 1$
- Compatibility = non-intersection
- If $v\{x, y\} = -1 \rightarrow$ chromatic polynomial

  ($\beta \rightarrow \infty$ with $J_{x\,y} < 0$, i.e. zero-temperature antiferromagnetic Potts)
- General v_{xy}: multivariate version of Tutte polynomial.

Geometrical polymer models

Previous examples: polymers formed by points of a set

These are the original polymer models of Gruber and Kunz:

- A set $\mathbb{V}$ (sites)
- A family $\mathcal{P}$ of finite subsets of $\mathbb{V}$ (grains, connected sets)
- Activity values $(z_\gamma)_{\gamma\in\mathcal{P}}$
- The relation $\gamma \sim \gamma' \iff \gamma \cap \gamma' = \emptyset$

More generally: $\gamma = (\underline{\gamma}, \text{decoration})$, $\underline{\gamma} \subset \mathbb{V}$

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Generalization : Continuous polymer systems

More generally,

$$\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n} \quad\longrightarrow\quad \frac{1}{n!}\int_{\mathcal{P}_\Lambda^n} d\gamma_1\cdots d\gamma_n$$

where $d\gamma_1\cdots d\gamma_n$ is an appropriate product measure

Also, for book-keeping purposes: $z_\gamma = z\,\xi_\gamma$

That is, we consider measures on $\sum_n \mathcal{P}_\Lambda^n$ with projections on $\mathcal{P}_\Lambda^n$

$$\frac{1}{\Xi_\Lambda}\,\frac{z^n}{n!}\,\xi_{\gamma_1}\xi_{\gamma_2}\cdots\xi_{\gamma_n}\prod_{j<k}1\!\!1_{\{\gamma_j\sim\gamma_k\}}\,d\gamma_1\cdots d\gamma_n$$

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Cluster expansions

Write $\Xi_\Lambda(\boldsymbol{z})$ as *formal* exponentials of a *formal* series

$$\Xi_\Lambda(\boldsymbol{z}) \stackrel{\mathrm{F}}{=} \exp\Big\{\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n}\phi^T(\gamma_1,\ldots,\gamma_n)\,z_{\gamma_1}\ldots z_{\gamma_n}\Big\}$$

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- The series between curly brackets is the *cluster expansion*
- $\phi^T(\gamma_1,\ldots,\gamma_n)$: *Ursell* or *truncated* functions (symmetric)
- *Clusters*: Families $\{\gamma_1,\ldots,\gamma_n\}$ s.t. $\phi^T(\gamma_1,\ldots,\gamma_n)\neq 0$
- Clusters are *connected* w.r.t. "$\not\sim$"

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Pinned expansions

Telescoping, it is enough to consider one-polymer ratios

$$\Big[\log \frac{\Xi_\Lambda}{\Xi_{\Lambda\setminus\{\gamma_0\}}}\Big](\boldsymbol{z}) \stackrel{\mathrm{F}}{=} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}_\Lambda^n \\ \exists i:\, \gamma_i=\gamma_0}} \phi^T(\gamma_1,\ldots,\gamma_n)\, z_{\gamma_1}\ldots z_{\gamma_n}$$

Algebraically simpler alternative:

$$\Big[\frac{\partial}{\partial z_{\gamma_0}} \log \Xi_\Lambda\Big](\boldsymbol{z}) \stackrel{\mathrm{F}}{=}$$

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Classical cluster-expansion strategy

Find a Λ-*independent* polydisc

$$\mathcal{R} \;=\; \Big\{\boldsymbol{z} : |z_\gamma| \le \rho_\gamma \,,\, \gamma \in \mathcal{P}\Big\}$$

where cluster expansions converge *absolutely*

Equivalently, find $\rho_\gamma > 0$ *independent of* Λ such that

$$\Theta_{\gamma_0}(\boldsymbol{\rho}) \;:=\; \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\substack{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n \\ \exists i:\, \gamma_i=\gamma_0}} \big|\phi^T(\gamma_1,\ldots,\gamma_n)\big| \,\rho_{\gamma_1}\ldots\rho_{\gamma_n}$$

or

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) \;:=\; 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} \big|\phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n)\big| \,\rho_{\gamma_1}\ldots\rho_{\gamma_n}$$

are finite

Consequences

Within $\mathcal{R}$

- No Ξ_Λ has a zero (no phase transitions!)
- Explicit series expressions for free energy and correlations
- Explicit mixing

$$\left|\frac{\mathrm{Prob}(\{\gamma_0\,,\,\gamma_x\})}{\mathrm{Prob}(\{\gamma_0\})\,\mathrm{Prob}(\{\gamma_x\})}-1\right| = \left|\mathrm{e}^{F[d(\gamma_0,\gamma_x)]}-1\right|$$

  with $F(d) \to 0$ as $d \to \infty$
- Central limit theorem

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- No Ξ_Λ has a zero (no phase transitions!)
- Explicit series expressions for free energy and correlations
- Explicit mixing

$$\left|\frac{\mathrm{Prob}(\{\gamma_0\,,\,\gamma_x\})}{\mathrm{Prob}(\{\gamma_0\})\,\mathrm{Prob}(\{\gamma_x\})}-1\right| = \left|\mathrm{e}^{F[d(\gamma_0,\gamma_x)]}-1\right|$$

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- System of linear coupled equations for the correlations
- Method used by Gruber and Kunz

Classical (1982–4):

- Based on *tree-graph bound*
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Tree-graph bound

Classical approach valid only for

geometrical translation-invariant polymers

Basic inequality

$$\left|\phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n)\right| \;\leq\; \left|\mathcal{T}_{\mathcal{G}_{(\gamma_0,\gamma_1,\ldots,\gamma_n)}}\right|$$

where $\mathcal{T}_{\mathcal{G}} = \{\text{connected spanning trees of } \mathcal{G}\}$

Hence:

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) \;\leq\; \sum_{n\geq 0} \frac{1}{n!}\,\overline{T}_n(\gamma_0)$$

where $\overline{T}_0 = 1$ and

$$\overline{T}_n(\gamma_0) \;=\; \sum_{(\gamma_1,\ldots,\gamma_n)} \;\sum_{\tau\in\mathcal{T}_{\mathcal{G}_{(\gamma_0,\gamma_1,\ldots,\gamma_n)}}} \rho_{\gamma_1}\cdots\rho_{\gamma_n}$$

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Interchanging sum over polymers with sum over trees:

$$\overline{T}_n(\gamma_0) = \sum_{\tau\in\mathcal{T}^0_{n+1}} \sum_{\substack{(\gamma_1,\dots,\gamma_n) \text{ s.t.} \\ \tau\subset\mathcal{G}_{(\gamma_0,\gamma_1,\dots,\gamma_n)}}} \rho_{\gamma_1}\cdots\rho_{\gamma_n}$$

$$= \sum_{\tau\in\mathcal{T}^0_{n+1}} \overline{T}_\tau(\gamma_0)$$

where

$$\mathcal{T}^0_{n+1} = \{\text{trees of vertices } 0, 1, \dots n, \text{rooted in } 0\}$$

Summing “from leaves down”

To compute $\overline{T}_\tau$ start summing over γ's at leaves:

$$\prod_{j=1}^{s_i} \sum_{\gamma_{(i,j)} \nsim \gamma_i} \rho_{\gamma_{(i,j)}} = \Big[\sum_{\gamma \nsim \gamma_i} \rho_\gamma\Big]^{s_i}$$

For translation-invariant geometrical polymers,

$$\sum_{\gamma \nsim \gamma_i} \rho_\gamma \le |\gamma_i| \sum_{\gamma \ni 0} \rho_\gamma$$

Then, for each γ_i that is ancestor of leaves

$$\rho_{\gamma_i} \longrightarrow \rho_{\gamma_i} |\gamma_i|^{s_i} \Big[\sum_{\gamma \ni 0} \rho_\gamma\Big]^{s_i}$$

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Iterate! The sum over successive ancestors yields

$$\overline{T}_\tau(\gamma_0) \;\leq\; |\gamma_0| \prod_{i=0}^{n} \Big[\sum_{\gamma \ni 0} \rho_\gamma \; |\gamma|^{s_i} \Big]$$

- This bound depends only on $s_0, s_1, \ldots, s_n$
- The sum over trees τ brings a factor

$$\begin{array}{r} \#\text{ trees with coord. nbers} \\ s_0, s_1+1, \ldots, s_n+1 \end{array} \;=\; \binom{n}{s_0+1 \;\; s_1 \;\; \ldots \;\; s_n}$$

(Cayley formula)

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Classical criterion

In consequence

$$\overline{T}_n(\gamma_0) \;\le\; |\gamma_0|\; n! \sum_{\substack{s_0,s_1,\ldots,s_n \\ \sum s_i = n-1}} \prod_{i=0}^{n} \Big[\sum_{\gamma \ni 0} \rho_\gamma \, \frac{|\gamma|^{s_i}}{s_i!}\Big]$$

Hence

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) \;\le\; |\gamma_0| \sum_{n\ge 0} \Big[\sum_{\gamma\ni 0} \rho_\gamma \, \mathrm{e}^{|\gamma|}\Big]^n$$

which converges if

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[Cammarota (1982), Brydges (1984)]

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The two stages of the inductive approach

1) The addition identity:

$$\Xi_{\Gamma\cup\{\gamma\}} \;=\; \Xi_\Gamma + z_\gamma\, \Xi_{\Gamma\setminus\{\gamma\}^*} \tag{1}$$

for $\gamma \notin \Gamma$

2) The right inductive hypothesis:

There exist $a(\gamma) > 0$ and $\rho_\gamma > 0$ such that

$$\rho_\gamma \exp\Big[\sum_{\widetilde\gamma \not\sim \gamma} a(\widetilde\gamma)\Big] \;\le\; \mathrm{e}^{a(\gamma)} - 1$$

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The main result

Theorem

If there exist $a(\gamma) > 0$ *and* $\rho_\gamma > 0$ *such that*

$$\rho_\gamma \exp\Big[\sum_{\widetilde{\gamma} \not\sim \gamma} a(\widetilde{\gamma})\Big] \;\le\; \mathrm{e}^{a(\gamma)} - 1 \tag{2}$$

for all γ, *then*

$$\Theta^{\Lambda}_{\gamma}(-\boldsymbol{\rho}) \;\le\; a(\gamma) \tag{3}$$

uniformly in Λ *for all* γ

Proof

Must show

$$\frac{\Xi_{\Lambda \cup \{\gamma\}}(-\boldsymbol{\rho})}{\Xi_{\Lambda}(-\boldsymbol{\rho})} \;\geq\; \mathrm{e}^{-a(\gamma)} \tag{4}$$

By the addition identity

$$\frac{\Xi_{\Lambda \cup \{\gamma\}}(-\boldsymbol{\rho})}{\Xi_{\Lambda}(-\boldsymbol{\rho})} \;=\; 1 - \rho_\gamma \, \frac{\Xi_{\Lambda \setminus \{\gamma\}^*}(-\boldsymbol{\rho})}{\Xi_{\Lambda}(-\boldsymbol{\rho})}$$

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Kotecký-Preiss criterion

Most popular: If $\exists\ b(\gamma) > 0$ such that

$$\sum_{\widetilde{\gamma} \not\sim \gamma} \rho_\gamma \, \mathrm{e}^{b(\widetilde{\gamma})} \ \leq \ b(\gamma)$$

then convergence for $|z_\gamma| \leq \rho_\gamma$

- Can be proven
  - By "defoliation" of $\mathbf{\Pi}$ (Procacci-Scoppola)
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Kirkwood-Salzburg approach

Strategy: Set up systems of linear equations for the functions

$$\Phi_\Lambda(\boldsymbol{z}, X) \;=\; \frac{\Xi_{\Lambda\setminus X}(\boldsymbol{z})}{\Xi_\Lambda(\boldsymbol{z})}$$

involving a *basically* Λ-*independent* operator K.

- Search for solutions in a suitable Banach space
- Solutions = fixed points
- K contraction uniform in Λ yields
  - Convergence with Λ
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Derivation of the equations

- For each $X \subset\subset \mathcal{P}_\Lambda$ choose some (first) $\gamma \in X$
- Write addition identity as *deletion identity*, with $\Lambda \to \Lambda \setminus X$

$$\Xi_{\Lambda\setminus X} \;=\; \Xi_{\Lambda\setminus(X\setminus\{\gamma\})} - z_\gamma\, \Xi_{\Lambda\setminus(X\cup\{\gamma\}^*)}$$

- Dividing by Ξ_Λ

$$\Phi_\Lambda(X) \;=\; \Phi_\Lambda(X\setminus\{\gamma\}) - z_\gamma\, \Phi_\Lambda(X\cup\{\gamma\}^*)$$

- The identity $\Phi_\Lambda(\emptyset) = 1$ is considered as inhomogenity
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Kirkwood-Salzburg equations

The equations are:

$$\Phi_\Lambda \;=\; \chi_\Lambda \alpha + \chi_\Lambda K_{\boldsymbol{z}} \Phi_\Lambda$$

with

$$\chi_\Lambda(X) \;=\; \begin{cases} 1 & \text{if } X \subset \Lambda \\ 0 & \text{otherwise} \end{cases} \quad , \quad \alpha(X) \;=\; \begin{cases} 1 & \text{if } |X| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and $K_{\boldsymbol{z}}$ the operator on $\mathbb{C}^{\{\text{non-empty fin parts of } \mathcal{P}\}}$

$$(K_{\boldsymbol{z}} f)(X) \;=\; \mathbb{1}_{\{|X|\geq 2\}}\, f(X \setminus \{\gamma\}) - z_\gamma\, f(X \cup \{\gamma\}^*_\Lambda)$$

Standard treatment

Aiming at factorized weights, introduce norms

$$\|f\|_{\boldsymbol a} \;=\; \sup_{X\subset\subset\mathcal{P}} \frac{|f(X)|}{\exp\big[\sum_{\widetilde\gamma\in X} a(\widetilde\gamma)\big]}$$

for $a(\widetilde\gamma) > 0$. Then

$$|(K_{\boldsymbol z} f)(X)| \;\le\; \|f\|_{\boldsymbol a} \exp\Big[\sum_{\substack{\widetilde\gamma\in X\\ \widetilde\gamma\neq\gamma}} a(\widetilde\gamma)\Big] + |z_\gamma|\;\|f\|_{\boldsymbol a} \exp\Big[\sum_{\substack{\widetilde\gamma\in(X\setminus\{\gamma\})\cup\{\gamma\}^*_\Lambda\\ \widetilde\gamma\in\Lambda}} a(\widetilde\gamma)\Big]$$

and

$$\|K_{\boldsymbol z}\|_{\boldsymbol a} \;\le\; \frac{1}{\mathrm{e}^{a(\gamma)}}\Big[1+|z_\gamma|\exp\Big(\sum_{\widetilde\gamma\not\sim\gamma} a(\widetilde\gamma)\Big)\Big]$$

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Gruber-Kunz condition

If for some $\rho_\gamma > 0$

$$\frac{1}{\mathrm{e}^{a(\gamma)}}\Big[1+\rho_\gamma \exp\Big(\sum_{\widetilde\gamma \not\sim \gamma} a(\widetilde\gamma)\Big)\Big] \;<\; 1 \qquad (5)$$

then, for $|z_\gamma| \le \rho_\gamma$, the operators $1-\xi_\Lambda K_{\boldsymbol z}$ are invertible and

$$\Phi_\Lambda \;=\; \big[1-\xi_\Lambda K_{\boldsymbol z}\big]^{-1} \chi_\Lambda \alpha \qquad (6)$$

is the only solution of the Λ-KS-equation.

Furthermore, as (5) is Λ-independent,

- The ratios converge

$$\Phi_\Lambda(X) \xrightarrow[\Lambda\to\mathbb{V}]{} \big([1-K_{\boldsymbol z}]^{-1}\alpha\big)(X)$$

- The ratios, and their limits have analytic dependence on $\boldsymbol z$

Gruber-Kunz condition

If for some $\rho_\gamma > 0$

$$\frac{1}{\mathrm{e}^{a(\gamma)}}\Big[1+\rho_\gamma \exp\Big(\sum_{\widetilde{\gamma}\not\sim\gamma} a(\widetilde{\gamma})\Big)\Big] \;<\; 1 \tag{5}$$

then, for $|z_\gamma| \le \rho_\gamma$, the operators $1 - \xi_\Lambda K_{\boldsymbol{z}}$ are invertible and

$$\Phi_\Lambda \;=\; \big[1-\xi_\Lambda K_{\boldsymbol{z}}\big]^{-1} \chi_\Lambda \alpha \tag{6}$$

is the only solution of the Λ-KS-equation.

Furthermore, as (5) is Λ-independent,

- The ratios converge

$$\Phi_\Lambda(X) \xrightarrow[\Lambda\to\mathbb{V}]{} \big([1-K_{\boldsymbol{z}}]^{-1}\alpha\big)(X)$$

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Find another way of making sense of the formula

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This series is term-by-term dominated by

$$\Phi_{\boldsymbol{\rho}} \;=\; \sum_{n\geq 0} K_{-\boldsymbol{\rho}}^n\,\alpha$$

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(As in cluster expansions: singularities at negative fugacities)

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Inductive-like bound

Find functions $\xi(X)$ such that

$$\big(\alpha + K_{-\boldsymbol{\rho}}\, \xi\big)(X) \;\le\; \xi(X) \tag{8}$$

Recursively this implies that

$$\sum_{n=0}^{N} K_{-\boldsymbol{\rho}}^{n}\, \alpha \;\le\; \xi$$

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Reciprocally, if $\Phi_{\boldsymbol{\rho}}$ is finite, (8) holds with $\xi = \Phi_{\boldsymbol{\rho}}$

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Why factorization

If X_1 and X_2 are disjoint,

$$\Phi_\Lambda(X_1 \cup X_2) \;=\; \frac{\Xi_{\Lambda\setminus(X_1\cup X_2)}}{\Xi_{\Lambda\setminus X_2}}\,\frac{\Xi_{\Lambda\setminus X_2}}{\Xi_\Lambda} \;=\; \Phi_{\Lambda\setminus X_2}(X_1)\,\Phi_\Lambda(X_2)\;.$$

In the limit $\Lambda \to \mathbb{V}$ we should obtain

$$\Phi(X_1 \cup X_2) \;=\; \Phi(X_1)\,\Phi(X_2)\;.$$

It is natural, to look for factorized majorizing functions.

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GK alla Dobrushin recovered

Postulating

$$\xi(X) \;=\; \prod_{\gamma\in X} \xi(\gamma)$$

(8) holds for all X iff it holds for a single-site:

$$\big(\alpha + K_{-\boldsymbol{\rho}}\,\xi\big)(\{\gamma\}) \;\le\; \xi(\{\gamma\})$$

Writing $\xi(\{\gamma\}) = \mathrm{e}^{a(\gamma)}$, this condition is

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Comparison so far

Classical $<$ KP $<$ Inductive $=$ KS (GK)

However, alternative KS leads to a sequence of bounds:

$$\left|\frac{\Xi_{\Lambda\setminus X}(\boldsymbol{z})}{\Xi_\Lambda(\boldsymbol{z})}\right| \leq \frac{\Xi_{\Lambda\setminus X}(-\boldsymbol{\rho})}{\Xi_\Lambda(-\boldsymbol{\rho})} \leq (\mathbb{T}_{\boldsymbol{\rho}})^m \boldsymbol{\xi}^X \leq (\mathbb{T}_{\boldsymbol{\rho}})^n \boldsymbol{\xi}^X \leq \boldsymbol{\xi}^X$$

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“Standard form” of the criteria

If we substitute

$$\begin{aligned} \mu_\gamma &= \rho_\gamma \, e^{a_\gamma} \quad \text{(Kotecký-Preiss)} \\ \mu_\gamma &= e^{a_\gamma} - 1 \quad \text{(Dobrushin)} \end{aligned}$$

We obtain convergence if there exists $\boldsymbol{\mu} \in [0,\infty)^{\mathcal{P}}$ such that

$$\begin{aligned} \rho_{\gamma_0} \; \exp\Big[\sum_{\gamma \not\sim \gamma_0} \mu_\gamma\Big] &\le \mu_{\gamma_0} \quad \text{(Kotecký-Preiss)} \\ \rho_{\gamma_0} \prod_{\gamma \not\sim \gamma_0} \big(1 + \mu_\gamma\big) &\le \mu_{\gamma_0} \quad \text{(Dobrushin)} \end{aligned}$$

Comparison D $\leftrightarrow$ KP

D improves KP because

$$\prod_{\gamma \nsim \gamma_0} \left(1 + \mu_\gamma\right) \;\leq\; \exp\Big[\sum_{\gamma \nsim \gamma_0} \mu_\gamma\Big]$$

Differences:

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- D exact for polymers with only self-exclusion

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- It looks as a hierarchy of approximations
- Why induction better than control of explicit series?
- Dobrushin extracts extra information **Which one?**
- Why the form
$$\rho_{\gamma 0}\, \varphi_{\gamma 0}(\boldsymbol{\mu}) \;\le\; \mu_{\gamma 0}\ ?$$
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Classical revisited: 1st ingredient

Penrose identity

$$\left|\phi^T(\gamma_0,\gamma_1,\ldots,\gamma_n)\right| \;=\; \left|\mathcal{T}^{\rm Pen}_{\mathcal{G}_{(\gamma_0,\gamma_1,\ldots,\gamma_n)}}\right|$$

A Penrose tree for $\mathcal{G}_{(\gamma_0,\ldots,\gamma_n)}$ is a spanning tree s.t.

(P1) Brothers are compatible

(P2) Cousins are compatible

(P3) Nephews compatible with uncles with smaller index

Hence, now

$$\Pi_{\gamma_0}(\boldsymbol{\rho}) \;=\; \sum_{n\geq 0}\frac{1}{n!}\,\overline{T}_n(\gamma_0)$$

with

$$T_n(\gamma_0) \;=\; \sum_{(\gamma_1,\ldots,\gamma_n)}\;\sum_{\tau\in\mathcal{T}_{\mathcal{G}_{(\gamma_0,\gamma_1,\ldots,\gamma_n)}}}\rho_{\gamma_1}\cdots\rho_{\gamma_n}$$

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Approximation

Retain only (P1): Brothers may not be linked in $\mathcal{G}$

If $\{i, i_1\}$ and $\{i, i_2\}$ are edges of τ, then $\gamma_{i_1} \sim \gamma_{i_2}$

In this way $\boldsymbol{\rho}\boldsymbol{\Pi}(\boldsymbol{\rho}) \leq \boldsymbol{\rho}^*$, with

$$\rho^*_{\gamma_0} := \rho_{\gamma_0}\Big[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} \sum_{\tau\in\mathcal{T}_n^0} \prod_{i=0}^{n} c_{s_i}(\gamma_i, \gamma_{i_1}, \ldots, \gamma_{i_{s_i}})\rho_{\gamma_{i_1}} \cdots \rho_{\gamma_{i_{s_i}}}\Big]$$

where $i_1, \ldots, i_{s_i}$ = descendants of i and

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2nd ingredient: Iterative generation of trees

Consider the function $\boldsymbol{T_\rho} : [0,\infty)^{\mathcal{P}} \to [0,\infty]^{\mathcal{P}}$ defined by

$$\Big(\boldsymbol{T_\rho}(\boldsymbol{\mu})\Big)_{\gamma_0} \;=\; \rho_{\gamma_0}\Big[1+\sum_{n\geq 1}\frac{1}{n!}\sum_{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n} c_n(\gamma_0,\gamma_1,\ldots,\gamma_n)\,\mu_{\gamma_1}\ldots\mu_{\gamma_n}\Big]$$

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Diagrammatically:

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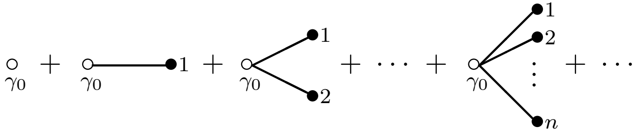

Summing "from the roots up"

The diagrams of the series

$$T_\rho(T_\rho(\mu)) \;=\; T_\rho^2(\mu)$$

have black dots replaced by each of the preceding diagrams.

$T_\rho^2(\mu)$ = sums over trees of up to two generations with • in 2nd generation

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and

$$T_\rho^n(\rho) \underset{n\to\infty}{\nearrow} \rho^*$$

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$\boldsymbol{T}_{\boldsymbol{\rho}}^{n}(\boldsymbol{\rho})$ converges *if, and only if*, exists $\boldsymbol{\mu}$ s.t.

$$\boldsymbol{T}_{\boldsymbol{\rho}}(\boldsymbol{\mu}) \ \leq \ \boldsymbol{\mu} \tag{9}$$

Indeed, (9) + positiveness $\Rightarrow \boldsymbol{T}_{\boldsymbol{\rho}}^{n}(\boldsymbol{\mu})$ decreasing and bdd below

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New criterion

For

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&= \rho_{\gamma_0}\,\Xi_{\{\gamma_0\}^*}(\boldsymbol{\mu})
\end{aligned}$$

Hence, convergence if

$$\rho_{\gamma_0}\,\Xi_{\{\gamma_0\}^*}(\boldsymbol{\mu}) \;\le\; \mu_{\gamma_0}$$

Furthermore,

$$\boldsymbol{\rho}\,\boldsymbol{\Pi} \;\le\; \boldsymbol{\rho}^* \;\le\; \boldsymbol{T}_{\boldsymbol\rho}^{n+1}(\boldsymbol\mu) \;\le\; \boldsymbol{T}_{\boldsymbol\rho}^{n}(\boldsymbol\mu) \;\le\; \boldsymbol\mu$$

Same for continuous polymers with $\sum_{\gamma\in\mathcal{P}_n} z_\gamma \to z^n\int_{\mathcal{P}_n} d\gamma$

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$$c_n(\gamma_0,\gamma_1,\ldots,\gamma_n) \;=\; \prod_{i=1}^{n} \mathbb{1}_{\{\gamma_0 \nsim \gamma_i\}} \prod_{j=1}^{n} \mathbb{1}_{\{\gamma_i \sim \gamma_j\}}$$

$$\begin{aligned}\Big(\boldsymbol{T_\rho}(\boldsymbol{\mu})\Big)_{\gamma_0} \;&=\; \rho_{\gamma_0}\Big[1+\sum_{n\ge 1}\frac{1}{n!}\sum_{\substack{(\gamma_1,\ldots,\gamma_n)\in\mathcal{P}^n\\ \gamma_0\nsim\gamma_i\,,\,\gamma_i\sim\gamma_j\,,\,1\le i,j\le n}} \mu_{\gamma_1}\cdots\mu_{\gamma_n}\Big]\\ &=\; \rho_{\gamma_0}\,\Xi_{\{\gamma_0\}^*}(\boldsymbol{\mu})\end{aligned}$$

Hence, convergence if

$$\rho_{\gamma_0}\,\Xi_{\{\gamma_0\}^*}(\boldsymbol{\mu}) \;\le\; \mu_{\gamma_0}$$

Furthermore,

$$\boldsymbol{\rho}\,\boldsymbol{\Pi} \;\le\; \boldsymbol{\rho}^* \;\le\; \boldsymbol{T}_{\boldsymbol{\rho}}^{n+1}(\boldsymbol{\mu}) \;\le\; \boldsymbol{T}_{\boldsymbol{\rho}}^{n}(\boldsymbol{\mu}) \;\le\; \boldsymbol{\mu}$$

Same for continuous polymers with $\sum_{\gamma\in\mathcal{P}_n} z_\gamma \to z^n \int_{\mathcal{P}_n} d\gamma$

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Explanation of the different criteria

If we replace $\gamma_i \sim \gamma_j$ by the weaker requirement $\gamma_i \neq \gamma_j$:

$$c_n^{\mathrm{Dob}}(\gamma_0, \gamma_1, \ldots, \gamma_n) = \prod_{i=1}^{n} \mathbb{1}_{\{\gamma_0 \nsim \gamma_i\}} \prod_{j=1}^{n} \mathbb{1}_{\{\gamma_i \neq \gamma_j\}}$$

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If requirement $\gamma_i \nsim \gamma_j$ is ignored altogether,

$$c_n^{\mathrm{KP}}(\gamma_0, \gamma_1, \ldots, \gamma_n) = \prod_{i=1}^{n} \mathbb{1}_{\{\gamma_0 \nsim \gamma_i\}}$$

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Comparison classical revisited $\leftrightarrow$ inductive

The improvement is expressed by the inequality

$$\Xi_{\mathcal{N}^*_{\gamma_0}}(\boldsymbol{\mu}) \;\leq\; \prod_{\gamma \not\sim \gamma_0} \big(1 + \mu_\gamma\big)$$

LHS contains only monomials of *mutually compatible* polymers

Sources of improvement:

(I1) $\Xi_{\{\gamma_0\}^*}$ has no triangle diagram (i.e. pairs of neighbors of γ_0 that are themselves neighbors)

(I2) In $\Xi_{\{\gamma_0\}^*}$, the only monomial containing μ_{γ_0} is μ_{γ_0} itself, (γ_0 is incompatible with all other polymers in $\{\gamma_0\}^*$)

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Directions for further research

- Incorporation of additional constraints in Penrose trees
- Use of other partition schemes
- Inductive proof?
- Extension to polymers with soft interactions (in progress)
- Uncountably many polymers (eg. quantum contours)
- Revisit “classical” results based on cluster expansions

Part II

Alternative probabilistic scheme

The alternative treatment has the following features:

- It is probabilistic, hence only positive activities
- Basic measures = invariant measures for point processes
- Larger region of validity, but no analyticity
- Yields a "universal" perfect simulation scheme

Probabilistic approach (with P. Ferrari and N. Garcia)

Basic measures are invariant for the following dynamics:

- Attach to each polymer γ a poissonian clock with rate z_γ
- When the clock rings, γ tries to be born
- It succeeds if no other γ' present with $\gamma \nsim \gamma'$
- Once born, the polymer has an exp(1) lifespan

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Alternative scheme

1st step: free process

- Generate first a *free process* where *all* birth are succesful
- Associate to each born polymer γ a space-time *cylinder*

$$C^\gamma = \left(\gamma, [\text{Birth}_{C^\gamma}, \text{Death}_{C^\gamma}]\right)$$

2nd step: cleaning

To decide whether a given cylinder C^γ remains alive, determine its *clan of ancestors*

$$\begin{aligned} \mathbf{A}_1(C^\gamma) &= \left\{C' : \text{Base}_{C'} \not\sim \gamma, \text{Birth}_{C^\gamma} \in [\text{Birth}_{C'}, \text{Death}_{C'}]\right\} \\ \mathbf{A}_{n+1}(C^\gamma) &= \mathbf{A}_1\left(\mathbf{A}_n(C^\gamma)\right) \\ \mathbf{A}(C^\gamma) &= \cup_n \mathbf{A}_n(C^\gamma) \end{aligned}$$

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\end{aligned}$$

Forward-forward scheme

If $\boldsymbol{A}(C^\gamma)$ is finite. do the cleaning starting from the "mother cylinder"

- Keep mother
- Erase first children
- Keep new mothers
- $\vdots$

This is a *forward-forward* scheme

Backward-forward scheme

Ancestors clan can be constructed backwards
(Poisson and exponential distributions are reversible)

To construct the clan of ancestors of a finite window Λ:

- Generate, backwards, marks at rate $z_\gamma \, \mathrm{e}^{-s}$ for each $\gamma \not\sim \Lambda$
- These are cylinders born at $-s$ and surviving up to 0
- Take the first mark; ignore the rest. If its basis is γ_1
- Repeat with

$$\Lambda \;\to\; \Lambda \cup \{\gamma_1\}$$

$$s \;\to\; s - \begin{cases} \mathrm{Birth}_{\gamma_1} & \text{if } \gamma \not\sim \gamma' \\ 0 & \text{if } \gamma \not\sim \Lambda, \gamma \sim \gamma_1 \end{cases}$$

- $\cdots \longrightarrow \mathbf{A}^{\Lambda}$

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Perfect simulation

If

$$\mathbb{P}(\{\mathbf{A}^\Lambda \text{ finite}\}) = 1 \tag{10}$$

cleaning leads *exactly* to a sample of the basic measure

Sufficient conditions for (10)?

- Clan of ancestors defines an *oriented percolation model*
- Lack of percolation $\Longrightarrow$ (10)
- Can dominate by a branching process:
  - branches = ancestors
  - branching rate = mean surface-area of cylinders:

$$\frac{1}{|\gamma|} \sum_{\theta \not\sim \gamma} |\theta| \; z_\theta \;\times\; 1$$

(geometrical case)

Extinction condition

Extinction of the branching process implies (10)

Hence, perfect simulation if

$$\frac{1}{|\gamma|}\sum_{\theta\not\sim\gamma}|\theta|\;z_\theta\;\leq\;1$$

Under this condition

- Prob $=\lim_\Lambda \text{Prob}_\Lambda$ exists
- Mixing properties

$$\Big|\text{Prob}(\{\gamma_0,\gamma_1\})-\text{Prob}(\{\gamma_0\})\,\text{Prob}(\{\gamma_1\})\Big|\;\leq\;\mathrm{e}^{-M\,\text{dist}(\gamma_0,\gamma_1)}$$

- CLT: If A depends on a finite # of polymers

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- Universal perfect simulation algorithm
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